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ON ALMOST COSYMPLECTIC LORENTZIAN HYPERSURFACES IMMERSED IN A LORENTZIAN MANIFOLD

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Let x: $V^{2n+1}\rightarrow V^{2n+2}$ be an immersion of a Lorentzian hypersurface into a Lorentzian manifold and let $2n+1$ be the time-like index of the metric of V^{2n+1}. This paper is concerned with a type of hypersurface V^{2n+1} (denoted by $\tilde{V}^{2n+1}$) such that the principal curvatures of $\tilde{V}^{2n+1}$ be related by $\sum_i k_i=-2nk_{2n+1}$ $(i=1, 2, \cdots, 2n)$. This condition corresponds to a certain geometrical property of the null real fields on $\tilde{V}^{2n+1}$. Next a certain almost cosymplectic structure $\boldsymbol{C}$ is considered on $\tilde{V}^{2n+1}$ and the necessary and sufficient conditions that the canonical field of $\boldsymbol{C}$ be concurrent over $\tilde{V}^{2n+1}$ (in the sense of K. Yano and B. Y. Chen) are established. Finally in the special case when the structure $\boldsymbol{C}$ is a Pfaffian structure, the infinitesimal automorphism of a null real field on $\tilde{V}^{2n+1}$ is investigated.

1. Preliminaries.

Let V^{2n+2} be a Lorentzian manifold (having a hyperbolic signature) and let x: $V^{2n+1}\rightarrow V^{2n+2}$ be an isometric immersion of an orientable Lorentzian hypersurface (V^{2n+1} has a Lorentzian structure in the tangent bundle [3]). Let $F(V^{2n+1})$ and $F(V^{2n+2})$ be the orthonormal frame bundles of V^{2n+1} and V^{2n+2} respectively, and $B\subset V^{2n+1}\times F(V^{2n+1})$ the principal fiber bundle of the adapted frames ($p\in V^{2n+1}$, $x(p)$, $e_1, \cdots, e_{2n+1}, e_{2n+2}$) such that e_α $(\alpha, \beta, \gamma=1, 2, \cdots, 2n+1)$ are unit tangent vectors and $e_{2n+2}=n$ is the normal unit vector at $x(p)$. Next we denote by e_r $(r, s, t=1, 2, \cdots, 2n, 2n+2)$ the space-like vectors of any frame $b\in B$ and by e_i $(i, j, k=1, 2, \cdots, 2n)$ the space like vectors of the tangent space $T_p(V^{2n+1})$ or V^{2n+1} at p. If ω^A and $\omega_B^A=\gamma_{BC}^A\omega^C$ $(A, B, C=1, 2, \cdots, 2n, 2n+1, 2n+2)$ are the 1-forms on B induced from the natural immersion $B\rightarrow F(V^{2n+2})$, we may write

$$dp=\omega^{2n+1}\otimes e_{2n+1}-\omega^i\otimes e_i. \tag{1}$$

The hypersurface V^{2n+1} is then structured by the connection

$$\begin{aligned}\nabla e_r&=\omega_r^A\otimes e_A,\\ \nabla e_{2n+1}&=-\omega_{2n+1}^A\otimes e_A\end{aligned} \tag{2}$$

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and both groups of structural equations are

$$
\begin{aligned}
&d\wedge\omega^r=\omega^A\wedge\omega^r_A,\\
&d\wedge\omega^{2n+1}=-\omega^A\wedge\omega^{2n+1}_A,\\
(3)\qquad &d\wedge\omega^{2n+1}_r=\Omega^{2n+1}_r+\omega^A_r\wedge\omega^{2n+1}_A,\\
&d\wedge\omega^r_s=\Omega^r_s+\omega^t_s\wedge\omega^r_t-\omega^{2n+1}_s\wedge\omega^r_{2n+1}
\end{aligned}
$$

where Ω^A_B are the curvature 2-forms.

