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VECTOR FIELDS IN A METRIC MANIFOLD WITH TORSION AND BOUNDARY

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The integral formulas and their applications in compact orientable Riemannian, Kahlerian or almost Hermitian manifolds have been studied by several authors (see [8]). Hsuing and Shahin [5] and Ishihara [6] have studied the integral formulas and their applications in certain compact orientable affinely connected manifolds.

The vector fields in compact orientable Riemannian or almost Hermitian manifolds with boundary have been studied, using integral formulas, by Ako [13], Hilt [2], Hsuing ([2], [3], [4]), Takahashi ([2], [12]), Yano ([10], [11], [12]) and others.

Hayden [1] introduced the metric connection with torsion in a Riemannian manifold. Yano and Bochner [9] have studied vector fields in a compact orientable Riemannian manifold with torsion by use of integral formulas.

The purpose of the present paper is to study systematically some special vector fields in a Riemannian manifold with torsion and with boundary.

§1. Metric manifolds with torsion.

We consider an n-dimensional differentiable manifold M on which there is given a positive definite metric

$$ds^2=g_{ji}dx^j dx^i \;{}^{1)}$$

and a metric connection $\Gamma_j{}^h{}_i$, that is, a connection such that

$$\nabla_j g_{ih}=0, \tag{1.1}$$

where ∇_j denotes the covariant differentiation with respect to $\Gamma_j{}^h{}_i$.

Denoting by $\{{}_j{}^h{}_i\}$ Christoffel symbols formed with g_{ji}, we can put

$$\Gamma_j{}^h{}_i=\left\{ {h \atop j\ i} \right\}+T_{ji}{}^h \tag{1.2}$$

where T is a tensor feild of type (1, 2).

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1) The indices $h, i, j, k, l, \cdots$ run over the range $1, 2, \cdots, n$. The so-called Einstein's summation convention is used with respect to this system of indices.

Equations (1. 1) and (1. 2) show that

$$\nabla_j g_{ih} = -T_{jih} - T_{jhi} = 0, \tag{1. 3}$$

where

$$T_{jih} = T_{ji}{}^t g_{th}.$$

The connection $\Gamma_j{}^h{}_i$ needs not be symmetric and we denote by

$$S_{ji}{}^h = \frac{1}{2}(\Gamma_j{}^h{}_i - \Gamma_i{}^h{}_j) \tag{1. 4}$$

the torsion tensor.

From (1. 2), (1. 3) and (1. 4), we find the metric connection $\Gamma_j{}^h{}_i$ has the form

$$\Gamma_j{}^h{}_i = \left\{ {h \atop j\ i} \right\} + S_{ji}{}^h + S^h{}_{ji} + S^h{}_{ij}, \tag{1. 5}$$

where $S^h{}_{ji} = g^{mh} g_{it} S_{mj}{}^t$, g^{mh} being contravariant components of the metric tensor.

The curvature tensor of the metric connection $\Gamma_j{}^h{}_i$ is given by

$$R_{kji}{}^h = \partial_k \Gamma_j{}^h{}_i - \partial_j \Gamma_k{}^h{}_i + \Gamma_k{}^h{}_t \Gamma_j{}^t{}_i - \Gamma_j{}^h{}_t \Gamma_k{}^t{}_i,$$

and we have

$$\begin{aligned} &R_{kjih} = -R_{jkih} = -R_{kjhi}, \\ &R_{kjih} + R_{jikh} + R_{ikjh} = 0, \\ &R_{kjih} = R_{ihkj}, \end{aligned} \tag{1. 6}$$

where

$$R_{kjih} = R_{kji}{}^t g_{th}.$$

The Ricci formulas for a contravariant vector field, for a covariant vector field, for a scalar and for a general tensor field, for example, of type (1, 2) are given, respectively by

$$\nabla_k \nabla_j v^h - \nabla_j \nabla_k v^h = R_{kji}{}^h v^i - 2S_{kj}{}^t \nabla_t v^h,$$

