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ON DEFICIENCIES OF AN ENTIRE ALGEBROID FUNCTION

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§ 1. Niino and Ozawa [1, 2] proved some interesting results for entire algebroid functions. A typical one is the following:

Let $f(z)$ be a two-valued entire transcendental algebroid function and a_1, a_2 and a_3 be different finite numbers satisfying

$$\sum_{j=1}^{3} \delta(a_j, f) > 2.$$

Then at least one of $\{a_j\}$ is a Picard exceptional value of f.

They also proved in the three- and four-valued cases that a more weaker condition on deficiencies, under a "non-proportionality" condition, implies the existence of Picard exceptional values (Theorem 1 in [2]).

In this paper we shall discuss the five-valued case and establish the similar conclusions as in Theorem 1 in [2] under a different assumption on deficiencies (see also Ozawa [3]). Those are the following:

THEOREM 1. *Let $f(z)$ be a five-valued transcendental entire algebroid function defined by an irreducible equation*

$$F(z, f) \equiv f^5 + A_4 f^4 + A_3 f^3 + A_2 f^2 + A_1 f + A_0 = 0,$$

where A_4, A_3, A_2, A_1 and A_0 are entire functions. Let a_j, $j=1, \cdots, 6$, be different finite numbers satisfying

$$\sum_{j=1}^{6} \delta(a_j, f) + \delta(a_m, f) + \delta(a_n, f) > 7$$

for every pair m, n $(m \neq n)$, $m, n = 1, \cdots, 6$, where $\delta(a_j, f)$ indicates the Nevanlinna-Selberg deficiency of f at a_j. Further assume that any four of $\{F(z, a_j)\}$ are not linearly dependent. Then one of $\{a_j\}_{j=1}^{6}$ is a Picard exceptional value of f.

THEOREM 2. *Let $f(z)$ be the same as in Theorem 1. Let $\{a_j\}_{j=1}^{7}$ be different finite numbers satisfying*

$$\sum_{j=1}^{6} \delta(a_j, f) + \delta(a_m, f) + \delta(a_n, f) > 7$$

for every pair m, n $(m \neq n)$, $m, n = 1, \cdots, 6$, and

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$$\sum_{\substack{j=1\\ j\neq 6}}^{7} \delta(a_j, f)+\delta(a_7, f)>6.$$

Further assume that any three of $\{F(z, a_j)\}$ *are not linearly dependent. Then at least two of* $\{a_j\}$ *are Picard exceptional values of* f.

THEOREM 3. *Let* $f(z)$ *be the same as in Theorem* 1. *Let* $\{a_j\}_{j=1}^{8}$ *be different finite numbers satisfying*

$$\sum_{j=1}^{6} \delta(a_j, f)+\delta(a_m, f)+\delta(a_n, f)>7$$

for every pair m, n $(m\neq n)$, $m, n=1, \cdots, 6$, *and*

$$\sum_{\substack{j=1\\ j\neq 6}}^{7} \delta(a_j, f)+\delta(a_k, f)>6$$

for every k, $k=1, 2, \cdots, 5, 7$, *and*

$$\sum_{\substack{j=1\\ j\neq 6,7}}^{8} \delta(a_j, f)>5.$$

Further assume that any two of $\{F(z, a_j)\}$ *are not proportional. Then at least three of* $\{a_j\}$ *are Picard exceptional values of* f.

Here we remark that Toda [4] proved that $\sum_{j=1}^{9} \delta(a_j, f)>8$ implies the existence of four Picard exceptional values among $\{a_j\}$.

§ 2. Proof of Theorem 1.

1. We put

$$g_j(z)=F(z, a_j), \qquad j=1, \cdots, 6,$$

and assume that all $g_j(z)$, $j=1, \cdots, 6$, are transcendental.

We first have

$$\sum_{j=1}^{6} \delta(a_j, f)>5$$

and

$$(1) \qquad \alpha_1 g_1+\alpha_2 g_2+\alpha_3 g_3+\alpha_4 g_4+\alpha_5 g_5+\alpha_6 g_6=1,$$

where

$$\alpha_j=1 \Big/ \prod_{\substack{k=1\\ k\neq j}}^{6} (a_j-a_k), \qquad j=1, \cdots, 6.$$

Applying the method in the proof of Theorem 1 in [1] to our case, we get the linear dependency of $\{g_j\}_{j=1}^6$, that is, for constants $\{\alpha'_j\}_{j=1}^6$ not all zero,

(2) $$\alpha'_1 g_1+\alpha'_2 g_2+\alpha'_3 g_3+\alpha'_4 g_4+\alpha'_5 g_5+\alpha'_6 g_6=0.$$

Here we may assume without any loss of generality that $\alpha'_5\alpha'_6 \neq 0$, $\alpha'_6=\alpha_6$. Eliminating g_6 from (1) and (2), we have

$$\sum_{j=1}^{5}(\alpha_j-\alpha'_j)g_j=1.$$

Since at least two of $\{\alpha_j-\alpha'_j\}$ are not zero, we study the following subcases:

1) $\alpha_1\neq\alpha'_1$, $\alpha_2\neq\alpha'_2$, $\alpha_3\neq\alpha'_3$, $\alpha_4\neq\alpha'_4$, $\alpha_5\neq\alpha'_5$,

2) $\alpha_1\neq\alpha'_1$, $\alpha_2\neq\alpha'_2$, $\alpha_3\neq\alpha'_3$, $\alpha_4\neq\alpha'_4$, $\alpha_5=\alpha'_5$,

(i) $\alpha'_1=\alpha'_2=\alpha'_3=\alpha'_4=0$,

(ii) $\alpha'_1=\alpha'_2=\alpha'_3=0$, $\alpha'_4\neq 0$,

(iii) $\alpha'_1=\alpha'_2=0$, $\alpha'_3\alpha'_4\neq 0$, $\alpha'_3\alpha_4-\alpha_3\alpha'_4\neq 0$,

(iv) $\alpha'_1=\alpha'_2=0$, $\alpha'_3\alpha'_4\neq 0$, $\alpha'_3\alpha_4-\alpha_3\alpha'_4=0$,

(v) $\alpha'_1=0$, $\alpha'_2\alpha'_3\alpha'_4\neq 0$, $(\alpha_2, \alpha_3, \alpha_4)\neq C(\alpha'_2, \alpha'_3, \alpha'_4)$ for any complex number C,

(vi) $\alpha'_1=0$, $\alpha'_2\alpha'_3\alpha'_4\neq 0$, $(\alpha_2, \alpha_3, \alpha_4)=C(\alpha'_2, \alpha'_3, \alpha'_4)$ for some complex number C,

(vii) $\alpha'_1\alpha'_2\alpha'_3\alpha'_4\neq 0$, $\dfrac{\alpha'_1}{\alpha_1}=\dfrac{\alpha'_2}{\alpha_2}=\dfrac{\alpha'_3}{\alpha_3}=\dfrac{\alpha'_4}{\alpha_4}$,

(viii) $\alpha'_1\alpha'_2\alpha'_3\alpha'_4\neq 0$, $\dfrac{\alpha'_{i_1}}{\alpha_{i_1}}=\dfrac{\alpha'_{i_2}}{\alpha_{i_2}}=\dfrac{\alpha'_{i_3}}{\alpha_{i_3}}$ for some (i_1, i_2, i_3), $1\leqq i_1, i_2, i_3\leqq 4$, but not (vii),

(ix) $\alpha'_1\alpha'_2\alpha'_3\alpha'_4\neq 0$, $\dfrac{\alpha'_{i_1}}{\alpha_{i_1}}=\dfrac{\alpha'_{i_2}}{\alpha_{i_2}}\neq\dfrac{\alpha'_{i_3}}{\alpha_{i_3}}=\dfrac{\alpha'_{i_4}}{\alpha_{i_4}}$ for some (i_1, i_2, i_3, i_4),

(x) $\alpha'_1\alpha'_2\alpha'_3\alpha'_4\neq 0$, not (vii), (viii), (ix),

3) $\alpha_1\neq\alpha'_1$, $\alpha_2\neq\alpha'_2$, $\alpha_3\neq\alpha'_3$, $\alpha_4=\alpha'_4$, $\alpha_5=\alpha'_5$,

(i) $\alpha'_1=\alpha'_2=\alpha'_3=0$,

(ii) $\alpha'_1=\alpha'_2=0$, $\alpha'_3\neq 0$,

(iii) $\alpha'_1=0$, $\alpha'_2\alpha'_3\neq 0$, $\alpha'_3\alpha_2-\alpha'_2\alpha_3\neq 0$,

(iv) $\alpha'_1=0$, $\alpha'_2\alpha'_3\neq 0$, $\alpha'_3\alpha_2-\alpha'_2\alpha_3=0$,

(v) $\alpha'_1\alpha'_2\alpha'_3\neq 0$, $(\alpha_1, \alpha_2, \alpha_3)=C(\alpha'_1, \alpha'_2, \alpha'_3)$ for some C,

(vi) $\alpha'_1\alpha'_2\alpha'_3\neq 0$, $(\alpha_1, \alpha_2, \alpha_3)\neq C(\alpha'_1, \alpha'_2, \alpha'_3)$ for any C,

4) $\alpha_1 \neq \alpha_1'$, $\alpha_2 \neq \alpha_2'$, $\alpha_3 = \alpha_3'$, $\alpha_4 = \alpha_4'$, $\alpha_5 = \alpha_5'$,

(i) $\alpha_1' = \alpha_2' = 0$,

(ii) $\alpha_1' = 0$, $\alpha_2' \neq 0$,

(iii) $\alpha_1'\alpha_2' \neq 0$, $\alpha_1\alpha_2' - \alpha_2\alpha_1' \neq 0$,

(iv) $\alpha_1'\alpha_2' \neq 0$, $\alpha_1\alpha_2' - \alpha_2\alpha_1' = 0$.