2. $\tilde{V}^{2n+1}$ hypersurfaces.

By means of a transformation of the group $\mathcal{L}(2n+1)$ it is possible to choose a frame $b\in B$, so as to bring the second fundamental form $\varphi=-\langle dn, dp\rangle$ associated with x, into diagonal form. The frame b is then called *principal* (e_α are tangent to the principal lines) and if k_α are the *principal curvatures* at p, we have

$$\omega^{2n+2}_\alpha=k_\alpha\omega^\alpha \qquad \text{(no summation)}$$

Assuming that the orientation of b is such that

$$[e_1, \cdots, e_{2n+1}]=in, \qquad i=\sqrt{-1}$$

we shall define following Amur [1] the *elementary symmtric functions* H_α of k_α by

$$(4)\qquad \text{“}\,\|\,\text{”}\underbrace{dn, \cdots, dn,}_{\alpha}\ \underbrace{dp, \cdots, dp}_{2n+1-\alpha}\text{“}\,\|\,\text{”}=i\alpha!\cdot(2n+1-\alpha)!\cdot\binom{2n+1}{\alpha}H_\alpha\eta n$$

In (4) “ || ” ··· “ || ” denotes the combined operation of exterior product and vector product in V^{2n+2} and η is the volume element of V^{2n+2}. By means of (1) and (2) one finds

$$(5)\qquad \binom{2n+1}{\alpha}H_\alpha=\sum\varepsilon_1k_1\cdots\varepsilon_\alpha k_\alpha; \qquad 1\leqq\alpha\leqq 2n+1$$

where

$$\varepsilon_i=1, \qquad \varepsilon_{2n+1}=-1$$

As in [1] an immediate consequence of (4) is that for a *compact*[1] hypersurface V^{2n+1} we have the integral equation

$$\int_{V^{2n+1}}H_\alpha\eta n=0.$$

1) See Techniques of differential topology in relativity by R. Penrose, Dept. of Math., Univ. of Pittsburg, U.S.A.

3. Since the tangent vector e_{2n+1} is time-like, we may express any *real null vector* field $I\subset T_p(V^{2n+1})$ by

$$(6)\qquad I=f\sum e_i \cos\theta_i \pm f e_{2n+1}; \qquad -\frac{\pi}{2}\leqq\theta_i\leqq\frac{\pi}{2}; \qquad f\in\mathcal{D}(V^{2n+1})$$

We shall now inquire, under what conditions a field I_b which pseudobissects the principal lines on V^{2n+1}, is an asymptotic direction of V^{2n+1} at p (not every hypersurface posseses a symptotic direction).

If ρ_b is the curvature of V^{2n+1} in the direction I_b, then as is known, the necessary and sufficient condition that I_b be an asymptotic direction is that ρ_b be null. By virtue of (6) and making use of (1) and (2) we find that $\rho_b=0$ for any null real field I_b on V^{2n+1} if and only if one has

$$\text{a)}\qquad \sum_i k_i+2nk_{2n+1}=0.$$

Hence we have the

THEOREM. *Let x: $V^{2n+1}\to V^{2n+2}$ be an immersion of a Lorentzian hypersurface into a Lorentzian manifold V^{2n+2}. If k_i $(i=1,\cdots,2n)$ and k_{2n+1} are the space-like and time-like index principal curvatures at $p\in V^{2n+1}$ respectively, then the necessary and sufficient condition that any null real vector field, pseudobissecting at p the principal lines of V^{2n+1}, be an asymptotic direction, is that condition* a) *holds.*

A Lorentzian hypersurface fulfilling condition a) will be denoted by $\tilde{V}^{2n+1}$, and we shall consider the following two cases

(i) If the immersion x is *umbilical* (i.e. φ is conformal to $ds^2=\langle dp, dp\rangle$) the number of principal curvatures of $\tilde{V}^{2n+1}$ in V^{2n+2} is two and taking account of condition a) we have

$$(7)\qquad k_1=k_2=\cdots=k_{2n}=-k_{2n+1}$$

In this case we easly see, that the *parallel map* $\mathcal{A}$ (or *dilatation*) defined by $\mathcal{A}$: $p\to p+cn$, (c=const.) is *conformal* and if the manifold V^{2n+2} is locally flat, then $\tilde{V}^{2n+1}$ is a Lorentzian *hypersphere*.

