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INVARIANT SUBMANIFOLDS OF AN f-MANIFOLD WITH COMPLEMENTED FRAMES

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Introduction. Recently, invariant hypersurfaces of a Kaehler manifold with constant holomorphic sectional curvature and invariant Einstein (or η-Einstein) submanifolds of normal contact or cosymplectic manifolds with constant ϕ-sectional curvature have been studied by several authors [2], [3], [4], [7]. Blair [1] has quite recently defined and studied $\mathcal{S}$-manifolds and $\mathcal{T}$-manifolds which reduce, in special cases, to normal contact manifolds and cosymplectic manifolds respectively.

Generalizing the notion of η-Einstein contact manifolds, we shall define, in §1, η-Einstein $\mathcal{S}$-manifolds and $\mathcal{T}$-manifolds and obtain some formulas giving curvature tensors for $\mathcal{S}$-manifolds and $\mathcal{T}$-manifolds with constant f-sectional curvature. In §2, we shall define f-invariant and invariant submanifolds in an $\mathcal{S}$-manifold or a $\mathcal{T}$-manifold and study invariant η-Einstein submanifolds of codimension 2 in an $\mathcal{S}$-manifold or a $\mathcal{T}$-manifold of constant f-sectional curvature. In the last section, we shall study f-invariant hypersurfaces in a certain $\mathcal{S}$-manifold or a $\mathcal{T}$-manifold. The authors wish to express their deep gratitude to Professor S. Hokari for his kind guidances and encouragement.

1. f-manifolds with complemented frames.

Let $\tilde{M}=\tilde{M}^{2n+s}$ be a manifold with an $\tilde{f}$-structure of rank $2n$. In the sequel, we assume that $n>1$. If there exist in $\tilde{M}$ vector fields $\underset{x}{\tilde{\xi}}$ $(x=1, \cdots, s)$ such that

$$
\begin{aligned}
&\underset{x}{\tilde{\eta}}(\underset{y}{\tilde{\xi}})=\delta_{xy},\\
&\tilde{f}\underset{x}{\tilde{\xi}}=0, \qquad \underset{x}{\tilde{\eta}}\circ\tilde{f}=0,\\
&\tilde{f}^2=-1+\sum_x \underset{x}{\tilde{\xi}}\otimes\underset{x}{\tilde{\eta}},
\end{aligned}
\tag{1.1}
$$

where $\underset{x}{\tilde{\eta}}$ are duals to $\underset{x}{\tilde{\xi}}$, then the $\tilde{f}$-structure is said to be with complemented frames $\underset{1}{\tilde{\xi}}, \cdots, \underset{s}{\tilde{\xi}}$ or simply to be with complemented frames. If $\tilde{M}$ has an $\tilde{f}$-structure with complemented frames, then there exists in $\tilde{M}$ a Riemannian metric $\tilde{G}$ such that

$$\tilde{G}(\tilde{X}, \tilde{Y})=\tilde{G}(\tilde{f}\tilde{X}, \tilde{f}\tilde{Y})+\tilde{\Phi}(\tilde{Y}, \tilde{Y}), \tag{1.2}$$

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where $\tilde{X}$ and $\tilde{Y}$ are vector fields in $\tilde{M}$ and $\tilde{\Phi}(\tilde{X}, \tilde{Y})=\sum_x \tilde{\eta}_x(\tilde{X})\tilde{\eta}_x(\tilde{Y})$. $\tilde{M}$ is then said to have a metric $\tilde{f}$-structure. The 2-form $\tilde{F}$ defined by

$$\tilde{F}(\tilde{X}, \tilde{Y})=\tilde{G}(\tilde{X}, \tilde{f}\tilde{Y}) \tag{1.3}$$

is called the fundamental 2-form in $\tilde{M}$. The $\tilde{f}$-structure is said to be normal if it has complemented frames and

$$N\equiv[\tilde{f}, \tilde{f}]+\sum_x \tilde{\xi}_x \otimes d\tilde{\eta}_x=0, \tag{1.4}$$

where $[\tilde{f}, \tilde{f}]$ is the Nijenhuis tensor of $\tilde{f}$.

A metric $\tilde{f}$-structure is called a $\mathcal{K}$-structure if it is normal and has closed fundamental 2-form. $\tilde{M}$ is then said to be a $\mathcal{K}$-manifold. A $\mathcal{K}$-manifold whose structure 1-forms $\tilde{\eta}_1, \cdots, \tilde{\eta}_s$ satisfy $d\tilde{\eta}_1=\cdots=d\tilde{\eta}_s$ and $\tilde{\eta}_1\wedge\cdots\wedge\tilde{\eta}_s\wedge(d\tilde{\eta}_x)^n\neq0$ is called an $\mathcal{S}$-manifold. A $\mathcal{K}$-manifold with $d\tilde{\eta}_x=0$ is called a $\mathcal{T}$-manifold. When $s=1$, a $\mathcal{K}$-manifold is an almost contact manifold, an $\mathcal{S}$-manifold is a normal contact manifold and a $\mathcal{T}$-manifold is a cosymplectic manifold.

Now, for later use, we shall list up the results given in [1], in the following two propositions:

PROPOSITION 1.1. *In a $\mathcal{K}$-manifold $\tilde{\xi}_x$'s are killing and*

$$d\tilde{\eta}_x(\tilde{X}, \tilde{Y})=-2(\tilde{\nabla}_{\tilde{Y}}\tilde{\eta}_x)(\tilde{X}) \tag{1.5}$$

holds, where $\tilde{\nabla}$ denotes covariant differentiation with respect to the Riemannian metric $\tilde{G}$. In an $\mathcal{S}$-manifold

$$\tilde{\nabla}_{\tilde{X}}\tilde{\xi}_x=-\frac{1}{2}\tilde{f}\tilde{X} \tag{1.6}$$

and in a $\mathcal{T}$-manifold

$$\tilde{\nabla}_{\tilde{X}}\tilde{\xi}_x=0. \tag{1.7}$$

PROPOSITION 1.2. *In an $\mathcal{S}$-manifold we have*

$$\begin{aligned}(\tilde{\nabla}_{\tilde{X}}\tilde{F})(\tilde{Y}, \tilde{Z})=&\frac{1}{2}\sum_x(\tilde{\eta}_x(\tilde{Y})\tilde{G}(\tilde{X}, \tilde{Z})-\tilde{\eta}_x(\tilde{Z})\tilde{G}(\tilde{X}, \tilde{Y}))\\ &-\frac{1}{2}\sum_{x,y}\tilde{\eta}_y(\tilde{X})(\tilde{\eta}_x(\tilde{Y})\tilde{\eta}_y(\tilde{Z})-\tilde{\eta}_x(\tilde{Z})\tilde{\eta}_y(\tilde{Y})).\end{aligned} \tag{1.8}$$

In an $\mathcal{S}$-manifold, (1.8) is equivalent to the condition

$$(\tilde{\nabla}_{\tilde{X}}\tilde{f})(\tilde{Y})=\frac{1}{2}\sum_{x}(\tilde{G}(\tilde{X},\tilde{Y})\underset{x}{\tilde{\xi}}-\underset{x}{\tilde{\eta}}(\tilde{Y})\tilde{X})$$

(1. 9)

$$-\frac{1}{2}\sum_{x,y}(\underset{y}{\tilde{\eta}}(\tilde{X})\underset{y}{\tilde{\eta}}(\tilde{Y})\underset{x}{\tilde{\xi}}-\underset{y}{\tilde{\eta}}(\tilde{X})\underset{x}{\tilde{\eta}}(\tilde{Y})\underset{y}{\tilde{\xi}}).$$

Let $\tilde{R}(\tilde{X},\tilde{Y})=\tilde{\nabla}_{[\tilde{X},\tilde{Y}]}-\tilde{\nabla}_{\tilde{X}}\tilde{\nabla}_{\tilde{Y}}+\tilde{\nabla}_{\tilde{Y}}\tilde{\nabla}_{\tilde{X}}$ and the $\tilde{S}(\tilde{X},\tilde{Y})$ be the curvature and the Ricci tensors of $\tilde{M}$ respectively. Then, by (1. 6) and (1. 9), we have in an $\mathcal{S}$-manifold

$$\tilde{G}(\tilde{R}(\tilde{X},\tilde{Y})\underset{z}{\tilde{\xi}},\tilde{Z})=\frac{1}{4}\sum_{x}\{\underset{x}{\tilde{\eta}}(\tilde{X})\tilde{G}(\tilde{f}\tilde{Y},\tilde{f}\tilde{Z})-\underset{x}{\tilde{\eta}}(\tilde{Y})\tilde{G}(\tilde{f}\tilde{X},\tilde{f}\tilde{Z})\}. \tag{1. 10}$$

Hence we have, from (1. 10),

$$\tilde{S}(\tilde{X},\underset{z}{\tilde{\xi}})=\frac{1}{2}n\sum_{x}\underset{x}{\tilde{\eta}}(\tilde{X}). \tag{1. 11}$$

PROPOSITION 1. 3. *There is no Einstein $\mathcal{S}$-manifold if $s\geqq 2$.*

Proof. If $\tilde{M}$ is Einstein, we have $\tilde{S}(\tilde{X},\tilde{Y})=k\tilde{G}(\tilde{X},\tilde{Y})$ for some constant k. Putting $\tilde{Y}=\underset{z}{\tilde{\xi}}$, we have $\tilde{S}(\tilde{X},\underset{z}{\tilde{\xi}})=k\tilde{G}(\tilde{X},\underset{z}{\tilde{\xi}})=k\underset{z}{\tilde{\eta}}(\tilde{X})$. This, together with (1. 11), shows that there is no Einstein $\mathcal{S}$-manifold, since $\underset{x}{\tilde{\xi}}$'s are linearly independent.