The cases 1), 2) (ii), (iii), (v), (viii), (x), 3) (ii), (iii), (vi), 4) (ii) and (iii) lead to an identity of the following type;

A) $$\lambda_1 g_1 + \lambda_2 g_2 + \lambda_3 g_3 + \lambda_4 g_4 + \lambda_5 g_5 = 1, \qquad \lambda_1\lambda_2\lambda_3\lambda_4\lambda_5 \neq 0,$$

The case 2) (i) leads to the following type;

B) $$\alpha_1 g_1 + \alpha_2 g_2 + \alpha_3 g_3 + \alpha_4 g_4 = 1, \qquad \alpha_5 g_5 + \alpha_6 g_6 = 0.$$

The case 2) (iv) leads to

C¹) $$\alpha_1 g_1 + \alpha_2 g_2 + \frac{\alpha_3' - \alpha_4}{\alpha_3}\alpha_5 g_5 + \frac{\alpha_3' - \alpha_3}{\alpha_3}\alpha_6 g_6 = 1,$$
$$\alpha_1 g_1 + \alpha_2 g_2 + (\alpha_3 - \alpha_3') g_3 + \frac{\alpha_4}{\alpha_3}(\alpha_3 - \alpha_3') g_4 = 1.$$

The cases 2) (vi) and 3) (iv) lead to

D) $$\alpha_1 g_1 + \frac{\alpha_3 - \alpha_3'}{\alpha_3}\alpha_2 g_2 + (\alpha_3 - \alpha_3') g_3 + \frac{\alpha_3 - \alpha_3'}{\alpha_3}\alpha_4 g_4 = 1,$$
$$\alpha_1 g_1 + \frac{\alpha_3' - \alpha_3}{\alpha_3}\alpha_5 g_5 + \frac{\alpha_3' - \alpha_3}{\alpha_3}\alpha_6 g_6 = 1.$$

The cases 2) (vii) and 4) (iv) lead to

E) $$\lambda_1 g_1 + \lambda_2 g_2 = 1, \qquad \lambda_3 g_3 + \lambda_4 g_4 + \lambda_5 g_5 + \lambda_6 g_6 = 1.$$

The case 2) (ix) leads to

C²) $$\left(1 - \frac{\alpha_1'}{\alpha_1}\right)\alpha_1 g_1 + \left(1 - \frac{\alpha_1'}{\alpha_1}\right)\alpha_2 g_2 + (\alpha_3 - \alpha_3') g_3 + \left(1 - \frac{\alpha_3'}{\alpha_3}\right)\alpha_4 g_4 = 1,$$
$$\left(1 - \frac{\alpha_1'}{\alpha_1}\cdot\frac{\alpha_3}{\alpha_3'}\right)\alpha_1 g_1 + \left(1 - \frac{\alpha_1'}{\alpha_1}\cdot\frac{\alpha_3}{\alpha_3'}\right)\alpha_2 g_2 + \left(1 - \frac{\alpha_3}{\alpha_3'}\right)\alpha_5 g_5 + \left(1 - \frac{\alpha_3}{\alpha_3'}\right)\alpha_6 g_6 = 1.$$

The case 3) (i) leads to

F) $$\alpha_1 g_1 + \alpha_2 g_2 + \alpha_3 g_3 = 1, \qquad \alpha_4 g_4 + \alpha_5 g_5 + \alpha_6 g_6 = 0.$$

The case 3) (v) leads to

G) $$\lambda_1 g_1+\lambda_2 g_2+\lambda_3 g_3=1, \qquad \lambda_4 g_4+\lambda_5 g_5+\lambda_6 g_6=1.$$

The case 4) (i) leads to

H) $$\alpha_1 g_1+\alpha_2 g_2=1, \qquad \alpha_3 g_3+\alpha_4 g_4+\alpha_5 g_5+\alpha_6 g_6=0.$$

2. By our assumption the cases B), C[1]), F) and H) may be omitted. We shall discuss the other cases.

In the first place we remark that Valiron [5] proved

$$T(r, f)=\mu(r, A)+O(1),$$

where

$$A=\max_{0\leqq j\leqq 4}(1, |A_j|)$$

and

$$5\mu(r, A)=\frac{1}{2\pi}\int_0^{2\pi}\log A\, d\theta.$$

Further we have

$$5\mu(r, A)=m(r, g)+O(1),$$

where $g=\max_{1\leqq j\leqq 5}(1, |g_j|)$.

The case A). In this case we have

$$\sum_{j=1}^{5}\delta(a_j, f)>4$$

and

$$5T(r, f)=m(r, g)+O(1)=m(r, g_1^*)+O(1),$$

where $g_1^*=\max_{1\leqq j\leqq 4}(1, |g_j|)$. By the same argument as in the proof of Theorem 1 in [2], we get the linear dependency of $\{g_j\}_{j=1}^5$, and hence we have one of the following:

A′) $$\mu_1 g_1+\mu_2 g_2+\mu_3 g_3+\mu_4 g_4=1, \qquad \mu_1\mu_2\mu_3\mu_4\neq 0,$$