(ii) By means of (5) one readily finds that the immersion is *minimal* if and only if the time-like index principal curvature k_{2n+1} is null.

4. Almost cosymplectic structure $C(\Omega\ \omega)$ on $\tilde{V}^{2n+1}$.

Assume now that

$$(8)\qquad \Omega=\omega^1\wedge\omega^2+\cdots+\omega^{2n-1}\wedge\omega^{2n},$$

$$(8')\qquad \omega=\omega^{2n+1}$$

define an *almost cosymplectic structure* $C(\Omega,\omega)$ on V^{2n+1} (If $(S_p(n,R)$ is the real *2n-dimensional symplectic group,* then an almost cosymplectic structure is a

$1\times S_p(n, R)$-structure [4]). According to Reeb's lemma, there exist uniquely a global vector field $E\in T(V^{2n+1})$, called the *canonical field* of $\boldsymbol{C}(\Omega, \omega)$, which is defined by

$$(8'') \qquad E \rfloor \omega=1, \qquad E \rfloor \Omega=0.$$

In the case under discussion, and in consequence of (8) and (8′) we easily get

$$E=e_{2n+1}$$

Suppose now that any field X in the direction of E is a *concurrent vector field* over V^{2n+1}. Following Yano and Chen [8] we must write

$$(9) \qquad dp+\nabla X=0.$$

Denoting by $\mu\in\mathcal{D}(V^{2n+1})$ a scalar factor, we get from (9) with the help of (1) and (2)

$$(10) \qquad \omega=-d\mu,$$

$$(11) \qquad \omega_i^{2n+1}=\omega^i/\mu,$$

$$(12) \qquad k_{2n+1}=0.$$

Equation (10) shows that the 1-form ω associated with $\boldsymbol{C}$ is a *coboundary*, and in consequence of (ii) it follows from (12) that the immersion x is *minimal*. On the other hand, (referring to the connection (2′) it is easy to see by means of (11) and (12), that ∇E is *conformal* to the projection dp_H of the line element dp, on the horizontal space H associated with $\boldsymbol{C}$.

Hence we may formulate the

THEOREM. *Let x: $\tilde{V}^{2n+1}\rightarrow V^{2n+2}$ be an immersion of a Lorentzian hypersurface fulfilling condition* a) *and let define on $\tilde{V}^{2n+1}$ an almost cosymplectic structure $\boldsymbol{C}(\Omega, \omega)$ such that the* 1*-form ω of $\boldsymbol{C}$ be the time-like index dual form associated with x. If E and H are the cannonical vector field and the horizontal space associated with $\boldsymbol{C}$ respectively, and dp_H the projection of the line element dp on it, then the necessary and sufficient condition that the field μE be concurrent over $\tilde{V}^{2n+1}$ is that*

$$\omega=-d\mu, \qquad \nabla E=-dp_H/\mu$$

and in this case the immersion x is minimal.

5. Sectional curvature.

According to what have been said at section 3, any horizontal vector field associated with the considered almost cosymplectic structure $\boldsymbol{C}$, may be expressed by

$$H=\textstyle\sum_i h_i e_i \qquad (i=1, 2, \cdots, n).$$

Under the assumption that the manifold V^{2n-2} is a 1-index Minkowski space, we consider the *sectional curvature* $K(\Pi)$ for the tangent plane element Π at $p\in\tilde{V}^{2n+1}$, spanned by H and any field E' in the direction of E. Setting according to Ōtsuki [6]