$$\nabla_k \nabla_j w_i - \nabla_j \nabla_k w_i = -R_{kji}{}^h w_h - 2S_{kj}{}^t \nabla_t w_i,$$

$$\nabla_j \nabla_i f - \nabla_i \nabla_j f = -2S_{ji}{}^h \nabla_h f,$$

$$\nabla_l \nabla_k T_{ji}{}^h - \nabla_k \nabla_l T_{ji}{}^h = R_{lkt}{}^h T_{ji}{}^t - R_{lkj}{}^t T_{ti}{}^h - R_{lki}{}^t T_{jt}{}^h - 2S_{lk}{}^t \nabla_t T_{ji}{}^h.$$

We consider a hypersurface B in the metric manifold M^n and represent it by parametric equations

$$x^h = x^h(u^a),$$

where here and in the sequel the indices $a, b, c, \cdots$ run over the range $1, 2, \cdots, n-1$. We put

$$B_a{}^h = \partial_a x^h,$$

where ∂_a denotes partial differentiation with respect to u^a. The $B_a{}^h$ represent $n-1$ linearly independent contravariant vectors tangent to the hypersurface. The induced metric g_{cb} of the hypersurface is given by

$$g_{cb} = g_{ji} B_c{}^j B_b{}^i. \tag{1.8}$$

The connection $\Gamma_c{}^a{}_b$ induced on the hypersurface is given by

$$\Gamma_c{}^a{}_b = B^a{}_h (\partial_c B_b{}^h + B_c{}^j B_b{}^i \Gamma_j{}^h{}_i), \tag{1.9}$$

where

$$B^a{}_h = B_d{}^t g^{ad} g_{th}.$$

Thus the torsion tensor of $\Gamma_c{}^a{}_b$ is given by

$$S_{cb}{}^a \overset{\text{def}}{=} \frac{1}{2} (\Gamma_c{}^a{}_b - \Gamma_b{}^a{}_c) = S_{ji}{}^h B_c{}^j B_b{}^i B^a{}_h,$$

Since the induced connection is metric and has the torsion $S_{cb}{}^a$, we have

$$\Gamma_c{}^a{}_b = \left\{ {a \atop c\ b} \right\} + S_{cb}{}^a + S^a{}_{cb} + S^a{}_{bc},$$

where $\{{}_c{}^a{}_b\}$ are Christoffel symbols formed with the induced metric g_{cb}.

§ 2. Stokes' theorem.

We assume that the n-dimensional metric manifold M is compact and is the closure of an open submanifold of an n-dimensional orientable metric manifold V^n of class C^∞. The Riemannian metric of M is given by $ds^2 = g_{ji} dx^j dx^i$ and is represented, in a neighborhood of each point lying on its boundary B by $x^n \geqq 0$ with respect to certain coodinates (x^h). It follows that B is an $(n-1)$-dimensional compact orientable submanifold (cf. [6]). We call the manifold M a compact orientable manifold with regular boundary B.

We shall represent B by parametric equations

$$x^h = x^h(u^a)$$

and put

$$\begin{aligned} B_b{}^h &= \partial_b x^h \qquad (\partial_b = \partial/\partial u^b), \\ g_{cb} &= g_{ji} B_c{}^j B_b{}^i, \end{aligned} \tag{2.1}$$

and denote by N^h the unit normal to B such that N^h and $B_1{}^h, B_2{}^h, \cdots, B_{n-1}{}^h$ form the positive sense of M. Then we have

$$g_{ji}N^jB_b{}^i=0, \qquad g_{ji}N^jN^i=1,$$

(2. 2)

$$\sqrt{g}\,|N^h, B_b{}^h|=\sqrt{'g}$$

where $|N^h, B_b{}^h|$ denotes the determinant formed by N^h and $B_1{}^h, \cdots, B_{n-1}{}^h$ and

$$g=|g_{ji}|, \qquad 'g=|g_{cb}|$$

are those formed by g_{ji} and g_{cb} respectively.