B′) $$\mu_1 g_1+\mu_2 g_2+\mu_3 g_3=1, \qquad \mu_4 g_4+\mu_5 g_5=1,$$

C′) $$\lambda_1 g_1+\lambda_2 g_2=1, \qquad \lambda_3 g_3+\lambda_4 g_4+\lambda_5 g_5=0,$$

D′) $$\lambda_1 g_1+\lambda_2 g_2+\lambda_3 g_3=1, \qquad \lambda_1 g_1+\lambda_4 g_4+\lambda_5 g_5=1,$$

E′) $$\lambda_1 g_1+\lambda_2 g_2+\lambda_3 g_3=1, \qquad \lambda_4 g_4+\lambda_5 g_5=0.$$

By our assumption the cases C′), D′) and E′) may be omitted. In the case A′) we have

$$\sum_{j=1}^{4}\delta(a_j, f)>3$$

and

$$5T(r, f) = m(r, g_1^*) + O(1) = m(r, g_2^*) + O(1),$$

where $g_2^* = \max_{1 \leq j \leq 3}(1, |g_j|)$. Therefore the reasoning in the proof of Theorem 2 in [1] leads to a contradiction. In the case B′) we have

$$5T(r, f) = m(r, g_3^*) + O(1),$$

where $g_3^* = \max_{2 \leq j \leq 4}(1, |g_j|)$. Hence we have a contradiction by virtue of the argument in the case (B) in the proof of Theorem 2 in [1].

The case C[2]). In this case we have

$$m(r, g_2^*) \leq \sum_{j=1}^{4} N(r; 0, g_j) + o\left(\sum_{j=1}^{4} m(r, g_j)\right)$$

with a negligible exceptional set, and

$$m(r, g_4^*) \leq \sum_{\substack{j=1 \\ j \neq 3, 4}}^{6} N(r; 0, g_j) + o\left(\sum_{\substack{j=1 \\ j \neq 3, 4}}^{6} m(r, g_j)\right),$$

where $g_4^* = \max(1, |g_1|, |g_5|, |g_6|)$. Evidently

$$\begin{aligned} m(r, g) &\leq m(r, g_2^*) + m(r, g_4^*) \\ &\leq \sum_{j=1}^{6} N(r; 0, g_j) + N(r; 0, g_1) + N(r; 0, g_2) + o(m(r, g)). \end{aligned}$$

On the other hand, for an arbitrary $\varepsilon > 0$,

$$N(r; 0, g_j) \leq \{1 - \delta(a_j, f) + \varepsilon\} m(r, g)$$

for $r \geq r_0$. Hence we have

$$m(r, g) \leq \left\{8 - \sum_{j=1}^{6} \delta(a_j, f) - \delta(a_1, f) - \delta(a_2, f) + \varepsilon\right\} m(r, g) + o(m(r, g)),$$

which leads to a contradictory inequality

$$\sum_{j=1}^{6} \delta(a_j, f) + \delta(a_1, f) + \delta(a_2, f) \leq 7.$$

The case D). We have

$$m(r, g_2^*) \leq \sum_{j=1}^{4} N(r; 0, g_j) + o\left(\sum_{j=1}^{4} m(r, g_j)\right)$$

and

$$m(r, g_4^*) \leq N(r; 0, g_1) + \sum_{j=5}^{6} N(r; 0, g_j) + o\left(m(r, g_1) + \sum_{j=5}^{6} m(r, g_j)\right).$$

Hence we have

$$m(r, g) \leqq \left\{7 - \sum_{j=1}^{6} \delta(a_j, f) - \delta(a_1, f) + \varepsilon\right\} m(r, g) + o(m(r, g)),$$

which contradicts the assumption

$$\sum_{j=1}^{6} \delta(a_j, f) + \delta(a_1, f) > 6.$$

The cases E) and G). In these cases we have

$$5T(r, f) = m(r, g_5^*) + O(1),$$

where $g_5^* = \max_{2 \leqq j \leqq 5}(1, |g_j|)$. Hence by virtue of the same argument as in the case (B) in the proof of Theorem 2 in [1] we have a contradiction.

Thus we have a contradiction in every case. Therefore at least one of $\{g_j\}_{j=1}^6$ must be a polynomial, that is, one of $\{a_j\}_{j=1}^6$ is a Picard exceptional value of f.

The proof of the theorem is completed.

§ 3. Proof of Theorem 2.

1. We shall use the same notations as in the proof of Theorem 1 and put $g_7(z) = F(z, a_7)$, and assume that all $g_j(z)$, $j = 1, \cdots, 7$, are transcendental. Then by the proof of Theorem 1 we have one of the following:

H[1]) $\alpha_1 g_1 + \alpha_2 g_2 = 1, \quad \alpha_3 g_3 + \alpha_4 g_4 + \alpha_5 g_5 + \alpha_6 g_6 = 0,$

H[2]) $\alpha_5 g_5 + \alpha_6 g_6 = 1, \quad \alpha_1 g_1 + \alpha_2 g_2 + \alpha_3 g_3 + \alpha_4 g_4 = 0.$