$$A(X)=\sum_{\alpha,\beta}A_{\alpha\beta}X_\beta e_\alpha$$

($X=\sum X_\alpha e_\alpha$ is any tangent vector field and $\omega_\alpha{}^{2n+2}=A_{\alpha\beta}\omega^\beta$), one has the general formula

$$K(\Pi)=P/G$$

where

$$P=\{\langle A(H), H\rangle\}\{\langle A(E'), E'\rangle\}-\{\langle A(H), E'\rangle\}^2,$$

$$G=\{||H||^2\}\{||E'||^2\}-\{\langle H, E'\rangle\}^2.$$

But in the case under discussion, H and E' are orthogonal,and

$$A_{\alpha\beta}=\begin{pmatrix} k_1 & & 0 \\ \vdots & \ddots & \\ 0 & & k_{2n+1}\end{pmatrix}.$$

It follows by straight forward calculation

$$K(\Pi)=k_{2n+1}\sum_i k_i h_i^2/\sum_i h_i^2.$$

From the above formula we find that in the case (i) section 3, one has

$$K(\Pi)=-(k_{2n+1})^2.$$

Hence we have the

THEOREM. *Suppose the almost cosymplectic hypersurface $\tilde{V}^{2n+1}$ defined at section 4, is immersed in a* 1*-index Minkowski space M^{2n+2}. Denote by H and E' any horizontal vector field and any vector field in direction of the canonical field associeted with the almost cosymplectic structure on $\tilde{V}^{2n+1}$, respectively. If the immersion x: $\tilde{V}^{2n+1}\to M^{2n+2}$ is umbilical, then the sectional curvature at any point $p\in\tilde{V}^{2n+1}$ spanned by H and E' is the negative of the square of the time-like index principal curvature of $\tilde{V}^{2n+1}$ at p.*

6 Immersion of the horizontal manifold of $\tilde{V}^{2n+1}$ associated with the structure $C(\Omega, \omega)$.

Since $\omega=-df$ we shall now consider the integral manifold V^{2n} of

(13) $$\omega=0$$

and the immersion $\bar{x}$: $V^{2n}\to V^{2n+2}$. The line element $d\bar{p}_H$ of V^{2n} being the restriction of dp_H to V^{2n}, we shall call V^{2n}, the 2-codimensional horizontal manifold associated with the almost cosymplectic structure $C(\Omega, \omega)$.

The second fundamental forms associated with $\bar{x}$ are

$$\bar{\varphi}_{2n+2} = -\langle d\bar{p}, \nabla e_{2n+1}\rangle = \sum_i \bar{k}_i(\bar{\omega}^i)^2, \tag{14}$$

$$\bar{\varphi}_{2n+1} = -\langle d\bar{p}, \nabla e_{2n+2}\rangle = -\sum_i(\bar{\omega}^i)^2/\bar{\mu} \tag{14'}$$

where $\bar{\omega}^i$, $\bar{k}_i$ and $\bar{\mu}$ are the induced values of ω^i, etc. by $\bar{x}$. Taking account of condition a), we have $\sum_i \bar{k}_i = 0$ and it follows from (14) and (14′) that the *mean quadratic from II* associated with $\bar{x}$ is

$$II = \frac{2n}{\bar{\mu}}\bar{\varphi}_{2n+1}.$$

But by virtue of (14), we see that II is conformal to the metric $\langle d\bar{p}_H, d\bar{p}_H\rangle$ of V^{2n}. Thus following a known theorem we conclude that the immersion $\bar{x}$ is *pseudo-umbilical* [5] Further a normal *non null* vector N is defined by

$$N = r(\operatorname{ch}\alpha\, e_{2n+1} + \operatorname{sh}\alpha\, e_{2n+2}); \qquad r, \alpha \in \mathcal{D}(V^{2n}) \tag{15}$$

and with the aid of (2) and (2′) we get

$$\begin{aligned} \nabla N = {} & r\sum_i\left(\frac{\operatorname{ch}\alpha}{\bar{\mu}} - \bar{k}_i \operatorname{sh}\alpha\right)\bar{\omega}^i e_i + r\left(\frac{dr}{r}\operatorname{ch}\alpha + d\alpha \operatorname{sh}\alpha\right)e_{2n+1} \\ & + r\left(\frac{dr}{r}\operatorname{sh}\alpha + d\alpha\operatorname{ch}\alpha\right)e_{2n+2}. \end{aligned} \tag{15'}$$

