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ON A GENERALIZED NOTION OF HARMONIC FUNCTIONS

BY RICHARD DELANGHE

§ 1. Introduction.

In [1], we considered functions $g: \mathbf{R}^n \to \mathcal{A}$ where $\mathcal{A}$ is the Clifford algebra constructed over an n-dimensional quadratic real vector space V with orthogonal basis $e=\{e_1, \cdots, e_n\}$; $(n>2)$. Since the elements $e_A = e_{i_1} \cdots e_{i_h}$ together with $e_\phi = e_0$ form a basis of $\mathcal{A}$,

$$g = \sum_A g_A e_A$$

where $g_A: \mathbf{R}^n \to \mathbf{R}$ for all $A \in \mathcal{P}N$ (Here $N=\{1, \cdots, n\}$ and all $A=\{i_1, \cdots, i_h\}$ are ordered in such a way that $1 \leqq i_1 < \cdots < i_h \leqq n$). Moreover, in $\mathcal{A}$, $e_i^2 = \varepsilon_i e_0$ ($\varepsilon_i \in \mathbf{R}$, $i=1, \cdots, n$) and $e_i e_j + e_j e_i = 0$ $(i \neq j)$. Throughout this paper we suppose that all $\varepsilon_i = +1$. Let now $g \in C^{(2)}(D)$ where D is an open non empty subset of $\mathbf{R}^n$; then, if we introduce the operator

$$\mathcal{M} = \sum_{i=1}^n e_i \frac{\partial}{\partial x_i},$$

we define $\mathcal{M}[g]$ as

$$\mathcal{M}[g] = \sum_{i, A} e_i e_A \frac{\partial g_A}{\partial x_i}$$

and

$$\square g = \sum_A \square g_A e_A$$

where

$$\square = \mathcal{M}^2 = e_0 \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}.$$

Furthermore, a function $g \in C^{(2)}(D)$ is called harmonic in D iff $\square g = 0$ in D.

§ 2. A generalized notion of harmonic functions.

In this section, we shall consider a class of functions called extended-harmonic, the notion of which is essentially derived from the following

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THEOREM 1. *Let* g: $\mathbf{R}^n \to \mathcal{A}$ *be of the class* $C^{(2)}$ *in* D *and satisfy* $\Box g=0$ *in* D; *then for all* $x \in D$ *and all* $I \in I(D\backslash\{x\})$,

$$\int_{\partial I} \frac{\bar{u}-\bar{x}}{\rho^n} d\sigma_u g(u) + \int_{\partial I} \frac{e_0}{(n-2)\rho^{n-2}} d\sigma_u \mathcal{M}[g] = 0.$$

Conversely, if g: $\mathbf{R}^n \to \mathcal{A}$ *is of the class* $C^{(1)}$ *in* D *such that for all* $x \in D$ *and all* $I \in I(D\backslash\{x\})$,

$$\int_{\partial I} \frac{\bar{u}-\bar{x}}{\rho^n} d\sigma_u g(u) + \int_{\partial I} \frac{e_0}{(n-2)\rho^{n-2}} d\sigma_u \mathcal{M}[g] = 0,$$

then $\Box g=0$ *in* D.

Here

$$d\sigma_u = \sum_{i=1}^{n} (-1)^{i-1} e_i d\hat{u}_i$$

with

$$d\hat{u}_i = du_1 \wedge \cdots \wedge du_{i-1} \wedge du_{i+1} \wedge \cdots \wedge du_n,$$

$$\bar{u} = \sum_{i=1}^{n} u_i e_i, \qquad \rho = +\sqrt{\sum_{i=1}^{n} (u_i - x_i)^2}$$

and $I(D)$ *denotes the set of closed intervals contained in* D.

$$(I = \{x\colon a_i \leqq x_i \leqq b_i,\ i=1, \cdots, n\} \subset D)$$

Proof. The first part of the Theorem is easily checked. Indeed, take $x \in D$ and $I \in (D\backslash\{x\})$; then in virtue of a Theorem stated in [1],

$$\int_{\partial I} \frac{\bar{u}-\bar{x}}{\rho^n} d\sigma_u g(u) = \int_I \left[\left(\frac{\bar{u}-\bar{x}}{\rho^n}\right)\mathcal{M} \cdot g(u) + \frac{\bar{u}-\bar{x}}{\rho^n} \cdot \mathcal{M}(g)\right] du^N$$

and

$$\int_{\partial I} \frac{e_0}{(n-2)\rho^{n-2}} d\sigma_u \mathcal{M}(g) = \int_I \left[\frac{\bar{x}-\bar{u}}{\rho^n} \cdot \mathcal{M}(g) + \frac{e_0}{(n-2)\rho^{n-2}} \cdot \Box g\right] du^N.$$