We denote by $\mathring{\nabla}_j$ the covariant differentiation in the metric manifold M with respect $\{{}_j{}^h{}_i\}$, and by $\mathring{\nabla}_c$ the covariant differentiation along the boundary B with respect to $\{{}_c{}^a{}_b\}$. We recall the equations of Gauss and those of Weingarten;

(2. 3)

$$\mathring{\nabla}_cB_b{}^h=\partial_cB_b{}^h+B_c{}^jB_b{}^i\left\{ {h \atop j\ i} \right\}-B_a{}^h\left\{ {a \atop c\ b} \right\}=\mathring{H}_{cb}N^h,$$

$$\mathring{\nabla}_cN^h=\partial_cN^h+B_c{}^jN^i\left\{ {h \atop j\ i} \right\}=-\mathring{H}_c{}^bB_b{}^h,$$

where $\mathring{H}_{cb}$ are components of the second fundamental tensor of the boundary B and $\mathring{H}_c{}^b=\mathring{H}_{ca}g^{ab}$.

We put

$$B^a{}_h=B_b{}^ig^{ba}g_{ih},$$

then we have

(2. 4) $$B^a{}_hB_b{}^h=\delta_b^a, \qquad B^a{}_hN^h=0, \qquad N_iN^h+B^a{}_iB_a{}^h=\delta_i^h.$$

We now state Stokes' theorem in the following form:

STOKES' THEOREM. *We consider a compact orientable Riemannian manifold M with regular boundary B. Then, for an arbitrary vector field v^h, we have an integral formula*

$$\int_M \mathring{\nabla}_iv^i d\sigma=\int_B v_iN^i d'\sigma,$$

where

$$d\sigma=\sqrt{g}\,dx^1\wedge dx^2\wedge\cdots\wedge dx^n$$

is the volume element of M and

$$d'\sigma=\sqrt{'g}\,du^1\wedge du^2\wedge\cdots\wedge du^{n-1}$$

the surface element of B.

§ 3. A generalization of Stokes' theorem.

We shall consider a compact orientable metric manifold M with torsion $S_{ji}{}^{h}$ and regular boundary B and assume that the torsion tensor $S_{ji}{}^{h}$ satisfies the condition

$$S_{jt}{}^{t}=0. \tag{3.1}$$

For any vector field v^h, we have by means of (1.5)

$$\nabla_j v^h=\mathring{\nabla}_j v^h+(S_{ji}{}^{h}+S^{h}{}_{ji}+S^{h}{}_{ij})v^i,$$

where ∇_j denotes the covariant differentiation respect to $\Gamma_j{}^h{}_i$ whose torsion tensor satisfies (3.1).

By virtue of (1.4), we get

$$\nabla_t v^t=\mathring{\nabla}_t v^t-2S_{jt}{}^{t}v^j.$$

By assuming (3.1), we have

$$\nabla_t v^t=\mathring{\nabla}_t v^t. \tag{3.2}$$

Therefore, we get the generalization of Stokes' theorem under the assumption (3.1);

GENERALIZED STOKES' THEOREM. *We consider a compact orientable metric manifold M with regular boundary B and torsion satisfying* (3.1). *Then, for an arbitrary vector field v^h, we have an integral formula*

$$\int_M \nabla_i v^i d\sigma=\int_B v_i N^i d'\sigma.$$

In the sequel, to apply the generalized Stokes' theorem, we assume that M is a compact orientable metric manifold with regular boundary B and the torsion tensor S satisfies (3.1).

§ 4. Pseudo-Killing vector fields.