Further we have

$$\beta_1 g_1 + \beta_2 g_2 + \beta_3 g_3 + \beta_4 g_4 + \beta_5 g_5 + \beta_7 g_7 = 1,$$

where

$$\beta_j = 1 \Big/ \prod_{\substack{k=1 \\ k \neq j, 6}}^{7} (a_j - a_k), \qquad j = 1, 2, \cdots, 5, 7.$$

If we have H[1]), then we get

$$\left(\beta_2 - \beta_1 \frac{\alpha_2}{\alpha_1}\right) g_2 + \beta_3 g_3 + \beta_4 g_4 + \beta_5 g_5 + \beta_7 g_7 = 1 - \frac{\beta_1}{\alpha_1}.$$

Here

$$5T(r, f) = m(r, g_6^*) + O(1), \qquad g_6^* = \max_{2 \leqq j \leqq 5}(1, |g_j|).$$

Hence it reduces to type A′), B′), C′), D′) or E′). Each of A′), B′), C′) and E′) leads to a contradiction. Hence we may consider the following:

(i) $\left(\beta_2-\beta_1\frac{\alpha_2}{\alpha_1}\right)g_2+\lambda_3g_3+\lambda_4g_4=1-\frac{\beta_1}{\alpha_1}$, $\left(\beta_2-\beta_1\frac{\alpha_2}{\alpha_1}\right)g_2+\lambda_5g_5+\lambda_7g_7=1-\frac{\beta_1}{\alpha_1}$,

(ii) $\beta_3g_3+\lambda_4g_4+\lambda_5g_5=1-\frac{\beta_1}{\alpha_1}$, $\beta_3g_3+\lambda_2g_2+\lambda_7g_7=1-\frac{\beta_1}{\alpha_1}$,

(iii) $\beta_3g_3+\lambda_5g_5+\lambda_7g_7=1-\frac{\beta_1}{\alpha_1}$, $\beta_3g_3+\lambda_2g_2+\lambda_4g_4=1-\frac{\beta_1}{\alpha_1}$,

(iv) $\beta_7g_7+\lambda_2g_2+\lambda_3g_3=1-\frac{\beta_1}{\alpha_1}$, $\beta_7g_7+\lambda_4g_4+\lambda_5g_5=1-\frac{\beta_1}{\alpha_1}$.

When (i) occurs, using $\alpha_1g_1+\alpha_2g_2=1$, we have

$$\left(\beta_1-\frac{\alpha_1}{\alpha_2}\beta_2\right)g_1+\lambda_3g_3+\lambda_4g_4=1-\frac{\beta_2}{\alpha_2},\qquad \left(\beta_2-\frac{\alpha_2}{\alpha_1}\beta_1\right)g_2+\lambda_5g_5+\lambda_7g_7=1-\frac{\beta_1}{\alpha_1}.$$

When (ii) occurs, we have

$$\alpha_1g_1+\alpha_2g_2=1,\qquad \beta_3g_3+\lambda_4g_4+\lambda_5g_5=1.$$

When (iii) occurs, we have $5T(r,f)=m(r,g_7^*)+O(1)$, $g_7^*=\max(1,|g_2|,|g_3|,|g_5|)$, and

$$\alpha_1g_1+\alpha_2g_2=1,\qquad \beta_3g_3+\lambda_5g_5+\lambda_7g_7=1-\frac{\beta_1}{\alpha_1}.$$

Finally when (iv) occurs, we have $5T(r,f)=m(r,g_8^*)+O(1)$, $g_8^*=\max(1,|g_2|,|g_4|,|g_5|)$, and

$$\alpha_1g_1+\alpha_2g_2=1,\qquad \lambda_4g_4+\lambda_5g_5+\lambda_7g_7=1-\frac{\beta_1}{\alpha_1}.$$

Thus in every case we get a contradiction.

If we have H^2), then we have

$$5T(r,f)=m(r,g_6^*)+O(1),$$

$$\left(\beta_2-\frac{\alpha_2}{\alpha_1}\beta_1\right)g_2+\left(\beta_3-\frac{\alpha_3}{\alpha_1}\beta_1\right)g_3+\left(\beta_4-\frac{\alpha_4}{\alpha_1}\beta_1\right)g_4+\beta_5g_5+\beta_7g_7=1,$$

and hence it is sufficient to consider the following:

(i) $\left(\beta_2-\frac{\alpha_2}{\alpha_1}\beta_1\right)g_2+\lambda_3g_3+\lambda_4g_4=1$, $\left(\beta_2-\frac{\alpha_2}{\alpha_1}\beta_1\right)g_2+\lambda_5g_5+\lambda_7g_7=1$,

(ii) $\left(\beta_2-\frac{\alpha_2}{\alpha_1}\beta_1\right)g_2+\lambda_3g_3+\lambda_5g_5=1$, $\left(\beta_2-\frac{\alpha_2}{\alpha_1}\beta_1\right)g_2+\lambda_4g_4+\lambda_7g_7=1$,

(iii) $\beta_5g_5+\lambda_2g_2+\lambda_3g_3=1$, $\beta_5g_5+\lambda_4g_4+\lambda_7g_7=1$,

(iv) $\beta_7g_7+\lambda_2g_2+\lambda_3g_3=1$, $\beta_7g_7+\lambda_4g_4+\lambda_5g_5=1$.