REMARK. If $\tilde{M}$ is a space of constant curvature, then $\tilde{M}$ is automatically Einstein so that there is no $\mathcal{S}$-manifold of constant curvature because of Proposition 1. 3.

PROPOSITION 1. 4. *In an $\mathcal{S}$-manifold, if the Ricci tensor has the form*

$$\tilde{S}(\tilde{X},\tilde{Y})=a(\tilde{G}(\tilde{X},\tilde{Y})+\sum_{x\neq y}\underset{x}{\tilde{\eta}}(\tilde{X})\underset{y}{\tilde{\eta}}(\tilde{Y}))+b(\tilde{\Phi}(\tilde{X},\tilde{Y})+\sum_{x\neq y}\underset{x}{\tilde{\eta}}(\tilde{X})\underset{y}{\tilde{\eta}}(\tilde{Y})), \tag{1. 12}$$

then a and b are necessarily constants.

Proof. Putting $\tilde{Y}=\underset{z}{\tilde{\xi}}$ in (1. 12), we have by virtue of (1. 1) and (1. 2)

$$\tilde{S}(\tilde{X},\underset{z}{\tilde{\xi}})=(a+b)\sum_{x}\underset{x}{\tilde{\eta}}(\tilde{X}).$$

Thus, comparing this with (1. 11), we have

$$a+b=\frac{1}{2}n,$$

since $\underset{x}{\tilde{\xi}}$'s are linearly independent. If we denote by $\tilde{r}$ the curvature scalar of $\tilde{M}$, it is given by

$$\tilde{r}=2an+s(a+b)$$

because of (1. 12). Hence we have

(1. 13) $$\tilde{V}_{\tilde{X}}\tilde{r}=2n\tilde{V}_{\tilde{X}}a.$$

On the other hand, if we denote by $\{E_i\}_{i=1,\dots,2n+s}$ an orthonormal basis, put $\tilde{U}=\tilde{Y}=E_i$, $\tilde{V}=\tilde{Z}=E_j$ in the second Bianchi identity

$$(\tilde{V}_{\tilde{X}}\tilde{R})(\tilde{U},\tilde{V},\tilde{Y},\tilde{Z})+(\tilde{V}_{\tilde{Y}}\tilde{R})(\tilde{U},\tilde{V},\tilde{Z},\tilde{X})+(\tilde{V}_{\tilde{Z}}\tilde{R})(\tilde{U},\tilde{V},\tilde{X},\tilde{Y})=0$$

$$\tilde{R}(\tilde{U},\tilde{V},\tilde{Y},\tilde{Z})=\tilde{G}(\tilde{R}(\tilde{U},\tilde{V})\tilde{Y},\tilde{Z})$$

and sum up with respect to i and j, we then have

$$\tilde{V}_{\tilde{X}}\tilde{r}=2\sum_i(\tilde{V}_{E_i}\tilde{S})(E_i,\tilde{X}).$$

On the other hand, using (1. 6), we have $(\tilde{V}_{E_i}\tilde{\eta}_x)(E_i)=\sum_{i}\sum_{x}\tilde{\eta}_x(E_i)(\tilde{V}_{E_i}\tilde{\eta}_y)(\tilde{X})=0$. Thus we get, from (1. 12),

$$\begin{aligned}
\tilde{V}_{\tilde{X}}\tilde{r}&=2\sum_i(\tilde{V}_{E_i}\tilde{S})(E_i,\tilde{X})\\
&=2\{\sum_i(\tilde{V}_{E_i}a)\tilde{G}(E_i,\tilde{X})+\sum_i(\tilde{V}_{E_i}b)\tilde{\Phi}(E_i,\tilde{X})+b\sum_i(\tilde{V}_{E_i}\tilde{\Phi})(E_i,\tilde{X})\\
&\quad+(a+b)\sum_i\sum_{x\neq y}((\tilde{V}_{E_i}\tilde{\eta}_x)(E_i)\tilde{\eta}_y(\tilde{X})+\tilde{\eta}_x(E_i)(\tilde{V}_{E_i}\tilde{\eta}_y)(\tilde{X}))\}\\
&=2\{\tilde{V}_{\tilde{X}}a+\sum_i(\tilde{V}_{E_i}b)\tilde{\Phi}(E_i,\tilde{X})+b\sum_i\sum_x((\tilde{V}_{E_i}\tilde{\eta}_x)(E_i)\tilde{\eta}_x(\tilde{X})+\tilde{\eta}_x(E_i)(\tilde{V}_{E_i}\tilde{\eta}_x)(\tilde{X}))\}\\
&=2(\tilde{V}_{\tilde{X}}a+\sum_x(\tilde{V}_{\tilde{\xi}_x}b)\tilde{\eta}_x(\tilde{X}))\\
&=2(\tilde{V}_{\tilde{X}}a-\sum_x(\tilde{V}_{\tilde{\xi}_x}a)\tilde{\eta}_x(\tilde{X})).
\end{aligned}$$

Thus, comparing this with (1. 13), we have

$$(n-1)\tilde{V}_{\tilde{X}}a=-\sum_x(\tilde{V}_{\tilde{\xi}_x}a)\tilde{\eta}_x(\tilde{X}).$$

Putting $\tilde{X}=\tilde{\xi}_z$ we have $\tilde{V}_{\tilde{\xi}_x}a=0$, which implies $\tilde{V}_{\tilde{X}}a=0$ since $n>1$. Hence a is constant and consequently b is also constant.

DEFINITION. An $\mathcal{S}$-manifold is said to be *η-Einstein* if the Ricci tensor of $\tilde{M}$ has the form (1. 12).

REMARK. By the definition above, we see that a $\mathcal{T}$-manifold is η-Einstein if the Ricci tensor has thè form $\tilde{S}(\tilde{X},\tilde{Y})=a\tilde{G}(\tilde{X},\tilde{Y})+b\tilde{\Phi}(\tilde{X},\tilde{Y})$.

A plane section π is called an $\tilde{f}$-section if it is determined by a vector $\tilde{X}\in\tilde{\mathcal{L}}(m)$, $m\in\tilde{M}$ such that $\{\tilde{X},\tilde{f}\tilde{X}\}$ is an orthonormal pair spanning the section, $\tilde{\mathcal{L}}$ being the distribution determined by the projection tensor $-\tilde{f}^2$. We now put $H(\tilde{X})=K(\tilde{X},\tilde{f}\tilde{X})$, where K denotes the sectional curvature, and call H the $\tilde{f}$-sectional curvature.

PROPOSITION 1.5. *If $\tilde{M}$ is an $\mathcal{S}$-manifold of constant $\tilde{f}$-sectional curvature $\tilde{c}$, then we have*

$$\begin{aligned}\tilde{G}(\tilde{R}(\tilde{X},\tilde{Y})\tilde{Z},\tilde{W})=&\left(\frac{\tilde{c}}{4}+\frac{3s}{16}\right)\{\tilde{G}(\tilde{X},\tilde{Z})\tilde{G}(\tilde{W},\tilde{Y})-\tilde{G}(\tilde{X},\tilde{W})\tilde{G}(\tilde{Y},\tilde{Z})-\tilde{G}(\tilde{X},\tilde{Z})\tilde{\Phi}(\tilde{W},\tilde{Y})\\&-\tilde{G}(\tilde{W},\tilde{Y})\tilde{\Phi}(\tilde{Z},\tilde{X})+\tilde{G}(\tilde{X},\tilde{W})\tilde{\Phi}(\tilde{Z},\tilde{Y})+\tilde{G}(\tilde{Y},\tilde{Z})\tilde{\Phi}(\tilde{X},\tilde{W})\\&+\tilde{\Phi}(\tilde{Z},\tilde{X})\tilde{\Phi}(\tilde{W},\tilde{Y})-\tilde{\Phi}(\tilde{X},\tilde{W})\tilde{\Phi}(\tilde{Z},\tilde{Y})\}+\left(\frac{\tilde{c}}{4}-\frac{s}{16}\right)\{\tilde{F}(\tilde{W},\tilde{X})\tilde{F}(\tilde{Y},\tilde{Z})\\&+\tilde{F}(\tilde{Y},\tilde{W})\tilde{F}(\tilde{X},\tilde{Z})-2\tilde{F}(\tilde{X},\tilde{Y})\tilde{F}(\tilde{W},\tilde{Z})\}-\frac{1}{4}\sum_{x,y}\{\tilde{\eta}_y(\tilde{W})\tilde{\eta}_x(\tilde{X})\tilde{G}(\tilde{f}\tilde{Z},\tilde{f}\tilde{Y})\\&-\tilde{\eta}_y(\tilde{W})\tilde{\eta}_x(\tilde{Y})\tilde{G}(\tilde{f}\tilde{Z},\tilde{f}\tilde{X})+\tilde{\eta}_y(\tilde{Y})\tilde{\eta}_x(\tilde{Z})\tilde{G}(\tilde{f}\tilde{W},\tilde{f}\tilde{X})-\tilde{\eta}_y(\tilde{Z})\tilde{\eta}_x(\tilde{X})\tilde{G}(\tilde{f}\tilde{W},\tilde{f}\tilde{Y})\}\end{aligned}\tag{1.14}$$