Since

$$\left(\frac{\bar{u}-\bar{x}}{\rho^n}\right)\mathcal{M} = 0$$

in $D\backslash\{x\}$ and $\Box g=0$ in D, we obtain that

$$\int_{\partial I} \frac{\bar{u}-\bar{x}}{\rho^n} d\sigma_u g(u) + \int_{\partial I} \frac{e_0}{(n-2)\rho^{n-2}} d\sigma_u \mathcal{M}(g) = 0.$$

Conversely, take $x \in D$; then there exists an $I \in I(D)$ such that $x \in \dot{I}$. Since $g \in C^{(1)}(D)$, for any $\varepsilon > 0$, there ought to exist an $\eta(\varepsilon) > 0$ such that $||h||_e < \eta(\varepsilon)$ implies $|g_A(x+h) - g_A(x)| < \varepsilon$ for all $A \in \mathcal{P}N$ ($||h||_e$ is the Euclidean norm of h). Consider a ball $B(x, r)$ with $0<r<\min(\varepsilon, \eta(\varepsilon))$ such that $\bar{B}(x, r)\subset \mathring{I}$ and construct a closed cube J with center x and edge s such that $J\subset \mathring{B}(x, r)$; moreover, divide $I\setminus\mathring{J}$ into m closed intervals I_j $(j=1, \cdots, m)$; then, since $I_j\in I(D\setminus\{x\})$ for all j,

$$\int_{\partial(I\setminus J)} \frac{\bar{u}-\bar{x}}{\rho^n} d\sigma_u g(u)+\frac{1}{n-2}\int_{\partial(I\setminus J)} \frac{e_0}{\rho^{n-2}} d\sigma_u \mathcal{M}(g)$$

$$=\sum_{j=1}^{m}\left[\int_{\partial I_j} \frac{\bar{u}-\bar{x}}{\rho^n} d\sigma_u g(u)+\frac{1}{n-2}\int_{\partial I_j} \frac{e_0}{\rho^{n-2}} d\sigma_u \mathcal{M}(g)\right]=0.$$

Hence,

$$\int_{\partial I} \frac{\bar{u}-\bar{x}}{\rho^n} d\sigma_u g(u)+\frac{1}{n-2}\int_{\partial I} \frac{e_0}{\rho^{n-2}} d\sigma_u \mathcal{M}(g)=\int_{\partial J} \frac{\bar{u}-\bar{x}}{\rho^n} d\sigma_u g(u)+\frac{1}{n-2}\int_{\partial J} \frac{e_0}{\rho^{n-2}} d\sigma_u \mathcal{M}(g).$$

But

$$\int_{\partial J} \frac{\bar{u}-\bar{x}}{\rho^n} d\sigma_u g(u)=\int_{\partial J} \frac{\bar{u}-\bar{x}}{\rho^n} d\sigma_u [g(u)-g(x)]+\left[\int_{\partial J} \frac{\bar{u}-\bar{x}}{\rho^n} d\sigma_u\right] g(x)$$

where

$$\int_{\partial J} \frac{\bar{u}-\bar{x}}{\rho^n} d\sigma_u=A_{n-1}=\frac{2\pi^{n/2}}{\Gamma(n/2)}$$

(see e.g. [2]).

By means of a norm which we introduced earlier on $\mathcal{A}$ (see e.g. [2]), and since on I, $|\phi_B|\leqq L$ for all B which appear in

$$\phi=\mathcal{M}(g)=\sum_B \phi_B e_B$$

and moreover on ∂J, $\rho\geqq s/2$,

$$\left\|\int_{\partial J} \frac{e_0}{(n-2)\rho^{n-2}} d\sigma_u \phi\right\|\leqq K\cdot\frac{1}{(s/2)^{n-2}}\cdot V(\partial J)$$

$$\leqq K'\cdot\varepsilon$$

where

$$K=\frac{n}{n-2}2^{3n/2}\cdot L, \qquad K'=n2^n K.$$

Moreover,

$$\left\|\int_{\partial J} \frac{\bar{u}-\bar{x}}{\rho^n} d\sigma_u [g(u)-g(x)]\right\|\leqq M\cdot\frac{1}{(s/2)^{n-1}}\cdot V(\partial J)\cdot\varepsilon$$

$$\leqq M'\cdot\varepsilon$$

since on ∂J,

$$\frac{|u_i-x_i|}{\rho^n} \leqq \frac{1}{\rho^{n-1}} \leqq \frac{1}{(s/2)^{n-1}}$$

and where $M=n^2 2^{3n/2}$, $M'=n2^n M$.

