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PERIODS OF DIFFERENTIALS AND RELATIVE EXTREMAL LENGTH, I

BY HISAO MIZUMOTO

Introduction.

Let R be an open Riemann surface and its ideal boundary be denoted by $\mathfrak{I}$. Let $\{A_j, B_j\}$ be a canonical homology basis modulo $\mathfrak{I}$ and $\{C_j\}$ be a homology basis of dividing cycles. Let Γ_h be the space of harmonic differentials on R with finite Dirichlet norm. It seems to be an important problem to decide when there exists a differential $\omega \in \Gamma_h$ which satisfies a period condition

$$\int_{A_j} \omega = a_j, \qquad \int_{B_j} \omega = b_j, \qquad \int_{C_j} \omega = c_j$$

for an arbitrarily given system of real numbers a_j, b_j, and c_j.

In the present paper we shall concern ourselves with the problem to decide the existence of such differentials in the important subspaces Γ_{h0}^*, Γ_{hm}^*, $\Gamma_{h0}^* \cap \Gamma_{hse}$ (cf. [3]) and further the more general subspaces Λ_{h0}^*, Λ_{hm}^* (see §1. **5**). In the terms of relative extremal length (see §1. **10** for the definition) we shall state a perfect condition in order that there exists the differential with given periods in each of these spaces (Theorems 2. 1, 2. 2, 2. 3 and 3. 1). Further it is shown that the differential gives an extremal metric of a certain relative extremal length problem.

In the subsequent paper II, some applications of the present consequences will be shown.

The problem concerning the existence of differentials with given periods has been studied by many authors: Virtanen [16], Kusunoki [6] and Sainouchi [15], etc. They are mainly based on such the algebraic method as the orthogonalization of differentials. Our present method is quite different from these and very geometrical.

§1. Preliminaries.

1. Canonical homology basis modulo the ideal boundary. Let R be an open Riemann surface. A singular cycle is said to be a *dividing cycle*, or *homologous to* 0 *modulo the ideal boundary*, if it is homologous to a singular cycle which lies outside of any given compact set. Let $\mathfrak{H}$ and $\mathfrak{H}_{\mathfrak{I}}$ be the groups formed by the homology classes of all singular cycles and dividing singular cycles respectively.

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The quotient group $\mathfrak{H}/\mathfrak{H}_{\mathfrak{J}}$ is the homology group modulo dividing cycles. It can also be called the *relative homology group modulo the ideal boundary.*

It is known that there exists a system of cycles $\{A_j, B_j\}_{j=1}^{g}$ $(g \leqq \infty)$ on R satisfying the following conditions (see [2], [3]):

(i) Each cycle C on R is homologous to a linear conbination of a finite number of the cycles A_j, B_j modulo $\mathfrak{J}$, that is

$$C \sim \sum_{j=1}^{\kappa} (x_j A_j + y_j B_j) \qquad (\text{mod } \mathfrak{J};\ \kappa < \infty),$$

where x_j, y_j $(j=1, \cdots, \kappa)$ are integers and $\mathfrak{J}$ denotes the ideal boundary of R;

(ii) The intersection numbers between them satisfy

$$A_j \times A_k = B_j \times B_k = 0, \qquad A_j \times B_k = \delta_{jk} \qquad (j, k = 1, \cdots, g),$$

where δ_{jk} denotes the Kronecker symbol.
Such a system is called a *canonical homology basis of R modulo the ideal boundary* $\mathfrak{J}$.

Let $\{R_n\}_{n=1}^{\infty}$ be an exhaustion of R. Then there exists a system of cycles $\{A_j, B_j\}_{j=1}^{g}$ furthermore satisfying the following condition (see [2]):

(iii) For every n there exists a finite number κ_n such that $A_1, B_1, \cdots, A_{\kappa_n}, B_{\kappa_n}$ form a relative homology basis of R_n mod ∂R_n, that is, each cycle $C \subset R_n$ is homologous to a linear combination of $A_1, B_1, \cdots, A_{\kappa_n}, B_{\kappa_n}$ modulo ∂R_n, where ∂R_n denotes the relative boundary of R_n.
We shall call such a system a *canonical homology basis modulo the ideal boundary* $\mathfrak{J}$ *associated to the exhaustion* $\{R_n\}$.

2. Canonical homology basis of dividing cycles modulo β. Let $\hat{R}$ be the Kerékjartó-Stoïlow compactification of R. Partition the ideal boundary $\mathfrak{J}$ of R into two disjoint sets α and β such that α is closed on $\hat{R}$. Let Ω be a subregion of R which is not relatively compact and of which the relative boundary $\partial\Omega$ consists of a finite number of Jordan analytic curves. Then Ω will be called a *non-compact regular region.* Further the subsets of α and β which are boundary components of Ω are denoted by $\alpha(\Omega)$ and $\beta(\Omega)$ respectively.

A dividing cycle is said to be *homologous to* 0 *modulo* β, if it is homologous to a cycle which lies on Ω with $\alpha(\Omega)=\phi$. Let $\mathfrak{H}_\beta$ be the group formed by the homology class of all dividing cycles homologous to 0 modulo β. The quotient group $\mathfrak{H}_{\mathfrak{J}}/\mathfrak{H}_\beta$ is called the *homology group of dividing cycles modulo* β.

