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PERIODS OF DIFFERENTIALS AND RELATIVE EXTREMAL LENGTH, II

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§ 4. Elementary differentials and relative extremal length.

1. Throughout **1** and **2** we shall preserve the notations in § 2. **1.** Let $\chi_{A_j}=\chi_{A_j}(\gamma)$ and $\chi_{B_j}=\chi_{B_j}(\gamma)$ $(j=1, \cdots, g;\ g\leqq\infty)$ be the functions on $\mathfrak{C}$ defined by

$$\chi_{A_j}(\gamma)=|A_j\times\gamma| \quad \text{and} \quad \chi_{B_j}(\gamma)=|\gamma\times B_j|$$

respectively. Let $\mathfrak{C}_{A_j}$ and $\mathfrak{C}_{B_j}$ $(j=1, \cdots, g)$ be the subclasses of $\mathfrak{C}$ consisting of curves γ such that $A_j\times\gamma\neq 0$ and $\gamma\times B_j\neq 0$ respectively. Then by Corollary 1.6 we have that

$$\text{(4. 1)} \qquad \begin{aligned} &\lambda(\mathfrak{C}, \chi_{A_j})=\lambda(\mathfrak{C}_{A_j}, \chi_{A_j}), \\ &\lambda(\mathfrak{C}, \chi_{B_j})=\lambda(\mathfrak{C}_{B_j}, \chi_{B_j}) \qquad (j=1, \cdots, g). \end{aligned}$$

We know (see [3][1)]) that there exist the differentials φ_{A_j} and φ_{B_j} in $\Gamma_{h0}^{*}\cap\Gamma_{hse}\subset\Lambda_{h0}^{*}$ uniquely determined by the period conditions:

$$\int_{A_k}\varphi_{A_j}=0, \qquad \int_{B_k}\varphi_{A_j}=\delta_{jk};$$

$$\int_{A_k}\varphi_{B_j}=-\delta_{jk}, \quad \int_{B_k}\varphi_{B_j}=0 \qquad (j, k=1, \cdots, g)$$

respectively. By Theorem 2.1 and (4.1) we have that

$$\lambda(\mathfrak{C}, \chi_{A_j})=\lambda(\mathfrak{C}_{A_j}, \chi_{A_j})=\|\varphi_{A_j}\|^{-2},$$

$$\lambda(\mathfrak{C}, \chi_{B_j})=\lambda(\mathfrak{C}_{B_j}, \chi_{B_j})=\|\varphi_{B_j}\|^{-2} \qquad (j=1, \cdots, g).$$

2. Let C_k be a generic element of $\{C_j\}_{j=1}^{N}$ $(N\leqq\infty)$. Let $\chi_{C_k}=\chi_{C_k}(\gamma)$ be the function on $\mathfrak{C}$ defined by

$$\chi_{C_k}(\gamma)=|\gamma\times C_k^{*}|.$$

Let $\mathfrak{C}_{C_k}$ be the subclass of $\mathfrak{C}$ consisting of curves γ such that $\gamma\times C_k^{*}\neq 0$. Then by Corollary 1.6 we have that

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1) See References of I (Kōdai Math. Sem. Rep. **21** (1969), 205-222).

$$\lambda(\mathfrak{C}, \chi_{C_k})=\lambda(\mathfrak{C}_{C_k}, \chi_{C_k}).$$

By Theorem 2.1 we see that there exists the unique differential $\varphi_{C_k}\in\Lambda_{h0}^*$ which satisfies the condition:

$$\int_{A_j}\varphi_{C_k}=\int_{B_j}\varphi_{C_k}=0 \qquad (j=1, \cdots, g),$$

$$\int_{C_j}\varphi_{C_k}=\delta_{jk} \qquad (j=1, \cdots, N)$$

if and only if $\lambda(\mathfrak{C}_{C_k}, \chi_{C_k})>0$, and provided any of these conditions is satisfied, the equality

$$\lambda(\mathfrak{C}_{C_k}, \chi_{C_k})=\|\varphi_{C_k}\|^{-2}$$

holds.