and, if $\tilde{M}$ is a $\mathcal{T}$-manifold of constant $\tilde{f}$-sectional curvature $\tilde{c}$, then

$$\begin{aligned}\tilde{G}(\tilde{R}(\tilde{X},\tilde{Y})\tilde{Z},\tilde{W})=&\frac{\tilde{c}}{4}\{\tilde{G}(\tilde{X},\tilde{Z})\tilde{G}(\tilde{W},\tilde{Y})-\tilde{G}(\tilde{X},\tilde{W})\tilde{G}(\tilde{Y},\tilde{Z})-\tilde{G}(\tilde{X},\tilde{Z})\tilde{\Phi}(\tilde{W},\tilde{Y})\\&-G(\tilde{W},\tilde{Y})\tilde{\Phi}(\tilde{Z},\tilde{X})+\tilde{G}(\tilde{X},\tilde{W})\tilde{\Phi}(\tilde{Z},\tilde{Y})+\tilde{G}(\tilde{Y},\tilde{Z})\tilde{\Phi}(\tilde{X},\tilde{W})\\&+\tilde{\Phi}(\tilde{Z},\tilde{X})\tilde{\Phi}(\tilde{W},\tilde{Y})-\tilde{\Phi}(\tilde{X},\tilde{W})\tilde{\Phi}(\tilde{Z},\tilde{Y})+\tilde{F}(\tilde{W},\tilde{X})\tilde{F}(\tilde{Y},\tilde{Z})\\&+\tilde{F}(\tilde{Y},\tilde{W})\tilde{F}(\tilde{X},\tilde{Z})-2\tilde{F}(\tilde{X},\tilde{Y})\tilde{F}(\tilde{W},\tilde{Z})\}\end{aligned}\tag{1.15}$$

for any vector fields $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$ and $\tilde{W}$ in $\tilde{M}$.

Proof. The proof of above proposition is given by a lengthy but straight computation, so that we shall show only the process how to obtain it. First, we put $B(\tilde{X},\tilde{Y})=\tilde{G}(\tilde{R}(\tilde{X},\tilde{Y})\tilde{X},\tilde{Y})$. Then, in general, we have

$$\begin{aligned}3\tilde{G}(\tilde{R}(\tilde{X},\tilde{Y})\tilde{Z},\tilde{W})=&B(\tilde{W}+\tilde{Y},\tilde{Z}+\tilde{X})+\frac{1}{2}B(\tilde{X}+\tilde{Y},\tilde{Z}+\tilde{W})-B(\tilde{W},\tilde{Z}+\tilde{X})\\&-B(\tilde{Y},\tilde{Z}+\tilde{X})-B(\tilde{X},\tilde{W}+\tilde{Y})-B(\tilde{Z},\tilde{W}+\tilde{Y})-\frac{1}{2}B(\tilde{X},\tilde{Z}+\tilde{W})\\&-\frac{1}{2}B(\tilde{Z},\tilde{X}+\tilde{Y})-\frac{1}{2}B(\tilde{W},\tilde{X}+\tilde{Y})-\frac{1}{2}B(\tilde{Y},\tilde{Z}+\tilde{W})+\frac{3}{2}B(\tilde{Z},\tilde{Y})\\&+B(\tilde{Z},\tilde{W})+B(\tilde{X},\tilde{Y})+\frac{3}{2}B(\tilde{X},\tilde{W})+\frac{1}{2}B(\tilde{Z},\tilde{X})+\frac{1}{2}B(\tilde{W},\tilde{Y}).\end{aligned}\tag{1.16}$$

By Lemma 2.4 of [1], we find

$$B(\tilde{X},\tilde{Y})=\frac{1}{32}\{3D(\tilde{X}+\tilde{f}\tilde{Y})+3D(\tilde{X}-\tilde{f}\tilde{Y})-D(\tilde{X}+\tilde{Y})-D(\tilde{X}-\tilde{Y})\tag{1.17}$$

$$-4D(\tilde{Y})-6sP(\tilde{X},\tilde{Y};\ \tilde{X},\tilde{f}\tilde{Y})\}$$

in an $\mathcal{S}$-manifold and

$$B(\tilde{X},\tilde{Y})=\frac{1}{32}\{3D(\tilde{X}+\tilde{f}\tilde{Y})+3D(\tilde{X}-\tilde{f}\tilde{Y})-D(\tilde{X}+\tilde{Y})-D(\tilde{X}-\tilde{Y})-4D(\tilde{X})-4D(\tilde{Y})\} \tag{1.18}$$

in a $\mathcal{T}$-manifold, where $\tilde{X},\tilde{Y}\in\tilde{L}(m)$, $D(\tilde{X})=B(\tilde{X},\tilde{f}\tilde{X})$ and $P(\tilde{X},\tilde{Y};\ \tilde{Z},\tilde{W})$ $=\tilde{F}(\tilde{X},\tilde{Z})\tilde{G}(\tilde{Y},\tilde{W})-\tilde{F}(\tilde{X},\tilde{W})\tilde{G}(\tilde{Y},\tilde{Z})-\tilde{F}(\tilde{Y},\tilde{Z})\tilde{G}(\tilde{X},\tilde{W})+\tilde{F}(\tilde{Y},\tilde{W})\tilde{G}(\tilde{X},\tilde{Z})$, Thus, substituting (1. 17) and (1. 18) into (1. 16) and taking account of $D(\tilde{X})=\tilde{c}||X||^4$, we have for $\tilde{X},\tilde{Y},\tilde{Z},\tilde{W}\in\tilde{L}(m)$,

$$\begin{aligned}\tilde{G}(\tilde{R}(\tilde{X},\tilde{Y})\tilde{Z},\tilde{W})=&\left(\frac{1}{4}\tilde{c}+\frac{3s}{16}\right)\{\tilde{G}(\tilde{X},\tilde{Z})\tilde{G}(\tilde{W},\tilde{Y})-\tilde{G}(\tilde{X},\tilde{W})\tilde{G}(\tilde{Y},\tilde{Z})\}\\&+\left(\frac{1}{4}\tilde{c}-\frac{s}{16}\right)\{\tilde{F}(\tilde{W},\tilde{X})\tilde{F}(\tilde{Y},\tilde{Z})+\tilde{F}(\tilde{X},\tilde{Z})\tilde{F}(\tilde{Y},\tilde{W})\\&-2\tilde{F}(\tilde{W},\tilde{Z})\tilde{F}(\tilde{X},\tilde{Y})\}\end{aligned} \tag{1.19}$$

in an $\mathcal{S}$-manifold and

$$\begin{aligned}\tilde{G}(\tilde{R}(\tilde{X},\tilde{Y})\tilde{Z},\tilde{W})=\frac{1}{4}\tilde{c}\{&\tilde{G}(\tilde{X},\tilde{Z})\tilde{G}(\tilde{W},\tilde{Y})-\tilde{G}(\tilde{X},\tilde{Z})\tilde{G}(\tilde{Y},\tilde{W})+\tilde{F}(\tilde{W},\tilde{X})\tilde{F}(\tilde{Y},\tilde{Z})\\&+\tilde{F}(\tilde{X},\tilde{Z})\tilde{F}(\tilde{Y},\tilde{W})-2\tilde{F}(\tilde{W},\tilde{Z})\tilde{F}(\tilde{X},\tilde{Y})\}\end{aligned} \tag{1.20}$$

in a $\mathcal{T}$-manifold. Therefore, since for vector fields $\tilde{X},\tilde{Y},\tilde{Z}$ and $\tilde{W}$ in $\tilde{M}$,

$$\tilde{X}-\sum_x\tilde{\eta}_x(\tilde{X})\tilde{\xi}_x,\qquad \tilde{Y}-\sum_x\tilde{\eta}_x(\tilde{Y})\tilde{\xi}_x,\ \tilde{Z}-\sum_x\tilde{\eta}_x(\tilde{Z})\tilde{\xi}_x\quad\text{and}\quad \tilde{W}-\sum_x\tilde{\eta}_x(\tilde{W})\tilde{\xi}_x$$

lie in $\tilde{L}$, substituting them into (1. 19) and (1. 20), we have (1. 14) and (1. 15) respectively.