When (i) occurs, we have B′)-type, and (ii), (iii) and (iv) lead to type A′). Hence we have a contradiction in every case.

Thus we conclude that one of $\{a_j\}_{j=1}^{7}$ is a Picard exceptional value of f.

2. Now we first suppose that this exceptional value is a_1, and that all g_j, $j=2, \cdots, 7$, are transcendental. We have only to consider when $1-\alpha_1 g_1 \equiv 0$. Then

$$\beta_2 g_2+\beta_3 g_3+\beta_4 g_4+\beta_5 g_5+\beta_7 g_7=1-\frac{\beta_1}{\alpha_1}$$

leads to type D′). Since we have

$$\alpha_2 g_2+\alpha_3 g_3+\alpha_4 g_4+\alpha_5 g_5+\alpha_6 g_6=0,$$

it is sufficient to consider the case

$$\beta_7 g_7+\lambda_2 g_2+\lambda_3 g_3=1-\frac{\beta_1}{\alpha_1}, \qquad \beta_7 g_7+\lambda_4 g_4+\lambda_5 g_5=1-\frac{\beta_1}{\alpha_1}.$$

But this contradicts the assumption

$$\sum_{\substack{j=2\\ j\neq 6}}^{7} \delta(a_j, f)+\delta(a_7, f)>5.$$

Hence we get two Picard exceptional values.

Next we suppose that the exceptional value is a_6. Similarly we have only to consider $1-\alpha_6 g_6 \equiv 0$. Then we have

$$\alpha_1 g_1+\alpha_2 g_2+\alpha_3 g_3+\alpha_4 g_4+\alpha_5 g_5=0,$$

and hence

$$\left(\beta_2-\frac{\beta_1}{\alpha_1}\alpha_2\right)g_2+\left(\beta_3-\frac{\beta_1}{\alpha_1}\alpha_3\right)g_3+\left(\beta_4-\frac{\beta_1}{\alpha_1}\alpha_4\right)g_4+\left(\beta_5-\frac{\beta_1}{\alpha_1}\alpha_5\right)g_5+\beta_7 g_7=1.$$

By the same reasoning as above we can conclude that there are at least two Picard exceptional values.

The proof of the theorem is completed.

§ 4. Proof of Theorem 3.

1. We set

$$g_j(z)=F(z, a_j), \qquad j=1, \cdots, 8,$$

and assume that all $g_j(z)$, $j=1, \cdots, 8$, are transcendental. Then by the proof of Theorem 1 we have one of the following:

A[1]) $$\lambda_1 g_1+\lambda_2 g_2+\lambda_3 g_3+\lambda_4 g_4+\lambda_5 g_5=1,$$

A[2]) $$\lambda_1 g_1+\lambda_2 g_2+\lambda_3 g_3+\lambda_4 g_4+\lambda_6 g_6=1,$$

F[1]) $$\alpha_1 g_1+\alpha_2 g_2+\alpha_3 g_3=1, \qquad \alpha_4 g_4+\alpha_5 g_5+\alpha_6 g_6=0,$$

F[2]) $$\alpha_4 g_4+\alpha_5 g_5+\alpha_6 g_6=1, \qquad \alpha_1 g_1+\alpha_2 g_2+\alpha_3 g_3=0,$$

H[1]) $$\alpha_1 g_1+\alpha_2 g_2=1, \qquad \alpha_3 g_3+\alpha_4 g_4+\alpha_5 g_5+\alpha_6 g_6=0,$$

H[2]) $$\alpha_5 g_5+\alpha_6 g_6=1, \qquad \alpha_1 g_1+\alpha_2 g_2+\alpha_3 g_3+\alpha_4 g_4=0.$$

2. We show that A[1]), A[2]) reduce to F[1]), F[2]), H[1]) or H[2]). Indeed, by our standard argument A[1]) reduces to

(i) $$\lambda_1 g_1+\lambda_2 g_2=1, \qquad \lambda_3 g_3+\lambda_4 g_4+\lambda_5 g_5=0.$$

Here if $(\lambda_3, \lambda_4, \lambda_5)=C(\alpha_3, \alpha_4, \alpha_5)$ for some complex number C, we get

$$\alpha_1 g_1+\alpha_2 g_2+\alpha_6 g_6=1, \qquad \alpha_3 g_3+\alpha_4 g_4+\alpha_5 g_5=0,$$

which is of type F[2]). If $(\lambda_3, \lambda_4, \lambda_5)\neq C(\alpha_3, \alpha_4, \alpha_5)$ for any complex number C, then we can eliminate one of g_j, $j=3, 4, 5$, and hence we have, for example,

$$\alpha_1 g_1+\alpha_2 g_2+\left(\alpha_4-\frac{\lambda_4}{\lambda_3}\alpha_3\right)g_3+\left(\alpha_5-\frac{\lambda_5}{\lambda_3}\alpha_3\right)g_5+\alpha_6 g_6=1.$$

Further we have

$$\lambda_1 g_1+\lambda_2 g_2=1.$$

It is easy to see that $\lambda_1=\alpha_1$, $\lambda_2=\alpha_2$ is only a non-contradictory case. Hence it reduces to H[1]). Other equations of type (i) also reduce to F[1]), F[2]), H[1]) or H[2]), as we can see easily.