3. We shall use the notation in § 1.2. A generic element $C_{j_1\cdots j_\nu}$ $(j_\nu>1)$ of the canonical homology basis of dividing cycles modulo β shall be also denoted by the simplified notation C_j $(C_j=C_{j_1\cdots j_\nu};\ j=1, \cdots, N;\ N\leqq\infty)$. The sequence of non-compact regular subregions $\{\Omega_k\}_{k=1}^{\infty}$ such that $\partial\Omega_1=-C_{j_1\cdots j_{\nu-1}1}$, $\partial\Omega_2=-C_{j_1\cdots j_{\nu-1}11}, \cdots$ defines an ideal boundary component α_{j0}. Further $\{\Omega_k'\}_{k=1}^{\infty}$ such that $\partial\Omega_1'=-C_{j_1\cdots j_\nu 1}$, $\partial\Omega_2'=-C_{j_1\cdots j_\nu 11}, \cdots$ defines an ideal boundary component α_j. Partition the ideal boundary $\mathfrak{J}$ of R into two disjoint sets $\alpha_{j0}\cup\alpha_j$ and $\mathfrak{J}-\alpha_{j0}\cup\alpha_j$. Let τ_{C_j} be the differential of the generalized harmonic measure with respect to $\Lambda_h(\alpha_{j0}\cup\alpha_j, \mathfrak{J}-\alpha_{j0}\cup\alpha_j)$ associated to C_j (cf. § 1.7). Assume that $\tau_{C_j}\neq0$. Then obviously $\tau_{C_j}\in\Lambda_{hm}(\alpha, \beta)$, by Corollary 1.1

$$\int_{C_j}\tau_{C_j}^*=\|\tau_{C_j}\|_R^2>0 \tag{4.2}$$

and further

$$\int_{C_k}\tau_{C_j}^*=0 \qquad (k\neq j). \tag{4.3}$$

Let C_j^* be the conjugate relative cycle of C_j, let $\mathfrak{C}$ be the class of curves in R defined in § 3.1 and let $\mathfrak{C}_{C_j}$ be the subclass of $\mathfrak{C}$ consisting of curves γ such that $\gamma\times C_j^*\neq0$. Let $\chi_{C_j}=\chi_{C_j}(\gamma)$ be the function on $\mathfrak{C}$ defined by

$$\chi_{C_j}(\gamma)=|\gamma\times C_j^*|.$$

Then by the definition of $\gamma\in\mathfrak{C}$ we see that

$$\chi_{C_j}(\gamma)=\begin{cases}1 & (\gamma\in\mathfrak{C}_{C_j}),\\ 0 & (\gamma\notin\mathfrak{C}_{C_j}).\end{cases} \tag{4.4}$$

Thus by Corollary 1.6, Theorem 3.1, (4.2), (4.3) and (4.4) we have that

$$\lambda(\mathfrak{C}, \chi_{C_j})=\lambda(\mathfrak{C}_{C_j}, \chi_{C_j})=\lambda(\mathfrak{C}_{C_j})=\|\tau_{C_j}\|_R^2.$$

The last equation is also valid for $\tau_{C_j}\equiv 0$.

§ 5. Applications.

1. Application of Theorem 2.1. Throughout the present number, we shall preserve the notations in § 2.1. We note that we can take an arbitrary subclass $\mathfrak{C}'$ of $\mathfrak{C}$ in place of $\mathfrak{C}$ in Theorem 2.1 provided $\lambda(\mathfrak{C}', \chi)=\lambda(\mathfrak{C}, \chi)$. By Corollary 1.6 we may assume that $\chi(\gamma)\neq 0$ for all $\gamma\in\mathfrak{C}'$. We shall fix such a class of curves $\mathfrak{C}'$.