As a direct corollary to Proposition 1. 5, we have

PROPOSITION 1. 6. *We have*

$$\begin{aligned}\tilde{S}(\tilde{X},\tilde{Y})=&\left\{\left(\frac{1}{4}\tilde{c}+\frac{3s}{16}\right)(2n-1)+\frac{3}{4}\tilde{c}+\frac{s}{16}\right\}\tilde{G}(\tilde{X},\tilde{Y})-\left\{\left(\frac{1}{4}\tilde{c}+\frac{3s}{16}\right)(2n-1)\right.\\&\left.+\frac{3}{4}\tilde{c}+\frac{s}{16}-\frac{n}{2}\right\}\tilde{\Phi}(\tilde{X},\tilde{Y})+\frac{n}{2}\sum_{x\neq y}\tilde{\eta}_x(\tilde{X})\tilde{\eta}_y(\tilde{Y})\end{aligned} \tag{1.21}$$

in an $\mathcal{S}$-manifold of constant $\tilde{f}$-sectional curvature $\tilde{c}$ and

$$\tilde{S}(\tilde{X},\tilde{Y})=\frac{n+1}{2}\tilde{c}\{\tilde{G}(\tilde{X},\tilde{Y})-\tilde{\Phi}(\tilde{X},\tilde{Y})\} \tag{1.22}$$

in a $\mathcal{T}$-manifold of constant $\tilde{f}$-sectional curvature $\tilde{c}$.

REMARK. We see easily that if $\tilde{M}$ is of constant $\tilde{f}$-sectional curvature, it is η-Einstein.

2. Invariant submanifolds of codimension 2 in an $\mathcal{S}$-manifold and in a $\mathcal{T}$-manifold.

Let M be a submanifold in an $\tilde{f}$-manifold with complemented frames $\underset{1}{\tilde{\xi}}, \cdots, \underset{s}{\tilde{\xi}}$ and i: $M \rightarrow \tilde{M}$ its imbedding.

DEFINITION. M is said to be an *$\tilde{f}$-invariant* submanifold of $\tilde{M}$ if the tangent space $T_p(i(M))$ is invariant by the linear map $\tilde{f}$ at each point p of $i(M)$.

DEFINITION. An $\tilde{f}$-invariant submanifold is said to be *invariant* if all of $\underset{x}{\tilde{\xi}}$ $(x=1, \cdots, s)$ are always tangent to $i(M)$.

Hereafter we assume that M is an $\tilde{f}$-invariant or invariant submanifold of $\hat{M}$. For arbitrary vector fields X and Y in M, we put

$$g(X, Y)=\tilde{G}(i_*X, i_*Y), \tag{2.1}$$

$$\tilde{\nabla}_{i_*X}i_*Y=i_*\nabla_X Y+T_X Y, \tag{2.2}$$

where $T_X Y$ is the normal component of $\tilde{\nabla}_{i_*X}i_*Y$. Then g is a Riemannian metric in M, ∇ is the covariant differentiation with respect to g and $T_X Y$ is the so called second fundamental tensor of the submanifold M. We next put

$$\tilde{f}i_*X=i_*fX, \tag{2.3}$$

where f is a (1. 1)-type tensor in M. We have the following Propositions 2. 1~2. 4, which are quite similar to those proved in contact cases, so that the proofs of them are omitted here (cf. [8].

PROPOSITION 2. 1. *An $\tilde{f}$-invariant submanifold M imbedded in an $\tilde{f}$-manifold with complemented frames $\underset{x}{\tilde{\xi}}$ $(x=1, \cdots, s)$ in such a way that $\underset{x}{\tilde{\xi}}$'s are never tangent to $i(M)$ is an almost complex manifold. If the $\tilde{f}$-structure is normal, then M is a complex manifold.*

PROPOSITION 2. 2. *An invariant submanifold M imbedded in an $\tilde{f}$-manifold with complemented frames is an f-manifold with complemented frames. If the $\tilde{f}$-structure is normal, then M is also normal.*

PROPOSITION 2. 3. *An $\tilde{f}$-invariant submanifold M imbedded in an $\mathcal{S}$-manifold $\tilde{M}$ in such a way that the vectors $\underset{x}{\tilde{\xi}}$ $(x=1, \cdots, s)$ are never tangent to $i(M)$ is a Kaehler manifold and minimal in $\tilde{M}$.*

PROPOSITION 2. 4. *An invariant submanifold M imbedded in an $\mathcal{S}$-manifold* *(resp. in a $\mathcal{T}$-manifold) is an $\mathcal{S}$-manifold (resp. a $\mathcal{T}$-manifold) and minimal in $\hat{M}$.*

Now, we confine our attention to the case where M is an invariant submanifold of codimension 2 in an $\mathcal{S}$-manifold of constant $\tilde{f}$-sectional curvature $\tilde{c}$. Let C be a field of unit normals defined on $i(M)$ such that $\tilde{G}(C, i_*X)=0$ and $G(\tilde{f}C, i_*X)=0$ for all vector fields X tangent to M. Since our submanifold is invariant, we may put

$$\tilde{\underset{x}{\xi}}=i_*\underset{x}{\xi} \tag{2.4}$$

for some tangent vectors $\underset{x}{\xi}$ $(x=1, \cdots, s)$ in M. Let $\underset{x}{\eta}$ $(x=1, \cdots, s)$ be duals to $\underset{x}{\xi}$, i.e. 1-forms satisfying (1.1). Then we have, by virtue of Proposition 2.4,

$$\begin{aligned}&\underset{x}{\eta}(\underset{y}{\xi})=\delta_{xy},\\ &f\underset{x}{\xi}=0, \qquad \underset{x}{\eta}\circ f=0,\\ &f^2=-1+\sum_x\underset{x}{\xi}\otimes\underset{x}{\eta},\end{aligned} \tag{2.5}$$

$$g(X, Y)=g(fX, fY)+\Phi(X, Y), \tag{2.6}$$

where we have put $\Phi(X, Y)=\sum_x\underset{x}{\eta}(X)\underset{x}{\eta}(Y)$. We have also, from (1.9),

$$(\nabla_X f)Y=\frac{1}{2}\sum_x(g(X, Y)\underset{x}{\xi}-\underset{x}{\eta}(Y)X)-\frac{1}{2}\sum_{x,y}(\underset{y}{\eta}(X)\underset{y}{\eta}(Y)\underset{x}{\xi}-\underset{y}{\eta}(X)\underset{x}{\eta}(Y)\underset{y}{\xi}). \tag{2.7}$$

Furthermore, since our submanifold is of codimension 2, we may put $T_X Y=H(X, Y)C+K(X, Y)\tilde{f}C$, where H and K are (0, 2)-type tensor fields in M. Hence we have

$$\tilde{\nabla}_{i_*X}i_*Y=i_*\nabla_X Y+H(X, Y)C+K(X, Y)\tilde{f}C, \tag{2.8}$$

$$\tilde{\nabla}_{i_*X}C=-i_*hX+s(X)\tilde{f}C, \tag{2.9}$$

$$\tilde{\nabla}_{i_*X}\tilde{f}C=-i_*kX-s(X)C, \tag{2.10}$$

where s is a 1-form in M and h, k are symmetric tensor fields of type (1, 1) in M satisfying the relation $g(hX, Y)=H(X, Y)$ and $g(kX, Y)=K(X, Y)$.

Moreover, using (1.9), we have

$$\begin{aligned}\tilde{\nabla}_{i_*X}\tilde{f}C&=(\tilde{\nabla}_{i_*X}\tilde{f})C+\tilde{f}\tilde{\nabla}_{i_*X}C\\ &=\frac{1}{2}\sum_x(\tilde{G}(i_*X, C)\tilde{\underset{x}{\xi}}-\tilde{\underset{x}{\eta}}(C)i_*X)-\frac{1}{2}\sum_{x,y}(\tilde{\underset{y}{\eta}}(i_*X)\tilde{\underset{y}{\eta}}(C)\tilde{\underset{x}{\xi}}-\tilde{\underset{y}{\eta}}(i_*X)\tilde{\underset{x}{\eta}}(C)\tilde{\underset{y}{\xi}})-fi_*hY+s(X)\tilde{f}^2C\\ &=-i_*fhX-s(X)C.\end{aligned}$$

Hence we have

$$k = f h, \qquad h = -f k, \tag{2.11}$$

$$fh + hf = 0, \qquad fk + kf = 0, \tag{2.12}$$

$$h\underset{x}{\xi} = 0, \qquad k\underset{x}{\xi} = 0. \tag{2.13}$$

Now, let $R(X, Y)Z$ be the curvature tensor of M. Then the fundamental equations of the submanifold M can be written as

$$\begin{aligned}\tilde{R}(i_*X, i_*Y)i_*Z = i_*[R(X, Y)Z &+ (H(Y, Z)hX - H(X, Z)hY) + (K(Y, Z)kX - K(X, Z)kY)]\\ &- g((\nabla_X h)Y - (\nabla_Y h)X - s(X)kY + s(Y)kX, Z)C\\ &- g((\nabla_X k)Y - (\nabla_Y k)X + s(X)hY - s(Y)hX, Z)\tilde{f}C,\end{aligned} \tag{2.14}$$

$$\begin{aligned}\tilde{R}(i_*X, i_*Y)C = i_*[&(\nabla_X h)Y - (\nabla_Y h)X - s(X)kY + s(Y)kX]\\ &- ((\nabla_X s)(Y) - (\nabla_Y s)(X) - K(X, hY) + K(Y, hX))\tilde{f}C.\end{aligned} \tag{2.15}$$