A[2]) can be dealt with similarly.

3. Now we consider the case F[1]). Eliminating g_1 from

$$\alpha_1 g_1+\alpha_2 g_2+\alpha_3 g_3=1$$

and

$$\beta_1 g_1+\beta_2 g_2+\beta_3 g_3+\beta_4 g_4+\beta_5 g_5+\beta_7 g_7=1,$$

we have

$$\left(\beta_2-\frac{\alpha_2}{\alpha_1}\beta_1\right)g_2+\left(\beta_3-\frac{\alpha_3}{\alpha_1}\beta_1\right)g_3+\beta_4 g_4+\beta_5 g_5+\beta_7 g_7=1-\frac{\beta_1}{\alpha_1}.$$

Here we have

$$5T(r, f)=m(r, g_5^*)+O(1), \qquad g_5^*=\max_{2\leq j\leq 5}(1, |g_j|).$$

Hence by our assumption only the following cases need to be discussed:

(i) $$\left(\beta_2-\frac{\alpha_2}{\alpha_1}\beta_1\right)g_2+\left(\beta_3-\frac{\alpha_3}{\alpha_1}\beta_1\right)g_3=1-\frac{\beta_1}{\alpha_1}, \qquad \beta_4g_4+\beta_5g_5+\beta_7g_7=0,$$

(ii) $$\left(\beta_2-\frac{\alpha_2}{\alpha_1}\beta_1\right)g_2+\beta_5g_5=1-\frac{\beta_1}{\alpha_1}, \qquad \left(\beta_3-\frac{\alpha_3}{\alpha_1}\beta_1\right)g_3+\beta_4g_4+\beta_7g_7=0,$$

(iii) $$\beta_4g_4+\beta_5g_5=1-\frac{\beta_1}{\alpha_1}, \qquad \left(\beta_2-\frac{\alpha_2}{\alpha_1}\beta_1\right)g_2+\left(\beta_3-\frac{\alpha_3}{\alpha_1}\beta_1\right)g_3+\beta_7g_7=0,$$

(iv) $$\beta_5g_5+\beta_7g_7=1-\frac{\beta_1}{\alpha_1}, \qquad \left(\beta_2-\frac{\alpha_2}{\alpha_1}\beta_1\right)g_2+\left(\beta_3-\frac{\alpha_3}{\alpha_1}\beta_1\right)g_3+\beta_4g_4=0,$$

(v) $$\left(\beta_2-\frac{\alpha_2}{\alpha_1}\beta_1\right)g_2+\beta_7g_7=1-\frac{\beta_1}{\alpha_1}, \qquad \left(\beta_3-\frac{\alpha_3}{\alpha_1}\beta_1\right)g_3+\beta_4g_4+\beta_5g_5=0.$$

Further we have

(1) $$\gamma_1g_1+\gamma_2g_2+\gamma_3g_3+\gamma_4g_4+\gamma_5g_5+\gamma_8g_8=1,$$

where

$$\gamma_j=1\Big/\prod_{\substack{k=1\\k\neq j,6,7}}^{8}(a_j-a_k), \qquad j=1, 2, \cdots, 5, 8.$$

Eliminating g_1 from (1) and $\alpha_1g_1+\alpha_2g_2+\alpha_3g_3=1$, we have

(2) $$\left(\gamma_2-\frac{\alpha_2}{\alpha_1}\gamma_1\right)g_2+\left(\gamma_3-\frac{\alpha_3}{\alpha_1}\gamma_1\right)g_3+\gamma_4g_4+\gamma_5g_5+\gamma_8g_8=1-\frac{\gamma_1}{\alpha_1}.$$

Each of (i), (ii), $\cdots$, (v) together with (2) leads to type A$'$) or B$'$), which implies that F^1) is contradictory. It is to be noted that

$$\begin{vmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ \beta_1 & \beta_2 & \beta_3\\ \gamma_1 & \gamma_2 & \gamma_3\end{vmatrix}\neq 0, \qquad \begin{vmatrix}\alpha_1 & \alpha_2 & 1\\ \beta_1 & \beta_2 & 1\\ \gamma_1 & \gamma_2 & 1\end{vmatrix}\neq 0.$$

F^2), H^1) and H^2) can be dealt with similarly, and hence we have a contradiction in every case.

Thus we conclude that at least one of $\{a_j\}$ is a Picard exceptional value of f.