We shall call a vector field v^h a pseudo-Killing vector field, if it satisfies the condition

$$\nabla_j v_i+\nabla_i v_j=0. \tag{4.1}$$

A pseudo-Killing vector field satisfies

$$\nabla_i v^i=0. \tag{4.2}$$

Differentiating (4.1) covariantly and taking account of (1.7) and (4.2), we get

$$g^{ji}\nabla_j\nabla_i v^h + K^h{}_i v^i - 2S_j{}^{hi}\nabla_i v^j = 0. \tag{4.3}$$

Now, by a straightforward computation we can prove

$$\begin{aligned}(g^{ji}\nabla_j\nabla_i v^h + K^h{}_i v^i - 2S_j{}^{hi}\nabla_i v^j)v_h + \frac{1}{2}(\nabla_j v_i + \nabla_i v_j)(\nabla^j v^i + \nabla^i v^j) - (\nabla_i v^i)^2 \\ = \nabla^j[(\nabla_j v_i + \nabla_i v_j)v^i - v_j(\nabla_i v^i)],\end{aligned} \tag{4.4}$$

which is valid for an arbitrary vector field v^h, where $\nabla^j = g^{ji}\nabla_i$. We assume that torsion $S_{ji}{}^h$ of M satisfies (3. 1) and integrate both members of (4. 4) on the whole manifold M and apply the generalized Stokes' theorem to the right hand member.

$$\begin{aligned}\int_M\Big[(g^{ji}\nabla_j\nabla_i v^h + K^h{}_i v^i - 2S_j{}^{hi}\nabla_i v^j)v_h + \frac{1}{2}(\nabla_j v_i + \nabla_i v_j)(\nabla^j v^i + \nabla^i v^j) - (\nabla_i v^i)^2\Big]d\sigma \\ = \int_B [(\nabla_j v_i + \nabla_i v_j)v^i - v_j(\nabla_i v^i)]N^j d'\sigma.\end{aligned} \tag{4.5}$$

Suppose now that v^h is a pseudo-Killing vector field. Then it satisfies

$$g^{ji}\nabla_j\nabla_i v^h + K^h{}_i v^i - 2S_j{}^{hi}\nabla_i v^j = 0 \quad \text{and} \quad \nabla_i v^i = 0 \quad \text{in} \quad M$$

and

$$(\nabla_j v_i + \nabla_i v_j)N^j v^i = 0 \qquad \text{on} \quad B.$$

Conversely, if a vector field v^h satisfies these conditions, then we have from (4. 5)

$$\frac{1}{2}\int_M (\nabla_j v_i + \nabla_i v_j)(\nabla^j v^i + \nabla^i v^j)d\sigma = 0,$$

from which

$$\nabla_j v_i + \nabla_i v_j = 0 \qquad \text{in} \quad M$$

and consequently v^h is a pseudo-Killing vector field. Thus we have

PROPOSITION 4. 1. *A necessary and sufficient condition for a vector field v^h in M with boundary B to be a pseudo-Killing vector field is that*

$$\begin{cases} g^{ji}\nabla_j\nabla_i v^h + K^h{}_i v^i - 2S_j{}^{hi}\nabla_i v^j = 0, \quad \nabla_i v^i = 0 & \text{in} \quad M, \\ (\nabla_j v_i + \nabla_j v_i)N^j v^i = 0 & \text{on} \quad B. \end{cases} \tag{4.6}$$

As a special case, if the vector field v^h vanishes on the boundary B, then the second condition in (4. 6) is automatically satisfied. Thus we have

COROLLARY 4. 1. *A necessary and sufficient condition for a vector field v^h in M with boundary B vanishing identically on B to be a pseudo-Killing vector field is that*

$$g^{ji}\nabla_j\nabla_i v^h + K^h{}_i v^i - 2S_j{}^{hi}\nabla_i v^j = 0 \qquad \nabla_i v^i = 0 \quad \text{in} \quad M.$$

Now we put, on the boundary B

$$v^h = v^a B_a{}^h + \alpha N^h \tag{4. 7}$$

then we have

$$B_b{}^i v_i = v_b, \qquad N^i v_i = \alpha. \tag{4. 8}$$

We consider the equations of Gauss and those of Weingarten with respect to $\Gamma_j{}^h{}_i$ and $\Gamma_c{}^a{}_b$ (cf. (1. 7), (2. 3)) [13].

