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ON A RIEMANNIAN MANIFOLD ADMITTING KILLING VECTORS WHOSE COVARIANT DERIVATIVES ARE CONFORMAL KILLING TENSORS

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§1. Let M^n be an n-dimensional Riemannian manifold with metric g_{ab}.[1] Let ∇_a denote the operator of covariant differentiation with respect to the Riemannian connection. We denote the Riemannian curvature tensor, the Ricci tensor and the scalar curvature respectively by $R_{abc}{}^e$, $R_{bc}=R_{ebc}{}^e$ and $R=g^{bc}R_{bc}$.

M^n is called a manifold of constant curvature if its Riemannian curvature tensor is given by

$$R_{abc}{}^e=(R/n(n-1))(\delta_a{}^e g_{bc}-g_{ac}\delta_b{}^e).$$

A vector field v^c is called a Killing vector if it satisfies

$$\nabla_b v_c+\nabla_c v_b=0, \qquad (v_c=v^e g_{ec}). \tag{1.1}$$

It is well known that a Killing vector v^c satisfies

$$\nabla_a\nabla_b v_c+R_{eabc}v^e=0. \tag{1.2}$$

A skew symmetric tensor field u_{bc} is called a *conformal Killing tensor*, if there exists a vector field p^c such that

$$\nabla_a u_{bc}+\nabla_b u_{ac}=2p_c g_{ab}-p_a g_{bc}-p_b g_{ac}.\text{[2]} \tag{1.3}$$

Such a vector field p^c is called an associated vector of u_{bc} and is given by

$$\nabla^e u_{ec}=(n-1)p_c. \tag{1.4}$$

Tachibana studied such a tensor and got the following:

THEOREM A. ([2]) *In a Riemannian manifold M^n of constant curvature, the covariant derivative $\nabla_b v_c$ of any Killing vector v_c is a conformal Killing tensor.*

It is well known that the set of all Killing vectors constitutes a Lie algebra L. We assume L to be transitive, i.e., there exists a Killing vector v^c satisfying $v^c(p)=V^c$ for any point p and for any direction V^c. Then, we know the converse of Theorem A is valid as follows.

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1) Indices $a, b, c, \cdots$ run over the range $1, 2, \cdots, n$.

2) This definition is primarily given by Tachibana in [2].

THEOREM 1. *In a Riemannian manifold M^n $(n>2)$, if the Lie algebra L of all Killing vectors v^c is transitive and the covariant derivative $\nabla_b v_c$ of any Killing vector v_c is a conformal Killing tensor, then M^n is a manifold of constant curvature.*

Proof. Taking $u_{bc}=\nabla_b v_c$ in (1. 4) and by making use of (1. 2), we find

$$p_c=-(1/(n-1))R_{ec}v^e. \tag{1. 5}$$

Again, taking $u_{bc}=\nabla_b v_c$ in (1. 3) and substituting (1. 2) and (1. 5) into what follows, we have

$$(R_{eabc}+R_{ebac})v^e=(1/(n-1))(2R_{ec}g_{ab}-R_{ea}g_{bc}-R_{eb}g_{ac})v^e.$$

Since the last equation is valid for any v^e, by the assumption, we have

$$R_{eabc}+R_{ebac}=(1/(n-1))(2R_{ec}g_{ab}-R_{ea}g_{bc}-R_{eb}g_{ac}). \tag{1. 6}$$

Contracting (1. 6) with g^{ec}, we get

$$R_{ab}=(R/n)g_{ab}.$$

By virtue of the last equation, (1. 6) becomes

$$R_{eabc}+R_{ebac}=(R/n(n-1))(2g_{ec}g_{ab}-g_{ea}g_{bc}-g_{eb}g_{ac}). \tag{1. 7}$$

Interchanging indices a, b, c in (1. 7) as $a\to b\to c\to a$, we have

$$R_{ebca}+R_{ecba}=(R/n(n-1))(2g_{ea}g_{bc}-g_{eb}g_{ca}-g_{ec}g_{ba}). \tag{1. 8}$$

If we form (1. 7)−(1. 8), we get

$$R_{ebac}=(R/n(n-1))(g_{ec}g_{ba}-g_{ea}g_{bc})$$

on taking account of the first Bianchi identity. Thus the proof is completed.

In the next section, we shall study analogous facts in a Kählerian manifold.