Let us define

$$\mathfrak{C}_{A_j}=\{\gamma\,|\,\gamma\times B_j\neq 0,\ \gamma\in\mathfrak{C}\},\qquad \mathfrak{C}_{B_j}=\{\gamma\,|\,A_j\times\gamma\neq 0,\ \gamma\in\mathfrak{C}\},$$

$$\mathfrak{C}_{C_k}=\{\gamma\,|\,\gamma\times C_k^*\neq 0,\ \gamma\in\mathfrak{C}\},$$

$$\mathfrak{C}'_{A_j}=\{\gamma\,|\,\gamma\times B_j\neq 0,\ \gamma\in\mathfrak{C}'\},\qquad \mathfrak{C}'_{B_j}=\{\gamma\,|\,A_j\times\gamma\neq 0,\ \gamma\in\mathfrak{C}'\},$$

$$\mathfrak{C}'_{C_k}=\{\gamma\,|\,\gamma\times C_k^*\neq 0,\ \gamma\in\mathfrak{C}'\},$$

$$m_{A_j}=\sup_{\gamma\in\mathfrak{C}'_{A_j}}\chi(\gamma),\qquad m_{B_j}=\sup_{\gamma\in\mathfrak{C}'_{B_j}}\chi(\gamma)\qquad\text{and}\qquad m_{C_k}=\sup_{\gamma\in\mathfrak{C}'_{C_k}}\chi(\gamma)$$

$$(j=1,\cdots,g;\ k=1,\cdots,N).$$

Obviously

$$\mathfrak{C}'\subset\bigcup_{a_j\neq 0}\mathfrak{C}'_{A_j}\cup\bigcup_{b_j\neq 0}\mathfrak{C}'_{B_j}\cup\bigcup_{c_k\neq 0}\mathfrak{C}'_{C_k},\ \ \mathfrak{C}'_{A_j}\subset\mathfrak{C}_{A_j},\ \ \mathfrak{C}'_{B_j}\subset\mathfrak{C}_{B_j}\ \ \text{and}\ \ \mathfrak{C}'_{C_k}\subset\mathfrak{C}_{C_k}$$

$$(j=1,\cdots,g;\ k=1,\cdots,N),$$

where by $\bigcup_{a_j\neq 0}$, etc. we denote the union for all j with $a_j\neq 0$ respectively. Thus by Lemmas 1.9, 1.10, 1.6 and the consequences of § 4.1 and § 4.2, we have that

$$\begin{aligned}\frac{1}{\lambda(\mathfrak{C},\chi)}&=\frac{1}{\lambda(\mathfrak{C}',\chi)}\\&\leqq\sum_{a_j\neq 0}\frac{1}{\lambda(\mathfrak{C}'_{A_j},\chi)}+\sum_{b_j\neq 0}\frac{1}{\lambda(\mathfrak{C}'_{B_j},\chi)}+\sum_{c_j\neq 0}\frac{1}{\lambda(\mathfrak{C}'_{C_j},\chi)}\\&\leqq\sum_{a_j\neq 0}\frac{m_{A_j}^2}{\lambda(\mathfrak{C}'_{A_j})}+\sum_{b_j\neq 0}\frac{m_{B_j}^2}{\lambda(\mathfrak{C}'_{B_j})}+\sum_{c_j\neq 0}\frac{m_{C_j}^2}{\lambda(\mathfrak{C}'_{C_j})}\\&\leqq\sum_{a_j\neq 0}\frac{m_{A_j}^2}{\lambda(\mathfrak{C}_{A_j})}+\sum_{b_j\neq 0}\frac{m_{B_j}^2}{\lambda(\mathfrak{C}_{B_j})}+\sum_{c_j\neq 0}\frac{m_{C_j}^2}{\lambda(\mathfrak{C}_{C_j})}\\&\leqq\sum_{a_j\neq 0}\frac{m_{A_j}^2}{\lambda(\mathfrak{C}_{A_j},\chi_{A_j})}+\sum_{b_j\neq 0}\frac{m_{B_j}^2}{\lambda(\mathfrak{C}_{B_j},\chi_{B_j})}+\sum_{c_j\neq 0}\frac{m_{C_j}^2}{\lambda(\mathfrak{C}_{C_j},\chi_{C_j})}\\&=\sum_{a_j\neq 0}m_{A_j}^2\|\varphi_{A_j}\|^2+\sum_{b_j\neq 0}m_{B_j}^2\|\varphi_{B_j}\|^2+\sum_{c_j\neq 0}m_{C_j}^2\|\varphi_{C_j}\|^2,\end{aligned}$$

where φ_{A_j}, φ_{B_j} and φ_{C_j} are the differentials defined in § 4. **1** and § 4. **2** respectively, and χ_{A_j}, χ_{B_j} and χ_{C_j} are the functions on $\mathfrak{C}$ defined in § 4. **1** and § 4. **2** respectively. Hence we obtain the corollary of Theorem 2. 1.