$$\begin{aligned} \nabla_c B_b{}^h &= \partial_c B_b{}^h + B_b{}^j B_c{}^i \Gamma_j{}^h{}_i - B_a{}^h \Gamma_c{}^a{}_b = H_{cb} N^h, \\ \nabla_c N^h &= \partial_c N^h + B_c{}^j N^i \Gamma_j{}^h{}_i = -H_c{}^a B_a{}^h \end{aligned} \tag{4. 9}$$

where H_{cb} are components of the second fundamental tensor of the boundary B with respect to $\Gamma_j{}^h{}_i$ and $H_c{}^a = H_{cb} g^{ba}$.

Differentiating the first equation of (4. 8) covariantly along the boundary B and taking account of (4. 8) and (4. 9), we find

$$\alpha H_{cb} + B_c{}^j B_b{}^i \nabla_j v_i = \nabla_c v_b,$$

from which, transvecting with g^{cb} and taking account of (2. 4),

$$\alpha H_a{}^a + \nabla_i v^i - (\nabla_j v_i) N^j N^i = \nabla_a v^a. \tag{4. 10}$$

Differentiating next the second equation of (4. 8) covariantly along the boundary and taking account of (4. 8) and (4. 9), we obtain

$$-H_c{}^b v_b + B_c{}^j N^i (\nabla_j v_i) = \nabla_c \alpha,$$

from which, transvecting with v^c,

$$-H_{cb} v^c v^b + (\nabla_j v_i) v^j N^i - \alpha (\nabla_j v_i) N^j N^i = v^c \nabla_c \alpha \tag{4. 11}$$

by virtue of (4. 7).

Eliminating $(\nabla_j v_i) N^j N^i$ from (4. 10) and (4. 11), we obtain

$$(\nabla_j v_i) v^j N^i = H_{cb} v^c v^b + \alpha^2 H_a{}^a + \alpha(\nabla_i v^i) - 2\alpha(\nabla_a v^a) + \nabla_a(\alpha v^a), \tag{4. 12}$$

from which

$$(4.13)\qquad (\nabla_j v_i+\nabla_i v_j)N^j v^i=(\nabla_j v_i)N^j v^i+H_{cb}v^c v^b+\alpha^2 H_a{}^a+\alpha(\nabla_i v^i)-2\alpha(\nabla_a v^a)+\nabla_a(\alpha v^a).$$

Thus we have

COROLLARY 4.2. *A necessary and sufficient condition for a vector field v^h in M with boundary B to be a pseudo-Killing vector field is that*

$$(4.14)\qquad \begin{aligned} &g^{ji}\nabla_j\nabla_i v^h+K^h{}_i v^i-2S_j{}^{hi}\nabla_i v^j=0, \qquad \nabla_i v^i=0 \quad \textit{in} \quad M, \\ &(\nabla_j v_i)N^j v^i+H_{cb}v^c v^b+\alpha^2 H_a{}^a-2\alpha(\nabla_a v^a)+\nabla_a(\alpha v^a)=0 \quad \textit{on} \quad B. \end{aligned}$$

Now if the vector v^h is tangential to B, then we have $\alpha=0$ and consequently we have

COROLLARY 4.3. *A necessary and sufficient condition for a vector field v^h in M tangential to the boundary B to be a pseudo-Killing vector field is that*

$$(4.15)\qquad \begin{aligned} &g^{ji}\nabla_j\nabla_i v^h+K^h{}_i v^i-2S_j{}^{hi}\nabla_i v^j=0, \qquad \nabla_i v^i=0 \quad \textit{in} \quad M, \\ &(\nabla_j v_i)N^j v^i+H_{cb}v^c v^b=0 \qquad\qquad \textit{on} \quad B. \end{aligned}$$