§2. A Kählerian manifold $\mathcal{M}^n$ is an even dimensioanl Riemannian manifold with a mixed tensor $F_a{}^b$ and with a Riemannian metric g_{ab} satisfying the following conditions

$$F_a{}^e F_e{}^b=-\delta_a{}^b, \qquad F_a{}^e F_b{}^r g_{er}=g_{ab},$$

$$\nabla_a F_b{}^c=0, \qquad F_{ab}=F_a{}^e g_{eb}=-F_{ba}.$$

It is well known that there holds the following relations:

$$R_{abce}F_d{}^e=R_{abde}F_c{}^e, \qquad R_{ab}=R_{er}F_a{}^e F_b{}^r,$$

(2. 1)

$$R_a{}^e F_e{}^b=-(1/2)R_{era}{}^b F^{er}.$$

If we define a tensor S_{ab} by

(2. 2) $$S_{ab}=(1/2)R_{aber}F^{er},$$

then we have

(2. 3) $$R_{ae}F_b{}^e=S_{ab}, \qquad S_{ae}F_b{}^e=-R_{ab} \qquad \text{and} \quad R=-S_{er}F^{er}.$$

$\mathcal{M}^n$ is called a manifold of constant holomorphic sectional curvature or a locally Fubinian manifold if its Riemannian curvature tensor is given by

(2. 4) $$R_{abc}{}^e=(R/n(n+2))(\delta_a{}^e g_{bc}-g_{ac}\delta_b{}^e+F_a{}^e F_{bc}-F_{ac}F_b{}^e-2F_{ab}F_c{}^e).$$

For a skew symmetric tensor field W_{bc} in $\mathcal{M}^n$, if there exists two vector fields p^c and q^c such that

(2. 5) $$\nabla_a w_{bc}+\nabla_b w_{ac}=2p_c g_{ab}-p_a g_{bc}-p_b g_{ac}+3(q_a F_{bc}+q_b F_{ac})$$

then, corresponding to a conformal Killing tensor u_{bc} in M^n, we shall call w_{bc} an *F-conformal Killing tensor* and p^c and q^c are associated vectors of w_{bc}.

Corresponding to Theorem A, we know the following

THEOREM B. ([1]) *In a manifold of constant holomorphic sectional curvature, the covariant derivative $\nabla_b v_c$ of any Killing vector v_c is an F-conformal Killing tensor.*

We know the converse case of Theorem B is also valid as follows.

THEOREM 2. *In a Kählerian manifold $\mathcal{M}^n$ ($n>2$), if the Lie algebra L of all Killing vectors v^c is transitive and the covariant derivative $\nabla_b v_c$ of any Killing vector v_c is an F-conformal Killing tensor, then $\mathcal{M}^n$ is a manifold of constant holomorphic sectional curvature.*

Proof. Taking $w_{bc}=\nabla_b v_c$ in (2. 5) and by making use of (1. 2) we know (2. 5) becomes

(2. 6) $$-(R_{eabc}+R_{ebac})v^e=2p_c g_{ab}-p_a g_{bc}-p_b g_{ac}+3(q_a F_{bc}+q_b F_{ac}).$$

Transvecting (2. 6) with g^{ab}, we get

(2. 7) $$-R_{ec}v^e=(n-1)p_c+3q^e F_{ec}.$$

On the other hand, transvecting (2. 6) with F^{bc}, we have

$$-S_{ea}v^e=p^e F_{ae}+(n+1)q_a.$$

Transvecting the last equation with $F_c{}^a$, we get

(2. 8) $$R_{ec}v^e=-p_c-(n+1)q^e F_{ec}.$$

If we form (2. 7)+(2. 8), we obtain

(2. 9) $$p_c = q^e F_{ec},$$

provided $n>2$. Consequently (2. 7) and (2. 8) imply

(2. 10) $$p_c = -(1/(n+2))R_{ec}v^e \quad \text{and} \quad q_c = -(1/(n+2))S_{ec}v^e.$$

Substituting (2. 10) into (2. 6), we have

$$(R_{eabc}+R_{ebac})v^e = (1/(n+2))[2R_{ec}g_{ab} - R_{ea}g_{bc} - R_{eb}g_{ac} + 3(S_{ea}F_{bc}+S_{eb}F_{ac})]v^e.$$

Since the last equation holds good for any v^e, therefore by the assumption, we obtain

(2. 11) $$R_{eabc}+R_{ebac} = (1/(n+2))[2R_{ec}g_{ab} - R_{ea}g_{bc} - R_{eb}g_{ac} + 3(S_{ea}F_{bc}+S_{eb}F_{ac})].$$

Transvecting (2. 11) with g^{ec} and taking account of (2. 3), we know

$$R_{ab} = (R/n)g_{ab}.$$

Substituting the last result into (2. 11), we get

(2. 12) $$R_{eabc}+R_{ebac} = (R/n(n+2))[2g_{ec}g_{ab} - g_{ea}g_{bc} - g_{eb}g_{ac} + 3(F_{ea}F_{bc}+F_{eb}F_{ac})].$$

Interchanging indices a, b, c in (2. 12) as $a \to b \to c \to a$, and then substracting what follows from (2. 12), we get the desired result

$$R_{ebac} = (R/n(n+2))(g_{ec}g_{ba} - g_{ea}g_{bc} + F_{ec}F_{ba} - F_{ea}F_{bc} - 2F_{eb}F_{ac}).$$

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