COROLLARY 5. 1. *If there exists the subclass* $\mathfrak{C}'$ *of* $\mathfrak{C}$ *for which the inequality*

$$\sum_{a_j\neq 0} m_{A_j}^2\|\varphi_{A_j}\|^2+\sum_{b_j\neq 0} m_{B_j}^2\|\varphi_{B_j}\|^2+\sum_{c_j\neq 0} m_{C_j}^2\|\varphi_{C_j}\|^2<\infty \tag{5. 1}$$

holds, then there exists the differential $\omega\in\Lambda_{h0}^*$ *which satisfies the period condition*:

$$\int_{A_j}\omega=a_j,\qquad \int_{B_j}\omega=b_j\qquad (j=1,\cdots,g),$$

$$\int_{C_j}\omega=c_j\qquad\qquad (j=1,\cdots,N).$$

REMARK. In the inequality (5. 1) it is assumed that there exist the differentials φ_{C_j} for every j with $c_j\neq 0$ respectively. Further the inequality (5. 1) implies that

$$m_{A_j}<\infty,\quad m_{B_j}<\infty\quad\text{and}\quad m_{C_j}<\infty \tag{5. 2}$$

for every j with $a_j\neq 0$, $b_j\neq 0$ and $c_j\neq 0$ respectively. If we take $\mathfrak{C}$ for $\mathfrak{C}'$ then (5. 2) is not satisfied. Furthermore for an arbitrarily given system $\{A_j, B_j, C_j\}$ of the homology basis modulo β there does not necessarily exist the $\mathfrak{C}'$ for which (5. 2) holds. Thus for the test of the criterion (5. 1) it is necessary to choose the system $\{A_j, B_j, C_j\}$ and the subclass $\mathfrak{C}'$ for which (5. 2) holds in the first place.

Similarly we obtain the corollaries of Theorems 2. 2 and 2. 3 analogous to Corollary 5. 1.

2. Application of Theorem 3. 1. Throughout the present number, we shall preserve the notations in § 3. **1.** Let $\mathfrak{C}_{C_j}$ $(j=1,\cdots,N)$ be the subclass of $\mathfrak{C}$ consisting of curves γ such that $\gamma\times C_j^*\neq 0$ respectively. Let $\mathfrak{C}''$ be the subclass of $\mathfrak{C}$ consisting of curves γ such that $\chi(\gamma)\neq 0$. Then

$$\mathfrak{C}''\subset\bigcup_{c_j\neq 0}\mathfrak{C}_{C_j}.$$

Set

$$m_{C_j}=\sup_{\gamma\in\mathfrak{C}_{C_j}}\chi(\gamma)\qquad (j=1,\cdots,N).$$

Then by Corollary 1. 6, Lemmas 1. 9, 1. 10 and the consequence of § 4. **3,** we have that

$$\frac{1}{\lambda(\mathfrak{C},\chi)}=\frac{1}{\lambda(\mathfrak{C}'',\chi)}\leqq\sum_{c_j\neq 0}\frac{1}{\lambda(\mathfrak{C}_{C_j},\chi)}\leqq\sum_{c_j\neq 0}\frac{m_{C_j}^2}{\lambda(\mathfrak{C}_{C_j})}=\sum_{c_j\neq 0}\frac{m_{C_j}^2}{\|\tau_{C_j}\|_R^2},$$

where τ_{C_j} is the differential defined in § 4. 3. Thus we obtain the corollary of Theorem 3. 1.

COROLLARY 5. 2. *If the series*

$$\sum_{c_j\neq 0}\frac{m_{C_j}^2}{\lambda(\mathfrak{C}_{C_j})}=\sum_{c_j\neq 0}\frac{m_{C_j}^2}{\|\tau_{C_j}\|_R^2}$$

is convergent, then there exists the differential $\omega\in\Lambda_{hm}^*$ *which satisfies the period condition*:

$$\int_{C_j}\omega=c_j\qquad(j=1,\cdots,N).$$

3. Another application. Throughout the present number, we shall preserve the notations in § 3. 1. We note that we can take an arbitrary subclass $\mathfrak{C}'$ of $\mathfrak{C}$ in place of $\mathfrak{C}$ in Theorem 3. 1 provided $\lambda(\mathfrak{C}',\chi)=\lambda(\mathfrak{C},\chi)$. By Corollary 1. 6 we may assume that $\chi(\gamma)\neq 0$ for all $\gamma\in\mathfrak{C}'$. We shall fix such a class of curves $\mathfrak{C}'$.

