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A PROOF OF THE BIEBERBACH CONJECTURE FOR THE FOURTH COEFFICIENT

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1. Introduction. Let $f(z)$ be a normalized regular function univalent in the unit circle $|z|<1$

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n.$$

So far as the present author knows, up to the present time, there have appeared five proofs of $|a_4|\leqq 4$; [2], [3], [4], [5], [7]. In this paper we shall give another proof of $|a_4|\leqq 4$.

THEOREM 1.

$$\Re\{a_4-2a_2a_3+a_2^3+2\alpha(a_3-a_2^2)+(1+\alpha)^2a_2\}\leqq 2(1+\alpha+\alpha^2)$$

for $\alpha\geqq 0$.

We shall give here two proofs of this theorem. One is due to Schiffer's variational method together with Bombieri's recent result [1] and the other is due to Grunsky's inequality [6]. Then we shall prove

THEOREM 2. $|a_4|\leqq 4$. *Equality occurs only for* $z/(1-e^{i\varepsilon}z)^2$, ε: *real.*

It is well known that Grunsky's inequality gives a quite easy proof of $|a_4|\leqq 4$ [2]. Hence our proof should be considered as a non-elementary proof from a methodological point of view. Our emphasis lies in the form of the corresponding Schiffer differential equation, which does not have any perfect square form.

2. Proof of Theorem 1. Let us consider the problem

$$\max \Re\{a_4-2a_2a_3+a_2^3+2\alpha(a_3-a_2^2)+(1+\alpha)^2a_2\}.$$

Then the image of $|z|=1$ by any extremal functions satisfies

$$\left(\frac{dw}{dt}\right)^2\frac{1}{w^5}[(1+\alpha)^2w^2+(a_2+2\alpha)w+1]+1=0$$

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for a suitable parameter t. Let $Q(\zeta)d\zeta^2$ be

$$-\frac{d\zeta^2}{\zeta}[(1+\alpha)^2+(a_2+2\alpha)\zeta+\zeta^2].$$

Let a_2 be x_2+iy_2. We may assume that $y_2>0$. What we want to prove here is $y_2=0$. Let ζ be real. Then

$$\Im\, Q(\zeta)d\zeta^2=-y_2d\zeta^2 \gtrless 0.$$

Hence by Bombieri's Theorem 1 and Corollary [1] the critical trajectory of $Q(\zeta)d\zeta^2$ intersects with the real axis at most once. Since $\zeta=0$ is the simple pole of $Q(\zeta)d\zeta^2$, the critical trajectory does not intersect with the real axis except at the origin which is an end point of the critical trajectory. Consider the tangent vector at the origin. This has the argument $-\arg(-(1+\alpha)^2)=+\pi$. Next we consider the tangent vector of the neighboring trajectories on the negative real axis. By the equation, putting $\zeta=\xi+i\eta$ and then putting $\eta=0$,

$$y_2\xi\left(\frac{d\eta}{d\xi}\right)^2+2((1+\alpha)^2+(x_2+2\alpha)\xi+\xi^2)\frac{d\eta}{d\xi}-y_2\xi=0.$$

Here we have two solutions. However by $d\eta/d\xi\to 0$ as $\xi\to -0$

$$\frac{d\eta}{d\xi}=\frac{\sqrt{U^2+y_2^2\xi^2}-U}{y_2\xi},\qquad U=\xi^2+(x_2+2\alpha)\xi+(1+\alpha)^2.$$

Here $U\geqq 0$. Equality occurs only for $x_2=2$ and $\xi=-(1+\alpha)$. Our interest lies in the case that $-\delta<\xi<0$. Hence $U>0$. Thus

$$\frac{d\eta}{d\xi}=\frac{y_2\xi}{\sqrt{U^2+y_2^2\xi^2}+U}<0.$$

Thus all the neighboring trajectories of the critical trajectory cross the negative real axis decreasingly. Hence the critical trajectory starts from the origin and enters into the second quadrant. Hence the image of $|z|=1$ by ζ lies in the upper half-plane. This contradicts

$$\frac{1}{2\pi}\int_0^{2\pi}\zeta(e^{i\theta})d\theta=-a_2.$$

Hence y_2 should be equal to zero.