Therefore, forming the inner product of i_*W with (2.14) and taking account of (1.14), we have

$$\begin{aligned}g(R(X, Y)Z, W) = \left(\frac{\tilde{c}}{4} + \frac{3s}{16}\right)\{&g(X, Z)g(W, Y) - g(X, W)g(Y, Z) - g(X, Z)\Phi(W, Y)\\ &- g(W, Y)\Phi(Z, X) + g(X, W)\Phi(Z, Y) + g(Y, Z)\Phi(X, W)\\ &+ \Phi(Z, X)\Phi(W, Y) - \Phi(X, W)\Phi(Z, Y)\} + \left(\frac{\tilde{c}}{4} - \frac{s}{16}\right)\{F(W, X)F(Y, Z)\\ &+ F(Y, W)F(X, Z) - 2F(X, Y)F(W, Z)\} - \frac{1}{4}\sum_{x, y}\{\underset{y}{\eta}(W)\underset{x}{\eta}(X)g(fZ, fY)\\ &- \underset{y}{\eta}(W)\underset{x}{\eta}(Y)g(fZ, fX) + \underset{y}{\eta}(Y)\underset{x}{\eta}(Z)g(fW, fX) - \underset{y}{\eta}(Z)\underset{x}{\eta}(X)g(fW, fY)\}\\ &- g(hY, Z)g(hX, W) + g(hX, Z)g(hY, W) - g(kY, Z)g(kX, W)\\ &+ g(kX, Z)g(kY, W),\end{aligned} \tag{2.16}$$

where we have put $F(X, Y) = g(X, fY)$. If we denote by $S(X, Y)$ the Ricci tensor of M, we have

$$\begin{aligned}S(X, Y) = &\left\{\left(\frac{\tilde{c}}{4} + \frac{3s}{16}\right)(2n-3) + \frac{3\tilde{c}}{4} + \frac{s}{16}\right\}g(X, Y) - \left\{\left(\frac{\tilde{c}}{4} + \frac{3s}{16}\right)(2n-3)\right.\\ &\left. + \frac{3\tilde{c}}{4} + \frac{s}{16} - \frac{n-1}{2}\right\}\Phi(X, Y) + \frac{n-1}{2}\sum_{x \neq y}\underset{x}{\eta}(X)\underset{y}{\eta}(Y) - 2g(h^2X, Y),\end{aligned} \tag{2.17}$$

since M is minimal by Proposition 2.4.

Assume that M is η-Einstein. Then the Ricci tensor of M has the form

(2. 18) $$S(X, Y)=ag(X, Y)+b\Phi(X, Y)+\frac{n-1}{2}\sum_{x\neq y}\underset{x}{\eta}(X)\underset{y}{\eta}(Y).$$

Thus, comparing this with (2. 17), we have

(2. 19) $$g(h^2X, Y)=\mu g(fX, fY),$$

since $a+b=(n-1)/2$, where we have put

$$\mu=\frac{1}{2}\left\{\left(\frac{\tilde{c}}{4}+\frac{3s}{11}\right)(2n-3)+\frac{3\tilde{c}}{4}+\frac{s}{16}-a\right\}.$$

Therefore, we have

(2. 20) $$\mu\geqq 0,$$

(2. 21) $$kh=\mu f.$$

Next, forming the inner products of C and $\tilde{f}C$ with (2. 14), we have respectively

(2. 22) $$(\nabla_X h)Y-(\nabla_Y h)X-s(X)kY+s(Y)kY=0,$$

(2. 23) $$(\nabla_X k)Y-(\nabla_Y k)X+s(X)hY-s(Y)hX=0.$$

LEMMA 2. 5. *If M is η-Einstein,*

(2. 24) $$(\nabla_X h)Y=s(X)kY-\frac{1}{2}\sum_x(\underset{x}{\eta}(X)kY+\underset{x}{\eta}(Y)kX+g(kX, Y)\underset{x}{\xi}),$$

(2. 25) $$(\nabla_X k)Y=-s(X)hY+\frac{1}{2}\sum_x(\underset{x}{\eta}(X)hY+\underset{x}{\eta}(Y)hX+g(hX, Y)\underset{x}{\xi}).$$

Proof. Differentiating the second equation of (2. 11) covariantly and taking account of (2. 13) and (2. 7), we have

$$(\nabla_X h)Y=(\nabla_X k)fY-\frac{1}{2}\sum_x\underset{x}{\eta}(Y)kX.$$

Putting $Y=\underset{y}{\xi}$, we have

$$(\nabla_X h)=-\frac{1}{2}kX.$$

Thus, putting $X=\underset{y}{\xi}$ in (2. 22), we have

$$(\nabla_{\underset{y}{\xi}} h)Y=s(\underset{y}{\xi})kY-\frac{1}{2}kY.$$

On the other hand, we have, from (2. 19), $h^2=\mu I$ in $L(m)$. Hence, using (2. 23) and similar method used in Proposition 7 of [5], we have for any $X', Y'\in\mathcal{L}$,

$$(\nabla_{X'}k)Y'=-s(X')hY'.$$

Therefore, using (2. 7), we have.

$$\begin{aligned}(\nabla_{X'}h)Y'&=\nabla_{X'}(-fk)Y'\\&=-(\nabla_{X'}f)kY'-f(\nabla_{X'}k)Y'\\&=-\frac{1}{2}\sum_x g(X', kY')\underset{x}{\xi}+s(X')kY'.\end{aligned}$$

Hence, for

$$X=X'+\sum_x \underset{x}{\eta}(X)\underset{x}{\xi},\qquad Y=Y'+\sum_x \underset{x}{\eta}(Y)\underset{x}{\xi}\qquad (X', Y'\in\mathcal{L})$$

we have

$$\begin{aligned}(\nabla_X h)Y&=(\nabla_{X'}h)Y'+(\nabla_{X'}h)(\sum_x \underset{x}{\eta}(Y)\underset{x}{\xi})+\sum_x \underset{x}{\eta}(X)(\nabla_{\underset{x}{\xi}}h)Y\\&=s(X')kY'-\frac{1}{2}\sum_x g(X', kY')\underset{x}{\xi}+\sum_x \underset{x}{\eta}(Y)\left(-\frac{1}{2}kX'\right)\\&\quad+\sum_x \underset{x}{\eta}(X)\left(s(\underset{x}{\xi})kY-\frac{1}{2}kY\right)\\&=s(X)kY-\frac{1}{2}\sum_x(\underset{x}{\eta}(Y)kX+\underset{x}{\eta}(X)kY+g(X, kY)\underset{x}{\xi}),\end{aligned}$$

which proves (2. 24). We can prove (2. 25) as follows:

$$\begin{aligned}(\nabla_X k)Y&=(\nabla_X fh)Y=(\nabla_X f)hY+f(\nabla_X h)Y\\&=\frac{1}{2}\sum_x g(X, hY)\underset{x}{\xi}+s(X)fkY-\frac{1}{2}\sum_x(\underset{x}{\eta}(Y)fkX+\underset{x}{\eta}(X)fkY)\\&=-s(X)hY+\frac{1}{2}\sum_x(\underset{x}{\eta}(Y)hX+\underset{x}{\eta}(X)hY+g(X, hY)\underset{x}{\xi}).\end{aligned}$$

Forming the inner product of $\tilde{f}C$ with (2. 15) and taking account of (2. 21), we have (cf. Lemma given in [3])

LEMMA 2. 6. *If M is η-Einstein, then*

$$(\nabla_X s)(Y)-(\nabla_Y s)(X)=\left(2\mu+\frac{\tilde{c}}{2}-\frac{s}{8}\right)F(X, Y). \tag{2. 26}$$

THEOREM 2. 7. *If M is an invariant η-Einstein submanifold of codimension* 2 *in an $\mathcal{S}$-manifold of constant $\tilde{f}$-sectional curvature $\tilde{c}$, then*

(I) *M is totally geodesic for $\tilde{c}\leqq -3s/4$,*

(II) *M is totally geodesic or η-Einstein with the scalar curvature*

$$(n-1)\left\{\tilde{c}(n-1)+\frac{s}{4}(3n-5)\right\} \qquad for \qquad \tilde{c}>-\frac{3s}{4}.$$

Proof. Differentiating (2. 24) covariantly, we have, for vector fields X and Y such that $[X, Y]=0$,

$$\begin{aligned}(\nabla_X\nabla_Y h)Z-(\nabla_Y\nabla_X h)Z=&(\nabla_X s)(Y)kZ-(\nabla_Y s)(X)kZ\\ &-\frac{1}{4}\sum_{x,y}\{\underset{x}{\eta}(Y)\underset{y}{\eta}(Z)hX+\underset{y}{\eta}(Y)g(hX, Z)\underset{x}{\xi}-\underset{x}{\eta}(X)\underset{y}{\eta}(Z)hY\\ &-\underset{y}{\eta}(X)g(hY, Z)\underset{x}{\xi}\}+\frac{s}{4}\{2F(Y, X)kZ+F(Z, X)kY\\ &+g(kY, Z)fX-F(Z, Y)kX-g(kX, Z)fY\}.\end{aligned}$$

