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A NOTE ON MINIMAL SUBMANIFOLDS WITH M-INDEX 2

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In [1], Ōtsuki gave some examples of minimal submanifolds with M-index 2 and geodesic codimension 3 in Euclidean spaces, spheres and hyperbolic non-Euclidean spaces, where the dimension of submanifolds is not less than 3 and the submanifolds satisfy the following conditions:

(1) principal asymptotic vector field $P\neq 0$,
(2) subprincipal asymptotic vector field $Q=0$,
(3) ϕ_v is of rank 1,
(α) $\tilde{\omega}\neq 0$ and $\sigma=\mu/\lambda$ is constant on W^2,
(β) W^2 is of constant curvature.

In this paper, the author replace (1) with (1)′ $P=0$ and prove that there are no minimal submanifolds with M-index 2 such that the dimension of submanifolds is not less than 3 and they satisfy the above conditions: (1)′, (2), (3), (α), (β). Moreover, in case of two dimension he will give an example of minimal surface with M-index 2 and geodesic codimension 3 in a sphere. This example is nothing but one that in the case the ambiant spaces are spheres in [1] we set $p=0$ and $n=2$ formally and solve the differential equation. In this paper, we use the notations and equations in [1].

§ 1. Minimal submanifolds with M-index 2, $P=0$, $Q=0$ and ϕ_v of rank 1.

Since $P=0$ and $Q=0$, on B, we have

(1.1) $$\omega_{ar}=0, \qquad \text{where} \quad a=1,2 \quad \text{and} \quad r=3,\cdots,n.$$

From (1.1) we get $0=d\omega_{ar}=-\bar{c}\omega_a\wedge\omega_r$, hence we get

(1.2) $$\bar{c}=0.$$

Then the equations in Lemmas 6 and 11 in [1] are written as follows

(1.3) $$\{d\log\lambda-i(2\omega_{12}-\sigma\tilde{\omega})\}\wedge(\omega_1+i\omega_2)=0,$$

(1.4) $$\{d\sigma+i(1-\sigma^2)\tilde{\omega}\}\wedge(\omega_1+i\omega_2)=0,$$

(1.5) $$d\omega_{12}=(\lambda^2+\mu^2)\omega_1\wedge\omega_2,$$

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$$d\tilde{\omega}=-\frac{1}{\lambda\mu}(2\lambda^2\mu^2-f^2-g^2)\omega_1\wedge\omega_2, \tag{1.6}$$

$$\begin{aligned}&\{d\log(f-ig)-d\log\lambda-\mathrm{i}\omega_{12}\}\wedge(\omega_1+i\omega_2)\\ &-\frac{i}{f-ig}\tilde{\omega}\wedge\left\{f\left(\left(2\sigma-\frac{1}{\sigma}\right)\omega_1+\frac{i}{\sigma}\omega_2\right)-ig\left(\frac{1}{\sigma}\omega_1+i\left(2\sigma-\frac{1}{\sigma}\right)\omega_2\right)\right\}=0.\end{aligned} \tag{1.7}$$

From (α) and (1. 4) we get $\sigma^2=1$, i.e. $\sigma=1$ or -1. We may suppose $\sigma=1$, then from (β) and (1.5) we get

$$c=-2\lambda^2, \tag{1.8}$$

which is a negative constant. Since λ is constant on W^2 from (1. 8), we have $\tilde{\omega}=2\omega_{12}$ from (1. 3). Then, from (1. 6), we get

$$f^2+g^2=2\lambda^2(\lambda^2-c)=6\lambda^4.$$

Hence we may put $f+ig=\sqrt{6}\,\lambda^2e^{i\varphi}$ on W^2. Using the relations above and (1. 7), we have $\omega_{12}=-(1/3)d\varphi$, and hence we get $c=0$, which contradicts to (1. 8). Thus there is no minimal submanifold with M-index 2, whose dimension is not less than 3, satisfing the conditions $(1)'$, (2), $\cdots$, (β).

§ 2. Minimal surfaces with M-index 2 and ϕ_v of rank 1.

In this section, we discuss a minimal surface M^2 with M-index 2 and ϕ_v of rank 1 in a Riemannian manifold $M^{2+\nu}$ with constant curvature $\bar{c}$. Using the notations in [1] for our case, we can choose a frame $b\in B_1$ such that

$$\begin{cases}\omega_{13}=\lambda\omega_1, & \omega_{23}=-\lambda\omega_2, \quad \omega_{i\alpha}=0, \quad i=1,2,\\ \omega_{14}=\mu\omega_2, & \omega_{24}=\mu\omega_1, \quad \lambda\neq0, \quad \mu\neq0, \quad 3\leqq\alpha\leqq2+\nu.\end{cases} \tag{2.1}$$

Since ϕ_v is of rank 1, we can choose a frame $b\in B_1$ such that $F=fe_5$ and $G=ge_5$, $f^2+g^2\neq0$. Let B_2 be the set of such $b\in B_1$ satisfing (2. 1). Then, using (2. 1) and the structure equations, we get

$$\begin{aligned}&\lambda\omega_{35}=f\omega_1+g\omega_2, \qquad \omega_{3\gamma}=0,\\ &\mu\omega_{45}=g\omega_1-f\omega_2, \qquad \omega_{4\gamma}=0, \qquad 5<\gamma.\end{aligned} \tag{2.2}$$