Hence,

$$\left\|\int_{\partial I} \frac{\bar{u}-\bar{x}}{\rho^n}\, d\sigma_u g(u) + \frac{1}{n-2}\int_{\partial I} \frac{e_0}{\rho^{n-2}}\, d\sigma_u \mathcal{M}(g) - A_{n-1}\cdot g(x)\right\|$$

$$=\left\|\int_{\partial J} \frac{\bar{u}-\bar{x}}{\rho^n}\, d\sigma_u g(u) + \frac{1}{n-2}\int_{\partial J} \frac{e_0}{\rho^{n-2}}\, d\sigma_u \mathcal{M}(g)\right\|$$

$$\leqq (K'+M')\cdot\varepsilon.$$

Since ε has been chosen arbitrarily,

$$\left\|\int_{\partial I} \frac{\bar{u}-\bar{x}}{\rho^n}\, d\sigma_u g(u) + \frac{1}{n-2}\int_{\partial I} \frac{e_0}{\rho^{n-2}}\, d\sigma_u \mathcal{M}(g) - A_{n-1}\cdot g(x)\right\| = 0$$

or

$$g(x) = \frac{1}{A_{n-1}}\int_{\partial I} \frac{\bar{u}-\bar{x}}{\rho^n}\, d\sigma_u g(u) + \frac{1}{A_{n-1}}\int_{\partial I} \frac{e_0}{(n-2)\rho^{n-2}}\, d\sigma_u \mathcal{M}(g)$$

and this relation clearly holds for any $x \in \mathring{I}$.

Consequently, $g \in C^{(2)}(\mathring{I})$ and in $\mathring{I}$,

$$\Box g = \frac{1}{A_{n-1}}\int_{\partial I} \Box\left(\frac{\bar{u}-\bar{x}}{\rho^n}\right) d\sigma_u g(u) + \frac{1}{(n-2)A_{n-1}}\int_{\partial I} \Box \frac{e_0}{\rho^{n-2}}\, d\sigma_u \mathcal{M}(g)$$

$$=0,$$

or g is harmonic in $\mathring{I}$.

Since x has been taken arbitrarily in D, $g \in C^{(2)}(D)$ and $\Box g = 0$ in D. Q.E.D.

Definition. Let g and f be integrable on ∂I for all $I \in I(D)$ and suppose furthermore that for all $x \in D$ and all $I \in I(D \setminus \{x\})$,

$$\int_{\partial I} \frac{\bar{u}-\bar{x}}{\rho^n}\, d\sigma_u g(u) + \int_{\partial I} \frac{e_0}{(n-2)\rho^{n-2}}\, d\sigma_u f(u) = 0;$$

then g is called *extended-harmonic* in D and f is said to be the *$\mathcal{M}$-derivative* of g and we put $f = \mathcal{M}(g)$.

Remark. Let f and f^* be integrable on ∂I for all $I \in I(D)$; then define the relation fRf^* iff for all $x \in D$ and all $I \in I(D \setminus \{x\})$,

$$\int_{\partial I} \frac{e_0}{\rho^{n-2}}\, d\sigma_u (f - f^*) = 0.$$

Since R is clearly an equivalence relation it is understood that in the above definition, when speaking of "f is the $\mathcal{M}$-derivative of g", the whole equivalence class $[f]$ of f is meant.

In a foregoing paper (see [1]), we remarked that if $g\in C^{(2)}(D)$ and $\Box g=0$ in D, then $\mathcal{M}(g)$ is regular in D. In [3], we introduced the definition of an extended-regular function. We now prove

THEOREM 2. *Let g be extended-harmonic in D; then $f=\mathcal{M}(g)$ is extended-regular in D.*

Proof. Take $x\in D$ and $I\in I(D\backslash\{x\})$; then, since g is extended-harmonic in D,

$$F(x)=\int_{\partial I}\frac{\bar{u}-\bar{x}}{\rho^n}\,d\sigma_u g(u)+\frac{1}{n-2}\int_{\partial I}\frac{e_0}{\rho^{n-2}}\,d\sigma_u f$$

vanishes for all $x\in D\backslash I$. Hence, in $D\backslash I$,

$$\begin{aligned}\mathcal{M}(F)&=\int_{\partial I}\mathcal{M}\left(\frac{\bar{u}-\bar{x}}{\rho^n}\right)d\sigma_u g+\int_{\partial I}\frac{\bar{u}-\bar{x}}{\rho^n}\,d\sigma_u f\\&=\int_{\partial I}\frac{\bar{u}-\bar{x}}{\rho^n}\,d\sigma_u f\\&=0.\end{aligned}$$

Consequently, for all $x\in D$ and all $I\in I(D\backslash\{x\})$,

$$\int_{\partial I}\frac{\bar{u}-\bar{x}}{\rho^n}\,d\sigma_u f=0.$$

But this means that f is extended-regular in D (see [3]). Q.E.D.

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SEMINARIE VOOR ALGEBRA,
RIJKSUNIVERSITEIT TE GENT,
22 JOZEF PLATEAUSTRAAT, 9000 GENT, BELGIË.