If the vector field v^h is normal to the boundary B, then we have $v^a=0$ and $v^h=\alpha H^h$ and consequently

$$(\nabla_j v_i+\nabla_i v_j)N^j v^i=2[\alpha^2 H_a{}^a+\alpha(\nabla_i v^i)],$$

by virtue of (4.10). Thus we have

COROLLARY 4.4. *A necessary and sufficient condition for a vector field v^h in M normal to the boundary B to be a psudo-Killing vector field is that*

$$(4.16)\qquad \begin{aligned} &g^{ji}\nabla_j\nabla_i v^h+K^h{}_i v^i-2S_j{}^{hi}\nabla_i v^j=0, \qquad \nabla_i v^i=0 \quad \textit{in} \quad M, \\ &\alpha H_a{}^a=0 \qquad\qquad \textit{on} \quad B. \end{aligned}$$

§ 5. Pseudo-harmonic vectors.

We shall call a vector field v^h a pseudo-harmonic vector field, if it satisfies the conditions

$$(5.1)\qquad \nabla_j v_i-\nabla_i v_j=0, \qquad \nabla_i v^i=0.$$

For a pseudo-harmonic vector field, we have

$$(5.2)\qquad g^{ji}\nabla_j\nabla_i v^h-K^h{}_i v^i-2S_j{}^{hi}\nabla_i v^j=0,$$

by virtue of (1.5) and (5.1).

By a straightforward computation, we can prove

$$(5.3)\qquad \begin{aligned} &(g^{ji}\nabla_j\nabla_i v^h - K^h{}_i v^i - 2S_j{}^{hi}\nabla_i v^j)v_h + \frac{1}{2}(\nabla_j v_i - \nabla_i v_j)(\nabla^j v^i - \nabla^i v^j) \\ &\qquad + (\nabla_i v^i)^2 = \nabla^j[(\nabla_j v_i - \nabla_i v_j)v^i + v_j(\nabla_i v^i)], \end{aligned}$$

which is valid for an arbitrary vector field v^h. So, we assume that the torsion of M satisfies (3.1), and integrate the both members of (5.3) on the whole M and apply generalized Stokes' theorem to the right hand member, then we get

$$(5.4)\qquad \int_M \left[(g^{ji}\nabla_j\nabla_i v^h - K^h{}_i v^i - 2S_j{}^{hi}\nabla_i v^j)v_h + \frac{1}{2}(\nabla_j v_i - \nabla_i v_j)(\nabla^j v^i - \nabla^i v^j) + (\nabla_i v^i)^2\right] d\sigma$$

$$= \int_B [(\nabla_j v_i - \nabla_i v_j)v^i + v_j(\nabla_i v^i)]N^j d'\sigma.$$

Suppose that v^h is a pseudo-harmonic vector field. Then it satisfies

$$g^{ji}\nabla_j\nabla_i v^h - K^h{}_i v^i - 2S_j{}^{hi}\nabla_i v^j = 0 \qquad \text{in} \quad M$$

and

$$[(\nabla_j v_i - \nabla_i v_j)v^i + v_j(\nabla_i v^i)]N^j = 0 \qquad \text{on} \quad B$$

Conversely if a vector field satisfies these conditions, then we have from (5.4)

$$\int_M \left[\frac{1}{2}(\nabla_j v_i - \nabla_i v_j)(\nabla^j v^i - \nabla^i v^j) + (\nabla_i v^i)^2\right] d\sigma = 0$$

from which

$$\nabla_j v_i - \nabla_i v_j = 0, \qquad \nabla_i v^i = 0 \qquad \text{in} \quad M.$$

Thus we have

PROPOSITION 5.1. *A necessary and sufficient condition for a vector field v^h in M with boundary B to be a pseudo-harmonic vector field is that*

$$(5.5)\qquad \begin{cases} g^{ji}\nabla_j\nabla_i v^h - K^h{}_i v^i - 2S_j{}^{hi}\nabla_i v^j = 0 & \text{in} \quad M, \\ [(\nabla_j v_i - \nabla_i v_j)v^i + v_j(\nabla_i v^i)]N^j = 0 & \text{on} \quad B. \end{cases}$$