Since $R(X, Y)\cdot h=-(\nabla_X\nabla_Y-\nabla_Y\nabla_X)h$ and $(R(X, Y)\cdot h)Z=R(X, Y)hZ-hR(X, Y)Z$, we have

$$\begin{aligned}&\left(\frac{\tilde{c}}{4}+\frac{3s}{16}-\mu\right)\{g(X, hZ)g(W, Y)-g(X, W)g(Y, hZ)-g(X, Z)g(hW, Y)+g(X, hW)g(Y, Z)\\ &+g(Y, hZ)\Phi(X, W)-g(X, hZ)\Phi(W, Y)-g(X, hW)\Phi(Z, Y)-g(hW, Y)\Phi(Z, X)\\ &+F(W, X)g(Y, kZ)+F(Y, W)g(X, hZ)-2F(X, Y)g(W, kZ)+F(Y, Z)g(X, kW)\\ &-g(Y, kW)F(X, Z)\}=0,\end{aligned}$$

by virtue of (2. 16), (2. 19), (2. 21) and (2. 26). Thus, taking the trace with respect to W and Y, we have

$$\left(\frac{\tilde{c}}{4}+\frac{3s}{16}-\mu\right)2n\, g(X, hZ)=0,$$

since M is minimal. Hence we see that M is totally geodesic except in the case where $\mu\neq\tilde{c}/4+3s/16$, which implies $\mu>0$ by (2. 20). Therefore Theorem 2. 7 is proved.

In the sequel, we assume that M is an invariant submanifold of codimension 2 in a $\mathcal{T}$-manifold of constant $\tilde{f}$-sectional curvature $\tilde{c}$. Then we have

$$\begin{aligned}g(R(X, Y)Z, W)=&\frac{\tilde{c}}{4}\{g(X, Z)g(W, Y)-g(X, W)g(Y, Z)+G(X, W)\Phi(Z, Y)\\ &+g(Y, Z)\Phi(W, X)-g(X, Z)\Phi(W, Y)-g(W, Y)\Phi(Z, X)\end{aligned}$$

(2.27) $$+\Phi(Z, X)\Phi(W, Y)-\Phi(X, W)\Phi(Z, Y)+F(W, X)F(Y, Z)$$
$$+F(Y, W)F(X, Z)-2F(X, Y)F(W, Z)\}-g(hY, Z)g(hX, W)$$
$$+g(hX, Z)g(hY, W)-g(kY, Z)g(kX, W)+g(kX, Z)g(kY, W),$$

by virtue of (2.14) and (1.14). Hence we have

(2.28) $$S(X, Y)=\frac{n\tilde{c}}{2}\{g(X, Y)-\Phi(X, Y)\}-2g(h^2X, Y).$$

Assume that M is η-Einstein. Then the Ricci tensor of M has the form

(2.29) $$S(X, Y)=ag(X, Y)+b\Phi(X, Y)$$

with $a+b=0$. Thus, comparing this with (2.28), we have

(2.30) $$g(h^2X, Y)=\lambda g(fX, fY),$$

where we have put $\lambda=(1/2)(n\tilde{c}/2-a)$. Hence we have

(2.31) $$\lambda\geqq 0,$$

(2.32) $$hk=\lambda f.$$

The proof of the following Lemma 2.8 is similar to that of Lemma 4.11 given in [2], so that the proof is omitted.

LEMMA 2.8. *If M is η-Einstein, then*

(2.33) $$(\nabla_X h)Y=s(X)kY,$$

(2.34) $$(\nabla_X k)Y=-s(X)hY.$$

The proof of the following Lemma 2.9 is similar to that of Lemma 2.6.

LEMMA 2.9. *If M is η-Einstein, then*

(2.35) $$(\nabla_X s)(Y)-(\nabla_Y s)(X)=\left(2\lambda+\frac{\tilde{c}}{2}\right)F(X, Y).$$

THEOREM 2.10. *If M is an invariant η-Einstein submanifold of codimension 2 in a $\mathcal{T}$-manifold of constant $\tilde{f}$-sectional curvature $\tilde{c}$, then*

(I) *M is totally geodesic for $\tilde{c}\leqq 0$,*

(II) *M is totally geodesic or η-Einstein with the scalar curvature $(n-1)^2\tilde{c}$ for $\tilde{c}>0$.*

Proof. First we have

$$(\nabla_Y\nabla_X h)Z-(\nabla_X\nabla_Y h)Z=\left(2\lambda+\frac{\tilde{c}}{2}\right)F(Y, X)kZ,$$

by virtue of (2. 32), (2. 33) and (2. 34). Thus, using the identity $(R(X, Y)\cdot h)Z=(\nabla_Y\nabla_X h-\nabla_X\nabla_Y h)Z$ for any vector fields X and Y in M such that $[X, Y]=0$, we have

$$\left(\frac{\tilde{c}}{4}-\lambda\right)\{g(X, hZ)g(W, Y)-g(X, W)g(Y, hZ)-g(X, Z)g(hW, Y)+g(X, hW)g(Y, Z)$$
$$+g(hZ, Y)\Phi(X, W)-g(X, hZ)\Phi(W, Y)+g(X, hW)\Phi(Z, Y)+g(hW, Y)\Phi(Z, X)$$
$$+F(W, X)F(Y, hZ)+F(Y, W)F(X, hZ)-2F(X, Y)F(W, hZ)\}=0.$$

Therefore, taking the trace with respect to W and Y, we have

$$\left(\frac{\tilde{c}}{4}-\lambda\right)2n\, g(X, hZ)=0,$$

from which we have Theorem 2. 10.

In closing this section, we state the following Theorems 2. 11 and 2. 12 which can be proved in a quite similar way for the corresponding theorems proved in the case $s=1$ (See [2]).

THEOREM 2. 11. *Let M be an invariant submanifold of codimension 2 in an $\mathcal{S}$-manifold or in a $\mathcal{T}$-manifold of constant $\tilde{f}$-sectional curvature. Then M is totally geodesic if and only if M is of constant f-sectional curvature.*

THEOREM 2. 12. *An invariant η-Einstein submanifold of codimension 2 in a $\mathcal{T}$-manifold of constant $\tilde{f}$-sectional curvature is locally symmetric.*

3. $\tilde{f}$-invariant hypersurfaces of $\tilde{M}^{2n+2}$.

Let M be an $\tilde{f}$-invariant hypersurface of an $\tilde{f}$-manifold $\tilde{M}^{2n+2}$ with complemented frames $\underset{1}{\tilde{\xi}}, \underset{2}{\tilde{\xi}}$ and $i: M\to\tilde{M}$ its imbedding. We denote the induced Riemannian metric of M by g, that is,

$$g(X, Y)=\tilde{G}(i_*X, i_*Y) \tag{3. 1}$$

for any vector fields X and Y tangent to M. Since M is $\tilde{f}$-invariant, we may put

$$\tilde{f}i_*X=i_*fX, \tag{3. 2}$$

where f is a tensor field of type (1, 1) in M.

We assume that M is orientable so that there exists a field of unit normals C to $i(M)$. Then, since $G(\tilde{f}C, i_*Y)=-G(C, \tilde{f}i_*X)=0$, we have $\tilde{f}C=0$. Hence we may put

(3.3) $$C=\alpha\underset{1}{\tilde{\xi}}+\beta\underset{2}{\tilde{\xi}},$$

where $\alpha=\underset{1}{\tilde{\eta}}(C)$, $\beta=\underset{2}{\tilde{\eta}}(C)$ and $\alpha^2+\beta^2=1$. If we define $\tilde{\xi}$ by

(3.4) $$\tilde{\xi}=-\beta\underset{1}{\tilde{\xi}}+\alpha\underset{2}{\tilde{\xi}},$$

then we see easily that $\tilde{\xi}$ is a unit tangent vector field to $i(M)$ and therefore we may put

(3.5) $$\tilde{\xi}=i_*\xi,$$

where ξ is a unit vector field in M. We denote by η the 1-form dual to ξ, that is,

(3.6) $$\eta(X)=g(X, \xi).$$

From (3.3) and (3.4), we have

(3.7) $$\underset{1}{\tilde{\xi}}=\alpha C-\beta\tilde{\xi},$$

(3.8) $$\underset{2}{\tilde{\xi}}=\beta C+\alpha\tilde{\xi}.$$

THEOREM 3.1. *An orientable $\tilde{f}$-invariant hypersurface of an $\tilde{f}$-manifold $\tilde{M}^{2n+2}$ with complemented frames admits an almost contact metric structure* (f, ξ, η, g) *defined by* (3.1), (3.2), (3.5) *and* (3.6).