The above expression of ∇N shows that there does not exist for $\bar{x}$: $V^{2n} \to V^{2n+2}$ a nowhere vanishing normal vector field N such that $\nabla N = 0$. Consequently $\bar{x}$: $V^{2n} \to V^{2n+2}$ is a *substantial immersion* [8], and we may formulate the

THEOREM. *Being given a hypersurface $\tilde{V}^{2n+1} \subset V^{2n+2}$ let $\boldsymbol{C}(\Omega, \omega)$ an almost cosymplectic structure on $\tilde{V}^{2n+1}$ such that the 1-form ω of $\boldsymbol{C}$ be the timelike index dual form of $\tilde{V}^{2n+1}$. Then the immersion $\bar{x}$: $V^{2n} \to V^{2n+2}$ of the horizontal 2-codimensional manifold associated with $\bar{x}$ is substantial and pseudo-umbilical.*

REMARK. X, Z being two tangent vector fields at $\bar{p}$ to V^{2n}, consider the *shape operator* $S_X(Z)$ [7] of V^{2n} in V^{2n+2}. Since

$$S_X(Z) = \bar{\gamma}_{ij}{}^{i^*} X^i Z^i e_{i^*}, \qquad (i^* = 2n+1, 2n+2)$$

one finds taking account of (11)

$$S_X(Z) = -\frac{1}{\mu}\langle X, Z\rangle e_{2n+1} + (\sum_i k_i Z^i X^i)e_{2n+2}. \tag{16}$$

Hence if X, Z are orthogonal, then $S(Z)$ is the direction of the normal to $\tilde{V}^{2n+1}$ at the homologous point p of $\bar{p}$. On the other hand, the orthogonal complement

$T_{\bar{p}}^{\perp}(V^{2n})$ of $T_{\bar{p}}(V^{2n})$ at $\bar{p}$ being a time-like 2-flat (or a Lorentzian 2-flat), it contains two null real vector fields, namely

$$N_1=\lambda_1(e_{2n+1}+e_{2n+2}), \qquad N_2=\lambda_2(e_{2n+1}-e_{2n+2}).$$

Thus we see from (16) that if

$$\sum k_i Z^i X^i=\varepsilon\langle X, Z\rangle/\mu, \qquad \varepsilon=\pm 1, \tag{17}$$

then the shape operator $S_X(Z)$ at $\bar{p}$ is in the direction of one of the two null real vectors which span the total normal plane of V^{2n} at $\bar{p}$. Consequently, for a given tangent field $X\in T_{\bar{p}}(V^{2n})$, the vectors Z such that condition (17) is fulfilled define a $(2n-1)$-subspace of $T_{\bar{p}}(V^{2n})$ and if the manifold V^{2n} is not totally geodetic the operation of *reflection* is possible [2].

7. Hypersurfaces $\tilde{V}^{2n+1}$ with Pfaffian structure and concurrent cannocial field.

Following a theorem of K. Yano and B. Y. Chen [8], being given the immersion x: $M^n\to R^m$ (M^n and R^m are Riemannian manifolds) if the normal field N is concurrent of M^n in R^m, then N has constant length and is parallel in the normal bundle and M^n is umbilical in the direction of N.