If $a_2=0$, then

$$\Re\{a_4-2a_2a_3+a_2^3+2\alpha(a_3-a_2^2)+(1+\alpha)^2a_2\}\leqq\frac{2}{3}+2\alpha<2(1+\alpha+\alpha^2),$$

since

$$|a_4-2a_2a_3+a_2^3|\leqq\frac{2}{3}$$

and

$$|a_3-a_2^2|\leqq 1.$$

If $a_2\neq 0$ but real, then there is no zero of $Q(\zeta)d\zeta^2$ on the real axis unless $a_2=2$. Hence if $a_2\neq 2$ then the image of $|z|=1$ by ζ should be a segment, which is certainly a contradiction. Thus $a_2=2$, which gives the desired result.

3. Proof of Theorem 2. It is sufficient to prove $\Re a_4\leqq 4$ for $|\arg a_2|\leqq\pi/3$ and $0\leqq\Re a_2\leqq 2$. Let

$$y+iy'=a_3-\frac{3}{4}a_2^2,$$

$$p+ix'=2-x+ix'=a_2.$$

Then by Theorem 1

$$\Re a_4\leqq 2+2\alpha+2\alpha^2-(1+2\alpha+\alpha^2)p+\frac{\alpha}{2}p^2+\frac{1}{2}p^3$$

$$+(2p-2\alpha)y-\frac{\alpha}{2}x'^2-\frac{3}{2}px'^2-2x'y'.$$

Here we put $\alpha=p$. Then

$$\Re a_4\leqq 4-x-2px'^2-2x'y'.$$

By the area theorem

$$-x\leqq-\frac{x^2}{4}-\frac{x'^2}{4}-\frac{3}{4}y'^2$$

we have

$$\Re a_4\leqq 4-\frac{x^2}{4}-\frac{1}{4}\{(8p+1)x'^2+8x'y'+3y'^2\}.$$

The last quadratic form is positive definite for $0\leqq x<35/24$. Thus

$$\Re a_4\leqq 4$$

for $0\leqq x\leqq 1.45$. Equality occurs only for $x=0$. If $0\leqq p\leqq 0.55$, then

$$\Re(a_4-2a_2a_3+a_2^3)\leqq\frac{2}{3}.$$

Then

$$\Re a_4 \leqq \frac{2}{3} + 2\Re a_2(a_3 - a_2^2) + \Re a_2^3.$$

Here $|a_2| \leqq 1.1$ and $|a_3 - a_2^2| \leqq 1$. Hence

$$\Re a_4 \leqq \frac{2}{3} + 2.2 + p^3 - 3px'^2 < 3.1.$$

Thus we have the desired result.

4. Another proof of Theorem 1. The following Grunsky inequality is very useful; [2], [6]:

$$\left| a_4 - 2a_2a_3 + \frac{13}{12} a_2^3 + 2\alpha\left(a_3 - \frac{3}{4} a_2^2\right) + \alpha^2 a_2 \right| \leqq \frac{2}{3} + 2\alpha^2.$$

We start from this inequality. Put $a_2 = Re^{i\varphi}$, $0 \leqq R \leqq 2$. Then

$$\begin{aligned} G &\equiv \Re\{a_4 - 2a_2a_3 + a_2^3 + 2\alpha(a_3 - a_2^2) + (1+\alpha)a_2\} \\ &= \Re\left\{a_4 - 2a_2a_3 + \frac{13}{12} a_2^3 + 2\alpha\left(a_3 - \frac{3}{4} a_2^2\right) + \alpha^2 a_2\right\} \\ &\quad + \Re\left\{(1+2\alpha)a_2 - \frac{\alpha}{2} a_2^2 - \frac{1}{12} a_2^3\right\} \\ &\leqq \frac{2}{3} + 2\alpha^2 + F, \end{aligned}$$

$$F = (1+2\alpha)R\cos\varphi - \frac{\alpha}{2} R^2 \cos 2\varphi - \frac{1}{12} R^3 \cos 3\varphi.$$