COROLLARY 5.1. *A necessary and sufficient condition for a vector field v^h in M with boundary B vanishing on B to be a pseudo-harmonic vector field is that*

$$g^{ji}\nabla_j\nabla_i v^h - K^h{}_i v^i - 2S_j{}^{hi}\nabla_i v^j = 0 \qquad \text{in} \quad M.$$

COROLLARY 5.2. *A necessary and sufficient condition for a vector field v^h in M with boundary B to be a pseudo-harmonic vector field is that*

$$
(5.6) \qquad \begin{aligned} & g^{ji}\nabla_j\nabla_i v^h - K^h{}_i v^i - 2S_j{}^{hi}\nabla_i v^j = 0 \qquad && in \quad M, \\ & (\nabla_j v_i)N^j v^i - H_{cb}v^c v^b - \alpha^2 H_a{}^a + 2\alpha(\nabla_a v^a) - \nabla_a(\alpha v^a) = 0 \qquad && on \quad B. \end{aligned}
$$

COROLLARY 5.3. *A necessary and sufficient condition for a vector field v^h in M tangential to the boundary B to be a pseudo-harmonic vector field in that*

$$
(5.7) \qquad \begin{aligned} & g^{ji}\nabla_j\nabla_i v^h - K^h{}_i v^i - 2S_j{}^{hi}\nabla_i v^j = 0 \qquad && in \quad M. \\ & (\nabla_j v_i)N^j v^i - H_{cb}v^c v^b = 0 \qquad && on \quad B. \end{aligned}
$$

COROLLARY 5.4. *A necessary and sufficient condition for a vector field v^h in M normal to the boundary B to be a pseudo-harmonic vector field is that*

$$
(5.8) \qquad \begin{aligned} & g^{ji}\nabla_j\nabla_i v^h - K^h{}_i v^i - 2S_j{}^{hi}\nabla_i v^j = 0 \qquad && in \quad M, \\ & \alpha(\nabla_i v^i) = 0 \qquad && on \quad B. \end{aligned}
$$

§6. Concluding remarks.

It is well known that in a compact orientable Riemannian manifold, the inner product of a Killing vector field and a harmonic vector field is constant. We remark that for a pseudo-Killing vector field and a pseudo-harmonic vector field, we get a similar result, that is,

PROPOSITION 6.1. *In a compact orientable metric manifold M whose torsion tensor S_{jih} is skew-symmetric with respect to all indices, the inner product of a pseudo-Killing vector field v^h and a pseudo-harmonic vector field w_h is constant.*

Proof. Computing the Laplacian of inner product $v^h w_h$ with respect to metric connection ∇ whose torsion is S_{jih}, we get

$$
(6.1) \qquad \Delta(w_h v^h) = \nabla^i\nabla_i w_h v^h + w_h \nabla^i\nabla_i v^h
$$

because $\nabla_j v_i$ is skew-symmetric and $\nabla_j w_i$ is symmetric. Using the Ricci formulas (1.7), (6.1) is rewritten as

$$
(6.2) \qquad \Delta(w_h v^h) = (K_{ji} - K_{ij})v^j w^i - 2S^{jih}(\nabla_h w_j \cdot v_i - \nabla_h v_j \cdot w_i).
$$

Since we have

$$
K_{ji} = \mathring{K}_{ji} + \mathring{\nabla}_t T_{ji}{}^t - T_{jt}{}^s T_{si}{}^t
$$

from the definition of the Ricci tensor of the metric connection $\Gamma_j{}^h{}_i$, (6.2) reduces to

$$\Delta(w_h v^h) = 2\nabla_h(S_{ji}{}^h v^j w^i) - 2(S^j{}_{ih} S^h{}_{jk} + S^j{}_{hi} S_{jk}{}^h) v^i w^k. \tag{6.3}$$

As M is compact and S_{jih} is skew-symmetric with respect to all indices, we get

$$\Delta(w_h v^h) = 0.$$

Since, for a function f, $\Delta f = \mathring{\Delta} f$, we have the proof of proposition 6.1.