Proof. First, we have

$$\eta(\xi)=g(\xi, \xi)=1,$$

$$i_*f\xi=\tilde{f}i_*\xi=\tilde{f}\tilde{\xi}=-\beta\tilde{f}\underset{1}{\tilde{\xi}}+\alpha\tilde{f}\underset{2}{\tilde{\xi}}=0,$$

$$(\eta\circ f)(X)=\eta(fX)=g(\xi, fX)=\tilde{G}(i_*\xi, i_*fX)=\tilde{G}(i_*\xi, \tilde{f}i_*X)$$
$$=-\tilde{G}(\tilde{f}i_*\xi, i_*X)=-\tilde{G}(i_*f\xi, i_*X)=0,$$

$$i_*f^2X=\tilde{f}^2i_*X=-i_*X+\sum_{x=1}^{2}\underset{x}{\tilde{\eta}}(i_*X)\underset{x}{\tilde{\xi}}=-i_*X+\underset{1}{\tilde{\eta}}(i_*X)\underset{2}{\tilde{\xi}}+\tilde{\eta}(i_*X)\underset{2}{\tilde{\xi}}$$
$$=-i_*X-\beta\eta(X)(\alpha C-\beta\tilde{\xi})+\alpha\eta(X)(\beta C+\alpha\tilde{\xi})$$
$$=-i_*X-\alpha\beta\eta(X)C+\beta^2\eta(X)\tilde{\xi}+\alpha\beta\eta(X)C+\alpha^2\eta(X)\tilde{\xi}$$
$$=-i_*X+\eta(X)\tilde{\xi}=i_*[-X+\eta(X)\xi].$$

We have also

$$g(fX, fY)=\tilde{G}(i_*fX, i_*fY)=\tilde{G}(\tilde{f}i_*X, \tilde{f}i_*Y)=G(i_*X, i_*Y)-\sum_{x=1}^{2}\underset{x}{\tilde{\eta}}(i_*X)\underset{x}{\tilde{\eta}}(i_*Y)$$
$$=g(X, Y)-\underset{1}{\tilde{\eta}}(i_*X)\underset{1}{\tilde{\eta}}(i_*Y)-\underset{2}{\tilde{\eta}}(i_*X)\underset{2}{\tilde{\eta}}(i_*Y)$$

$$=g(X, Y)-\beta^2\eta(X)\eta(Y)-\alpha^2\eta(X)\eta(Y)$$

$$=g(X, Y)-\eta(X)\eta(Y).$$

Thus, (f, ξ, η, g) is an almost contact metric structure on M.

REMARK. If we define $\tilde{f}'$ by $\tilde{f}'\tilde{X}=\tilde{f}\tilde{X}$ for $\tilde{X}\in\tilde{\mathcal{L}}$. $\tilde{f}'\underset{1}{\tilde{\xi}}=\underset{2}{\tilde{\xi}}$ and $\tilde{f}'\underset{2}{\tilde{\xi}}=-\underset{1}{\tilde{\xi}}$, then $\tilde{f}'$ is an almost complex structure on $\tilde{M}^{2n+2}$ so that the existence of an almost contact structure on an orientable hypersurface of $\tilde{M}^{2n+2}$ is clear (See [6]).

Next, since M is of codimension 1, we may put

$$\tilde{\nabla}_{i_*X}i_*Y=i_*\nabla_XY+H(X, Y)C, \tag{3.9}$$

$$\tilde{\nabla}_{i_*X}C=-i_*hX, \tag{3.10}$$

where h is a symmetric tensor field of type (1, 1) in M satisfying $H(X, Y)=g(hX, Y)$.

THEOREM 3.2. *An orientable $\tilde{f}$-invariant hypersurface of an $\mathcal{S}$-manifold $\tilde{M}^{2n+2}$ is a normal contact manifold and totally geodesic in $\tilde{M}^{2n+2}$.*

Proof. We have here

$$\tilde{\nabla}_{i_*X}C=(X\alpha)\underset{1}{\tilde{\xi}}+\alpha\tilde{\nabla}_{i_*X}\underset{1}{\tilde{\xi}}+(X\beta)\underset{2}{\tilde{\xi}}+\beta\tilde{\nabla}_{i_*X}\underset{2}{\tilde{\xi}} \qquad \text{(by (3.3))}$$

$$=(X\alpha)\underset{1}{\tilde{\xi}}-\frac{1}{2}\alpha\tilde{f}i_*X+(X\beta)\underset{2}{\tilde{\xi}}-\frac{1}{2}\beta\tilde{f}i_*X \qquad \text{(by (1.6))}$$

$$=(X\alpha)(\alpha C-\beta\tilde{\xi})-\frac{1}{2}\alpha i_*fX+(X\beta)(\beta C+\alpha\tilde{\xi})-\frac{1}{2}\beta i_*fX \qquad \text{(by (3.7) and (3.8))}$$

$$=\{(X\alpha)\alpha+(X\beta)\beta\}C-i_*\left[\frac{1}{2}(\alpha+\beta)fX+((X\alpha)\beta-(X\beta)\alpha)\xi\right]$$

$$=-i_*\left[\frac{1}{2}(\alpha+\beta)fX+((X\alpha)\beta-(X\beta)\alpha)\xi\right].$$

Thus, comparing this with (3.10), we have

$$hX=\frac{1}{2}(\alpha+\beta)fX+((X\alpha)\beta-(X\beta)\alpha)\xi. \tag{3.11}$$

Putting $X=\xi$ here, we have

$$h\xi=\gamma\xi,$$

where we have put $\gamma=(\xi\alpha)\beta-(\xi\beta)\alpha$. Thus for $Y'\in L$ we have

$$g(hY', \xi)=g(Y', h\xi)=\gamma g(Y', \xi)=0.$$

Hence we have by (3. 11)

$$hY'=\frac{1}{2}(\alpha+\beta)fY'.$$

But, since h is symmetric and f is skew-symetric with respect to g, we must have

$$\frac{1}{2}(\alpha+\beta)fY'=0,$$

which implies $\alpha+\beta=0$ and consequently (3. 11) becomes $hX=0$, since $\alpha=-\beta=1/\sqrt{2}$ or $\alpha=-\beta=-1/\sqrt{2}$. Thus, M is totally geodesic. We also have

$$[\tilde{f}, \tilde{f}](i_*X, i_*Y)=i_*[f, f](X, Y)$$

and

$$\begin{aligned}\sum_{x=1}^{2} d\underset{x}{\tilde{\eta}}(i_*X, i_*Y)\underset{x}{\tilde{\xi}} &= \{\tilde{\nabla}_{i_*X}(\underset{1}{\tilde{\eta}}(i_*Y))-\tilde{\nabla}_{i_*Y}(\underset{1}{\tilde{\eta}}(i_*X))-\underset{1}{\tilde{\eta}}(i_*[X, Y])\}\underset{1}{\tilde{\xi}}\\ &\quad+\{\tilde{\nabla}_{i_*X}(\underset{2}{\tilde{\eta}}(i_*Y))-\tilde{\nabla}_{i_*Y}(\underset{2}{\tilde{\eta}}(i_*X))-\underset{2}{\tilde{\eta}}(i_*[X, Y])\}\underset{2}{\tilde{\xi}}\\ &=\{\tilde{\nabla}_{i_*X}(-\beta\eta(Y))-\tilde{\nabla}_{i_*Y}(-\beta\eta(X))+\beta\eta([X, Y])\}\underset{1}{\tilde{\xi}}\\ &\quad+\{\tilde{\nabla}_{i_*X}(\alpha\eta(Y))-\tilde{\nabla}_{i_*Y}(\alpha\eta(X))-\alpha\eta([X, Y])\}\underset{2}{\tilde{\xi}}\\ &=-\beta d\eta(X, Y)\underset{1}{\tilde{\xi}}+\alpha d\eta(X, Y)\underset{2}{\tilde{\xi}}\\ &=d\eta(X, Y)i_*\xi.\end{aligned}$$

Thus, we have $[f, f](X, Y)+d\eta(X, Y)\xi=0$, that is, M is an almost normal contact manifold. Finally, we have

$$\begin{aligned}F(X, Y)\equiv g(X, fY)&=\tilde{G}(i_*X, i_*fY)=\tilde{G}(i_*X, \tilde{f}i_*Y)=\tilde{F}(i_*X, i_*Y)\\ &=d\underset{2}{\tilde{\eta}}(i_*X, i_*Y)=\alpha d\eta(X, Y).\end{aligned}$$

Thus, to show that M is normal, it is sufficient to prove the following Lemma 3. 3:

LEMMA 3. 3. *Let M be an almost normal contact manifold with an (ϕ, ξ, η, g)-structure and fundamental 2-form F. If $F(X, Y)=kd\eta(X, Y)$, where k is a non-zero contant, then M is a normal contact manifold.*