Coming back to the immersion x: $\tilde{V}^{2n+1}\to V^{2n+2}$ and putting $N=\lambda e_{2n+2}$, $\lambda\in\mathcal{D}(\tilde{V}^{2n+1})$ for the normal vector field at $p\in\tilde{V}^{2n+1}$, we get from $dp+\nabla N=0$

$$\lambda=\text{const}, \tag{18}$$

$$k_1=k_2=\cdots=-k_{2n+1}=-1/\lambda. \tag{18$'$}$$

If we refer to the case (i) from section 3, conditions (18) and (18′) show that the above theorem is also valid for the immersion x: $\tilde{V}^{2n+1}\to V^{2n+2}$ which satisfies the additional condition that the curvatures k_a are all constant.

REMARK. Making use of equations (4), one readly finds that if conditions (18′) and (18′) fulfilled, then all transversal curvature forms $\Omega_\alpha{}^{2n+2}$ vanish. Further we shall assume that the almost cosymplectic structure $\boldsymbol{C}(\Omega, \omega)$ defined at section (4) is a Pfaffian structure, (denoted by $\boldsymbol{C}_p$) that is

$$\Omega=d\wedge\omega. \tag{19}$$

The canonical field E becomes now the dynamical vector field associated ted with the Pfaffian structure $\boldsymbol{C}_p(d\wedge\omega, \omega)$.

Any tangential vector field X of $\tilde{V}^{2n+1}$ may be written

$$X=fE+HX \tag{20}$$

where HX is the horizontal component of X. As is known [4] X is an *infinitesimal automorphism* of the Pfaffian structure $\boldsymbol{C}_p$ if

$$L_X\omega=0. \tag{21}$$

Taking account of (20) one gets

$$L_X\omega=df+HX\rfloor(d\wedge\omega) \tag{22}$$

and in this case $f=X\rfloor\omega$ is the *basic function* of X, that is $E\rfloor df=0$ [4].

If we write $HX=\Sigma_i h_i e_i$ we find from (22) and (8)

$$df+h_1\omega^2+\cdots+h_{2n-1}\omega^{2n}-h_2\omega^1-\cdots-h_{2n}\omega^{2n-1}=0. \tag{23}$$

It follows from (23) that

$$\partial_a f=h_{a+1}, \qquad \partial_{\bar a} f=-h_{\bar a-1} \tag{24}$$

where ∂_i denotes the Pfaffian derivate and the numbers $1\leqq a\leqq 2n-1$ and $2\leqq \bar a\leqq 2n$ are odd and even respectively. Consequently an infinitesimal automorphism $X(f)$ for the considered Pfaffian structure $\boldsymbol{C}_p$ on $\tilde V^{2n+1}$ is expressed by

$$X=fe_{2n+1}+\textstyle\sum_a\partial_a fe_{a+1}-\sum_{\bar a}\partial_{\bar a} fe_{\bar a-1}. \tag{25}$$

From (25) it follows that the necessary and sufficient condition that $X(f)$ be a null real vector field is that

$$d\lg f=\frac{1}{\sqrt{n}}\sum_i \varepsilon_i\omega^i=\tilde\omega. \tag{26}$$

Hence we may state the

THEOREM. *If the almost cosymplectic structure on an $\tilde V^{2n+1}$ hypersurface is a Pfaffian structure $\boldsymbol{C}_p$ then the necessary and sufficient condition that an infinitesimal automorphism $X(f)$ be a null real vector field is that there exist a* 1*-form $\tilde\omega$ associated with a horizontal principal pseudobissecting line on $\tilde V^{2n+1}$ such that $\tilde\omega$ be a coboundary.*

REMARK. $X(f)$ and $y(f)$ being two infinitesimal automorphism of $\boldsymbol{C}_p$, ne has [4]

$$[X, Y]\rfloor\omega=X\rfloor dg-gE\rfloor df=HX\rfloor dg$$

and if $X(f)$ and $Y(g)$ are both null real fields, we deduce from (26)

$$[X, Y]=(*)fg$$

where $(*)$ is a constant factor.

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