We divide $\mathfrak{C}'$ into homology classes $\mathfrak{C}_n'$ $(n=1,\cdots,\nu;\ \nu\leqq\infty)$ by the homology relation modulo β. Further let $\mathfrak{C}_n$ $(n=1,\cdots,\nu)$ be the subclasses of $\mathfrak{C}$ which consist of all elements of $\mathfrak{C}$ which are homologous modulo β to an element of $\mathfrak{C}_n'$ respectively. Then obviously

$$\mathfrak{C}'=\bigcup_{n=1}^{\nu}\mathfrak{C}_n'\subset\bigcup_{n=1}^{\nu}\mathfrak{C}_n\subset\mathfrak{C}.$$

The function $\chi(\gamma)$ takes a constant value k_n on each $\mathfrak{C}_n$ respectively. Hence by Lemmas 1. 9, 1. 10 and 1. 6 we have that

$$\frac{1}{\lambda(\mathfrak{C},\chi)}=\frac{1}{\lambda(\mathfrak{C}',\chi)}\leqq\sum_{n=1}^{\nu}\frac{1}{\lambda(\mathfrak{C}_n',\chi)}=\sum_{n=1}^{\nu}\frac{k_n^2}{\lambda(\mathfrak{C}_n')}\leqq\sum_{n=1}^{\nu}\frac{k_n^2}{\lambda(\mathfrak{C}_n)}.\tag{5. 3}$$

Let σ_n $(n=1,\cdots,\nu)$ be the differentials of the generalized harmonic measures with respect to $\Lambda_h(\alpha,\beta)$ associated to $\gamma\in\mathfrak{C}_n$ respectively. Here σ_n does not depend on a particular choice of an element γ of $\mathfrak{C}_n$ for each n. Then we find that

$$\lambda(\mathfrak{C}_n)=\|\sigma_n\|^2\qquad(n=1,\cdots,\nu)\tag{5. 4}$$

(e. g. see Theorem III. 3. 1. of [8]).

By (5. 3) and (5. 4) we obtain the corollary of Theorem 3. 1.

COROLLARY 5. 3. *If there exists the subclass* $\mathfrak{C}'$ *of* $\mathfrak{C}$ *for which the series*

$$\sum_{n=1}^{\nu}\frac{k_n^2}{\lambda(\mathfrak{C}_n)}=\sum_{n=1}^{\nu}\frac{k_n^2}{\|\sigma_n\|^2}\tag{5. 5}$$

is convergent, then there exists the differential $\omega\in\Lambda_{hm}^*$ *which satisfies the period condition*:

$$\int_{C_j} \omega = c_j \qquad (j=1, \cdots, N).$$

REMARK. If we can find the subclass $\mathfrak{C}'$ of $\mathfrak{C}$ for which the homology classes $\mathfrak{C}'_n$ $(n=1, \cdots, \nu)$ are families of curves in disjoint open sets Ω_n in R respectively, then by Corollary 1.4 and (5.3)

$$\frac{1}{\lambda(\mathfrak{C}, \chi)} = \frac{1}{\lambda(\mathfrak{C}', \chi)} = \sum_{n=1}^{\nu} \frac{k_n^2}{\lambda(\mathfrak{C}'_n)}.$$

Thus by Theorem 3.1 if the series

$$\sum_{n=1}^{\nu} \frac{k_n^2}{\lambda(\mathfrak{C}'_n)}$$

is convergent, then there exists the differential ω in Corollary 5.3 and

$$||\omega||_R^2 = \sum_{n=1}^{\nu} \frac{k_n^2}{\lambda(\mathfrak{C}'_n)}.$$

Similarly we obtain the corollaries of Theorems 2.1, 2.2 and 2.3 analogous to Corollary 5.3.

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