Proof. We now put

$$\hat{\xi}=k\xi,$$

$$\hat{\eta}=\frac{1}{k}\eta,$$

$$\hat{\phi}=\phi, \tag{3.13}$$
$$\hat{g}(X, Y)=\frac{1}{k^2}g(X, Y).$$

Then, (3.13) gives a normal contact metric structure on M. Indeed, we have

$$\hat{\eta}(\hat{\xi})=\hat{g}(\hat{\xi}, \hat{\xi})=\frac{1}{k^2}g(k\xi, k\xi)=1,$$

$$\hat{\eta}\circ\hat{\phi}=\frac{1}{k}\eta\circ\phi=0, \qquad \hat{\phi}\hat{\xi}=k\phi\xi=0,$$

$$\hat{\phi}^2X=\phi^2X=-X+\eta(X)\xi=-X+\hat{\eta}(X)\hat{\xi},$$

$$\hat{g}(X, Y)=\frac{1}{k^2}g(X, Y)=\frac{1}{k^2}g(\phi X, \phi Y)+\frac{1}{k}\eta(X)\frac{1}{k}\eta(Y)$$
$$=\hat{g}(\hat{\phi}X, \hat{\phi}Y)+\hat{\eta}(X)\hat{\eta}(Y),$$

$$[\hat{\phi}, \hat{\phi}](X, Y)+d\hat{\eta}(X, Y)\hat{\xi}=[\phi, \phi](X, Y)+\{X(\hat{\eta}(Y))-Y(\hat{\eta}(X))-\hat{\eta}([X, Y])\}\hat{\xi}$$
$$=[\phi, \phi](X, Y)+\{X(\eta(Y))-Y(\eta(X))-\eta([X, Y])\}$$
$$=[\phi, \phi](X, Y)+d\eta(X, Y)\xi=0,$$

and, if we denote by $\hat{F}$ the fundamental 2-form corresponding to $\hat{\phi}$, we have

$$\hat{F}(X, Y)=\hat{g}(X, \hat{\phi}Y)=\frac{1}{k^2}g(X, \phi Y)=\frac{1}{k^2}F(X, Y)=\frac{1}{k}d\eta(X, Y)=d\hat{\eta}(X, Y),$$

which shows that M is a normal contact manifold.

Next, we shall prove

THEOREM 3.4. *If M is an $\tilde{f}$-invariant hypersurface of an $\mathcal{S}$-manifold $\tilde{M}^{2n+2}$ of constant $\tilde{f}$-sectional curvature $\tilde{c}$, then M is η-Einstein.*

Proof. Since M is totally geodesic, by Theorem 3.2, we have

$$\tilde{R}(i_*X, i_*Y)i_*Z=i_*R(X, Y)Z.$$

Thus, by the formula of Proposition 1.5 with $s=2$, noticing that

$$\tilde{\Phi}(i_*X, i_*Y)=\tilde{\eta}_1(i_*X)\tilde{\eta}_1(i_*Y)+\tilde{\eta}_2(i_*X)\tilde{\eta}_2(i_*Y)$$
$$=\beta^2\eta(X)\eta(Y)+\alpha^2\eta(X)\eta(Y)$$
$$=\eta(X)\eta(Y)$$

and

$$\underset{1}{\tilde{\eta}}(i_*X)\underset{2}{\tilde{\eta}}(i_*Y)+\underset{2}{\tilde{\eta}}(i_*X)\underset{1}{\tilde{\eta}}(i_*Y)=-\alpha\beta\eta(X)\eta(Y)-\alpha\beta\eta(X)\eta(Y)$$
$$=\eta(X)\eta(Y),$$

we then have

$$g(R(X,\ Y)Z,\ W)=\left(\frac{\tilde{c}}{4}+\frac{3}{8}\right)\{g(X,\ Z)g(W,\ Y)-g(X,\ W)g(Y,\ Z)\}$$
$$+\left(\frac{\tilde{c}}{4}-\frac{1}{8}\right)\{-g(X,\ Z)\eta(W)\eta(Y)-g(W,\ Y)\eta(Z)\eta(X)+g(X,\ W)\eta(Z)\eta(Y)$$
$$+g(Y,Z)\eta(X)\eta(W)+F(W,\ X)F(Y,\ Z)+F(Y,\ W)F(X,\ Z)$$
$$-2F(X,\ Y)F(W,\ Z)\}.$$

Thus, taking the trace with respect to Y and W, we have

$$S(X,\ Z)=\left\{\left(\frac{\tilde{c}}{4}+\frac{3}{8}\right)2n+2\left(\frac{\tilde{c}}{4}-\frac{1}{8}\right)\right\}g(X,\ Z)-\left(\frac{\tilde{c}}{4}-\frac{1}{8}\right)(2n+2)\eta(X)\eta(Z),$$

which shows that M is η-Einstein.

COROLLARY 3. 5. *If M is an $\tilde{f}$-invariant hypersurface of an $\mathcal{S}$-manifold M^{2n+2} of constant $\tilde{f}$-sectional curvature* 1/2, *then M is of constant curvature* 2.

In the last step, we consider the case where $\tilde{M}^{2n+2}$ is a $\mathcal{T}$-manifold. We shall now prove

THEOREM 3. 6. *An orientable $\tilde{f}$-invariant hypersurface of a $\mathcal{T}$-manifold is a cosympletic manifold.*

Proof. Putting $Y=\xi$ in (3. 9), we have

$$\tilde{\nabla}_{i_*X}i_*\xi=i_*\nabla_X\xi+H(X,\ \xi)C.$$

On the other hand, using (3. 4) and (1. 7), we have

$$\tilde{\nabla}_{i_*X}i_*\xi=\tilde{\nabla}_{i_*X}\tilde{\xi}=-(X\beta)\underset{1}{\tilde{\xi}}+(X\alpha)\underset{2}{\tilde{\xi}}$$
$$=-(X\beta)(\alpha C-\beta\tilde{\xi})+(X\alpha)(\beta C+\alpha\tilde{\xi})$$
$$=\{-\alpha(X\beta)+\beta(X\alpha)\}C.$$

Hence we have $\nabla_X\xi=0$. Thus, we have here

$$d\eta(X,\ Y)=X(\eta(Y))-Y(\eta(X))-\eta([X,\ Y])$$
$$=g(\nabla_XY,\ \xi)-g(\nabla_YX,\ \xi)-g([X,\ Y],\ \xi)$$
$$=0,$$

which shows that M is a cosymplectic manifold.

THEOREM 3.7. *If M is an orientable $\tilde{f}$-invariant hypersurface of a $\mathcal{T}$-manifold $\tilde{M}^{2n+2}$ of constant $\tilde{f}$-sectional curvature $\tilde{c}$, then M is η-Einstein.*

Proof. Using (1.7), we have

$$\begin{aligned}\tilde{\nabla}_{i_*X}C=\tilde{\nabla}_{i_*X}(\alpha\underset{1}{\tilde{\xi}}+\beta\underset{2}{\tilde{\xi}})&=(X\alpha)\underset{1}{\tilde{\xi}}+(X\beta)\underset{2}{\tilde{\xi}}\\&=(X\alpha)(\alpha C-\beta\tilde{\xi})+(X\beta)(\beta C+\alpha\tilde{\xi})\\&=-i_*[(X\alpha)\beta-(X\beta)\alpha]\xi.\end{aligned}$$

Thus we have, by (3.10),

$$hX=\{(X\alpha)\beta-(X\beta)\alpha\}\xi,$$

from which we have

$$hX'=0 \qquad (\text{for} \quad X'\in L) \tag{3.14}$$

$$h\xi=\gamma\xi. \tag{3.15}$$

On the other hand, we have the equation of Gauss

$$\tilde{G}(\tilde{R}(i_*X, i_*Y)i_*Z, i_*W)=g(R(X, Y)Z, W)+g(hY, Z)g(hX, W)-g(hX, Z)g(hY, W).$$

Thus, by the formula of Proposition 1.5 with $s=2$, we have

$$\begin{aligned}g(R(X, Y)Z, W)=\frac{\tilde{c}}{4}\{&g(X, Z)g(W, Y)-g(X, W)g(Y, Z)-g(X, Z)\eta(W)\eta(Y)\\&-g(W, Y)\eta(Z)\eta(X)+g(X, W)\eta(Z)\eta(Y)+g(Y, Z)\eta(X)\eta(W)\\&+F(W, X)F(Y, Z)+F(Y, W)F(X, Z)-2F(X, Y)F(W, Z)\}\\&-g(hY, Z)g(hX, W)+g(hX, Z)g(hY, W).\end{aligned}$$

Therefore, taking account of (3.14) and (3.15), we have

$$\begin{aligned}S(X, Y)&=\frac{n\tilde{c}}{2}\{g(X, Z)-\eta(X)\eta(Z)\}-g(hX, hZ)+g(hX, Z)\,\text{trace}\,h\\&=\frac{n\tilde{c}}{2}\{g(X, Z)-\eta(X)\eta(Z)\}-g(hX, \xi)g(hZ, \xi)+g(hX, \xi)g(Z, \xi)\\&=\frac{n\tilde{c}}{2}\{g(X, Z)-\eta(X)\eta(Z)\}-\gamma^2\eta(X)\eta(Z)+\gamma^2\eta(X)\eta(Z)\\&=\frac{n\tilde{c}}{2}\{g(X, Z)-\eta(X)\eta(Z)\}.\end{aligned}$$

Thus M is η-Einstein.

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