We now consider the problem $\max F$. Let (R, φ) be the maximum point of F. Assume first that $0 < R < 2$. Then at (R, φ)

$$0 = \frac{\partial F}{\partial R} = (1+2\alpha)\cos\varphi - \alpha R \cos 2\varphi - \frac{1}{4} R^2 \cos 3\varphi,$$

$$0 = \frac{\partial F}{\partial \varphi} = -(1+2\alpha)R\sin\varphi + \alpha R^2 \sin 2\varphi + \frac{1}{4} R^2 \sin 3\varphi.$$

Hence

$$(1+2\alpha)e^{i\varphi} = \alpha R e^{2i\varphi} + \frac{1}{4} R^2 e^{3i\varphi},$$

which leads to

$$1+2\alpha=\alpha R\cos\varphi+\frac{1}{4}R^2\cos 2\varphi,$$

$$0=\alpha R\sin\varphi+\frac{1}{4}R^2\sin 2\varphi.$$

If $\sin\varphi=0$, $\cos\varphi=-1$, then

$$1+2\alpha=-\alpha R+\frac{1}{4}R^2,$$

$$(1+\alpha)^2=\left(\alpha-\frac{1}{2}R\right)^2.$$

This gives $R=2+4\alpha$. Since $\alpha\geqq 0$ and $R\leqq 2$, we have $R=2$ and $\alpha=0$, which was excluded already. If $\cos\varphi=1$, $\sin\varphi=0$, then

$$(1+\alpha)^2=\left(\alpha+\frac{R}{2}\right)^2,$$

which also gives $R=2$. This is a contradiction. If $R\cos\varphi=-2\alpha$, then

$$1+2\alpha=-2\alpha^2+\frac{2}{4}R^2\cos^2\varphi-\frac{1}{4}R^2$$

implies

$$0<1+2\alpha=-\frac{1}{4}R^2<0.$$

This is a contradiction. Therefore F does not attain its maximum in $(0, 2)$.

Next we consider the boundary part. If $R=0$, then $F=0$. If $R=2$, then

$$F=2(1+2\alpha)\cos\varphi-2\alpha\cos 2\varphi-\frac{2}{3}\cos 3\varphi.$$

At the maximum point

$$0=\frac{\partial F}{\partial\varphi}.$$

Then

$$(1+2\alpha)\sin\varphi=2\alpha\sin 2\varphi+\sin 3\varphi.$$

This holds for either $\sin\varphi=0$ or $2\cos^2\varphi+2\alpha\cos\varphi-1-\alpha=0$. If $\sin\varphi=0$, then

$$F=\frac{4}{3}+2\alpha \qquad \text{or} \qquad -\frac{4}{3}-6\alpha.$$

If $2\cos^2\varphi+2\alpha\cos\varphi=1+\alpha$, then

$$F=-\frac{8}{3}\cos^3\varphi-4\alpha\cos^2\varphi+(4+4\alpha)\cos\varphi+2\alpha$$

$$=-\frac{4}{3}\alpha\cos^2\varphi+\frac{8}{3}(1+\alpha)\cos\varphi-\frac{4}{3}+\frac{2}{3}\alpha,$$

which is increasing for $\cos\varphi$. Hence

$$F\leqq\max F=F(\cos\varphi=1)=\frac{4}{3}+2\alpha.$$

However the solution of $2\cos^2\varphi+2\alpha\cos\varphi=1+\alpha$ is less than 1, that is, $\cos\varphi<1$. Hence in this case

$$\max F<\frac{4}{3}+2\alpha.$$

Thus summing up the results we have

$$\max F=\frac{4}{3}+2\alpha$$

and hence

$$G\leqq 2+2\alpha+2\alpha^2.$$

Equality occurs only for $a_2=2$.

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