Analogously to Theorem 1 in [1], we can verify that the geodesic codimension becomes 3. On B_2, we have the equations

$$\{d\log\lambda-i(2\omega_{12}-\sigma\tilde{\omega})\}\wedge(\omega_1+i\omega_2)=0, \tag{2.3}$$

$$\{d\sigma+i(1-\sigma^2)\tilde{\omega}\}\wedge(\omega_1+i\omega_2)=0, \tag{2.4}$$

$$d\omega_{12}=-(\bar{c}-\lambda^2-\mu^2)\omega_1\wedge\omega_2, \tag{2.5}$$

$$d\tilde{\omega}=-\frac{1}{\lambda\mu}(2\lambda^2\mu^2-f^2-g^2)\omega_1\wedge\omega_2, \tag{2.6}$$

$$\begin{aligned}&\{d\log(f-ig)-d\log\lambda-i\omega_{12}\}\wedge(\omega_1+i\omega_2)\\&-\frac{i}{f-ig}\tilde{\omega}\wedge\left\{f\left(\left(2\sigma-\frac{1}{\sigma}\right)\omega_1+\frac{i}{\sigma}\omega_2\right)-ig\left(\frac{1}{\sigma}\omega_1+i\left(2\sigma-\frac{1}{\sigma}\right)\omega_2\right)\right\}=0,\end{aligned} \tag{2.7}$$

where $\tilde{\omega}=\omega_{34}$ and $\sigma=\mu/\lambda$.

Now, we suppose that

$$\tilde{\omega}\neq 0 \quad\text{and}\quad \sigma=\text{constant on } M^2, \tag{α}$$

$$M^2 \text{ is of constant curvature } c. \tag{β}$$

Then we have $\sigma^2=1$, $\tilde{\omega}=2\omega_{12}$, $c=0$ and $2\lambda^2=\bar{c}$. Hence we may suppose that $\sigma=1$ and $\omega_{12}=d\theta$, then we get

$$\tilde{\omega}=2d\theta, \tag{2.8}$$

$$dx=\Re((e_1^*+ie_2^*)d\bar{z}), \tag{2.9}$$

$$\bar{D}(e_1^*+ie_2^*)=\lambda(e_3^*+ie_4^*)d\bar{z}, \tag{2.10}$$

$$\bar{D}(e_3^*+ie_4^*)=-(e_1^*+ie_2^*)\lambda dz+\sqrt{2}\,\lambda e_5 d\bar{z}, \tag{2.11}$$

$$\bar{D}e_5=-\sqrt{2}\,\lambda\Re((e_3^*+ie_4^*)dz), \tag{2.12}$$

where $e_1^*+ie_2^*=e^{i\theta}(e_1+ie_2)$, $e_3^*+ie_4^*=e^{2i\theta}(e_3+ie_4)$ and z is an isothermal coordinate of M^2 such that $\omega_1+i\omega_2=e^{-i\theta}dz$. Since $\bar{c}$ is positive constant, we may suppose $\bar{c}=1$, i.e. $\bar{M}^5=S^5$ (unit sphere). Then we have that $\lambda^2=1/2$. When $\lambda=1/\sqrt{2}$, considering as $S^5\subset E^6$, we get the Frenet formulas:

$$\begin{cases} dx=\Re(e_1^*+ie_2^*)d\bar{z},\\ d(e_1^*+ie_2^*)=\dfrac{1}{\sqrt{2}}(e_3^*+ie_4^*)d\bar{z}-e_6dz,\\ d(e_3^*+ie_4^*)=-\dfrac{1}{\sqrt{2}}(e_1^*+ie_2^*)dz+e_5d\bar{z}\\ de_5=-\Re((e_3^*+ie_4^*)dz).\end{cases} \tag{2.13}$$

These formulas are nothing but ones when formally we put $p=0$ and $n=2$ in the case $\bar{M}^{n+3}=S^{n+3}$ in [1]. Analogously to the case $\bar{M}^{n+3}=E^{n+3}$ in [1], we can give a solution of (2.13) as follows

$$
\begin{aligned}
x = & A_1 \exp \frac{i(u_1+\sqrt{3}\,u_2)}{\sqrt{2}} + \bar{A}_1 \exp \frac{-i(u_1+\sqrt{3}\,u_2)}{\sqrt{2}} + A_2 \exp \sqrt{2}\,iu_1 \\
& + \bar{A}_2 \exp(-\sqrt{2}\,iu_1) + A_3 \exp \frac{i(u_1-\sqrt{3}\,u_2)}{\sqrt{2}} + \bar{A}_3 \exp \frac{-i(u_1-\sqrt{3}\,u_2)}{\sqrt{2}},
\end{aligned} \tag{2.14}
$$

where $z=u_1+iu_2$ and A_1, A_2, and A_3 are fixed vectors in C^3 satisfing

$$
A_iA_j = A_i\bar{A}_k = 0, \qquad \sum_{j=1}^{3} A_j\bar{A}_j = \frac{1}{2} \quad \text{and} \quad A_j\bar{A}_j = \frac{1}{6}, \qquad (i, j, k = 1, 2, 3,\ i \neq k).
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