Now, we remark that similar considerations can be applied to pseudo-conformal Killing vector fields.

We call a vector field v^h pseudo-conformal Killing vector, if it satisfies the condition

$$\nabla_j v_i + \nabla_i v_j = 2\phi g_{ji}, \tag{6.4}$$

for a certain scalar function ϕ. This function ϕ is found to be $(1/n)(\nabla_i v^i)$ and consequently (6.4) can be written as

$$\nabla_j v_i + \nabla_i v_j - \frac{2}{n} g_{ji}(\nabla_t v^t) = 0. \tag{6.5}$$

For pseudo-conformal Killing vector fields we get the following results:

Proposition 6.2. *A necessary and sufficient condition for a vector field v^h in M with boundary B to be a pseudo-conformal Killing vector field is that*

$$\begin{aligned} & g^{ji}\nabla_j\nabla_i v^h + K^h{}_i v^i - 2S_j{}^{hi}\nabla_i v^j + \frac{n-2}{n}\nabla^h(\nabla_i v^i) = 0 && \text{in } M, \\ & \left[\nabla_j v_i + \nabla_i v_j - \frac{2}{n} g_{ji}(\nabla_t v^t)\right] N^j v^i = 0 && \text{on } B. \end{aligned} \tag{6.6}$$

If the vector v^h vanishes on the boundary B, then the second condition of (6.6) is automatically satisfied. Thus we have

Corollary 6.1. *A necessary and sufficient condition for a vector field v^h in M with boundary B vanishing identically on B to be a pseudo-conformal Killing vector field that*

$$g^{ji}\nabla_j\nabla_i v^h + K^h{}_i v^i - 2S_j{}^{hi}\nabla_i v^j + \frac{n-2}{n}\nabla^h(\nabla_i v^i) = 0 \quad \text{in } M.$$

Now from (4.13), we have the following Corollaries.

Corollary 6.2. *A necessary and sufficient condition for a vector field v^h in M with boundary B to be a pseudo-conformal Killing vector field is that*

$$g^{ji}\nabla_j\nabla_i v^h + K^h{}_i v^i - 2S_j{}^{hi}\nabla_i v^j + \frac{n-2}{n}\nabla^h(\nabla_i v^i) = 0 \quad \text{in } M,$$

(6.7)

$$(\nabla_j v_i)N^j v^i + H_{cb}v^c v^b + \alpha^2 H_a{}^a + \frac{n-2}{n}\alpha(\nabla_i v^i) - 2\alpha(\nabla_a v^a) + \nabla_a(\alpha v^a) = 0 \qquad on \quad B.$$

COROLLARY 6.3. *A necessary and sufficient condition for a vector field v^h in M tangential to B to be a pseudo-conformal Killing vector field is that*

$$(6.8)\qquad \begin{aligned} &g^{ji}\nabla_j\nabla_i v^h + K^h{}_i v^i - 2S_j{}^{hi}\nabla_i v^j + \frac{n-2}{n}\nabla^h(\nabla_i v^i) = 0 && in \quad M, \\ &(\nabla_j v_i)N^j v^i + H_{cb}v^c v^b = 0 && on \quad B. \end{aligned}$$

COROLLARY 6.4. *A necessary and sufficient condition for a vector field v^h in M normal to B to be a pseudo-conformal Killing vector is that*

$$(6.9)\qquad \begin{aligned} &g^{ji}\nabla_j\nabla_i v^h + K^h{}_i v^i - 2S_j{}^{hi}\nabla_i v^j + \frac{n-2}{n}\nabla^h(\nabla_i v^i) = 0 && in \quad M, \\ &\alpha^2 H_a{}^a + \frac{n-1}{n}\alpha(\nabla_i v^i) = 0 && on \quad B. \end{aligned}$$

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