4. We first suppose that g_1 is a polynomial and the remaining g's are transcendental. We may suppose $(1-\beta_1g_1)(1-\gamma_1g_1)\neq 0$. Then

$$\beta_2g_2+\beta_3g_3+\beta_4g_4+\beta_5g_5+\beta_7g_7=1-\beta_1g_1$$

leads to either of the following:

(i) $\beta_2 g_2+\beta_3 g_3=1-\beta_1 g_1, \qquad \beta_4 g_4+\beta_5 g_5+\beta_7 g_7=0,$

(ii) $\beta_5 g_5+\beta_7 g_7=1-\beta_1 g_1, \qquad \beta_2 g_2+\beta_3 g_3+\beta_4 g_4=0.$

Further we have

$$\gamma_2 g_2+\gamma_3 g_3+\gamma_4 g_4+\gamma_5 g_5+\gamma_8 g_8=1-\gamma_1 g_1.$$

Hence, eliminating g_2 (or g_3), we get a contradiction in every case.

Next we suppose that g_6 is a polynomial and that the remaining g's are transcendental. If $1-\alpha_6 g_6\equiv 0$, then by the same argument as in 3, we get a contradiction. If $1-\alpha_6 g_6\not\equiv 0$, then

$$\alpha_1 g_1+\alpha_2 g_2+\alpha_3 g_3+\alpha_4 g_4+\alpha_5 g_5=1-\alpha_6 g_6$$

leads to

$$\alpha_4 g_4+\alpha_5 g_5=1-\alpha_6 g_6, \qquad \alpha_1 g_1+\alpha_2 g_2+\alpha_3 g_3=0.$$

Again, by the same argument as in 3, we get a contradiction.

Next we consider the case that g_7 is a polynomial. In this case we have

$$\beta_1 g_1+\beta_2 g_2+\beta_3 g_3+\beta_4 g_4+\beta_5 g_5=1-\beta_7 g_7.$$

Further

$$\alpha_1 g_1+\alpha_2 g_2+\alpha_3 g_3+\alpha_4 g_4+\alpha_5 g_5+\alpha_6 g_6=1$$

leads to one of F^1), F^2), H^1) and H^2). In every case we get an equation of type A$'$), hence we get a contradiction.

The case that g_8 is a polynomial is quite similar as above.

Thus two of $\{g_j\}$ are polynomials, that is, there are two Picard exceptional values among $\{a_j\}$.

5. Now we show that there is one more Picard exceptional value. We distinguish several cases: (i) g_1 and g_2 are polynomials, (ii) g_1 and g_6, (iii) g_1 and g_7, (iv) g_1 and g_8, (v) g_6 and g_7, (vi) g_6 and g_8, (vii) g_7 and g_8.

We suppose that in every case other g's are transcendental.

Case (i). Since

$$\begin{vmatrix} \alpha_1 & \alpha_2 & 1 \\ \beta_1 & \beta_2 & 1 \\ \gamma_1 & \gamma_2 & 1 \end{vmatrix} \neq 0,$$

we may assume that

$$\alpha_3 g_3+\alpha_4 g_4+\alpha_5 g_5+\alpha_6 g_6=1-\alpha_1 g_1-\alpha_2 g_2\not\equiv 0.$$

This implies a contradictory inequality

$$\sum_{j=3}^{6}\delta(a_j, f)\leqq 3.$$

Case (ii). If $1-\alpha_1 g_1-\alpha_6 g_6 \not\equiv 0$, then we have obviously a contradiction. If $1-\alpha_1 g_1-\alpha_6 g_6 \equiv 0$, then we have

$$\alpha_2 g_2+\alpha_3 g_3+\alpha_4 g_4+\alpha_5 g_5=0.$$

We may assume that $1-\beta_1 g_1 \not\equiv 0$. Hence, eliminating g_2 from

$$\beta_2 g_2+\beta_3 g_3+\beta_4 g_4+\beta_5 g_5+\beta_7 g_7=1-\beta_1 g_1,$$

we have a contradiction.

Case (iii) and case (iv). Similarly as above.

Case (v). If both of $1-\alpha_6 g_6$ and $1-\beta_7 g_7$ are not constantly zero, we have a contradiction, eliminating one of g_j, $j=1, \cdots, 5$, from

$$\alpha_1 g_1+\alpha_2 g_2+\alpha_3 g_3+\alpha_4 g_4+\alpha_5 g_5=1-\alpha_6 g_6$$

and

$$\beta_1 g_1+\beta_2 g_2+\beta_3 g_3+\beta_4 g_4+\beta_5 g_5=1-\beta_7 g_7.$$

If both of them are constantly zero, we eliminate g_1 and g_2 from

$$\alpha_1 g_1+\alpha_2 g_2+\alpha_3 g_3+\alpha_4 g_4+\alpha_5 g_5=0,$$

$$\beta_1 g_1+\beta_2 g_2+\beta_3 g_3+\beta_4 g_4+\beta_5 g_5=0$$

and

$$\gamma_1 g_1+\gamma_2 g_2+\gamma_3 g_3+\gamma_4 g_4+\gamma_5 g_5+\gamma_8 g_8=1.$$

Then we have a contradiction, too.

Case (vi) and (vii). Similarly as above.

Thus we have a contradiction in every case. Therefore at least three of $\{g_j\}_{j=1}^8$ are polynomials, that is, at least three of $\{a_j\}_{j=1}^8$ are Picard exceptional values of f.

The proof of the theorem is completed.

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