Let K be a relatively compact subregion of R. We shall call K a (compact) *regular region,* if ∂K consists of a finite number of Jordan analytic curves, $\partial K = \partial(R-K)$ and all components of $R-K$ are noncompact. Furthermore a regular region K is called *canonical* if each component of $R-K$ has a single contour. An exhaustion $\{R_n\}$ of R is called to be *canonical* if each R_n is canonical.

Let C be a generic component of ∂K. Partition the set of all components of ∂K into two disjoint sets α_K and β_K in such a manner that $C \in \alpha_K$ if $\alpha(\Omega) \neq \phi$ for Ω with $\partial\Omega = -C$ and otherwise $C \in \beta_K$.

Let $\{R_n\}_{n=1}^{\infty}$ be a canonical exhaustion of R. For simplicity we set $\alpha_n=\alpha_{R_n}$ and $\beta_n=\beta_{R_n}$. Let the elements of α_1 be denoted by C_j $(j=1,\cdots,s)$. We denote by Q_j $(j=1,\cdots,s)$ the component of R_2-R_1 which has C_j in a common boundary component with R_1. The boundary components of Q_j belonging to α_2 are denoted by C_{jk} $(1\leqq k\leqq s_j)$. Then $\partial Q_j=\sum_{k=1}^{s_j}C_{jk}-C_j+\sum$ (comp. of $\partial Q_j\in\beta_2$). Next, we denote by Q_{jk} the component of R_3-R_2 which has C_{jk} in a common boundary component with Q_j.

When we continue in this way we obtain the symbol $C_{j_1\cdots j_n}$ for each element of α_n with the subscript j_n running from 1 to a number $s_{j_1\cdots j_{n-1}}$. The component of $R_{n+1}-R_n$ which has $C_{j_1\cdots j_n}$ in a common boundary component with $Q_{j_1\cdots j_{n-1}}$ has a name $Q_{j_1\cdots j_n}$ and

$$\partial Q_{j_1\cdots j_n}=\sum_j C_{j_1\cdots j_n j}-C_{j_1\cdots j_n}+\sum \text{(comp. of } \partial Q_{j_1\cdots j_n}\in\beta_{n+1}).$$

LEMMA 1.1. (Cf. [3] for the case $\beta=\phi$ and also [9].) *A homology basis modulo β of an open Riemann surface R is formed by the combined system of all cycles A_j, B_j of a canonical homology basis modulo the ideal boundary associated to a canonical exhaustion $\{R_n\}$ and all cycles $C_{j_1\cdots j_n}$ with $j_n>1$.*

Such a system of cycles will be called a *canonical homology basis modulo β associated to the exhaustion $\{R_n\}$*. Each cycle C satisfies a unique homology relation

$$C\sim\sum_j (x_jA_j+y_jB_j)+\sum_{j_n>1} z_{j_1\cdots j_n}C_{j_1\cdots j_n} \quad (\text{mod } \beta),$$

where x_j, y_j and $z_{j_1\cdots j_n}$ are integers, the sum is finite, and C is a dividing cycle if and only if all x_j and y_j are 0.

The system of cycles $C_{j_1\cdots j_n}$ $(j_n>1)$ determines a basis for $\mathfrak{H}_{\mathfrak{J}}/\mathfrak{H}_{\beta}$, which is called a *canonical homology basis of dividing cycles modulo β associated to the exhaustion $\{R_n\}$*.

If $\beta=\phi$ and thus $\alpha=\mathfrak{J}$, then the term "modulo β" in the above may be omitted.

For simplicity of notation, a canonical homology basis $C_{j_1\cdots j_\nu}$ $(\nu=1,\cdots,n;\ j_\nu>1)$ of dividing cycles modulo β_n of R_n is also denoted by $C_1,\cdots,C_{\iota_n}$ with the changed subindices.

3. Conjugate relative cycle of $C_{j_1\cdots j_n}$. Let C^* be a one-dimensional singular chain. We shall say that C^* is a *relative cycle* if and only if the boundary $\partial C^*=0$. The group of relative cycles contains the subgroup of cycles.

Consider a point on $C_{j_1\cdots j_n}$ $(j_n>1)$. It can be joined by a simple curve in $Q_{j_1\cdots j_n}$ to a point on $C_{j_1\cdots j_n 1}$. This point can be joined to a point on $C_{j_1\cdots j_n 11}$ by a simple curve in $Q_{j_1\cdots j_n 1}$, and so on. In the opposite direction we can pass from the point on $C_{j_1\cdots j_n}$ through $Q_{j_1\cdots j_{n-1}}$ to a point on $C_{j_1\cdots j_{n-1}1}$, then through $Q_{j_1\cdots j_{n-1}1}$ to a point on $C_{j_1\cdots j_{n-1}11}$, and so on. Here we take the curves so that each of them does not intersect any element A_j or B_j of canonical homology basis modulo $\mathfrak{J}$

associated to the exhaustion $\{R_n\}$, which is possible. Then the product of the curves between consecutive points is a relative cycle, which we denote by $C^*_{j_1\cdots j_n}$ and we call the *conjugate relative cycle of* $C_{j_1\cdots j_n}$. Its direction can be fixed so that $C_{j_1\cdots j_n}\times C^*_{j_1\cdots j_n}=1$. Then $C_{j_1\cdots j_n 1\cdots 1}\times C^*_{j_1\cdots j_n}=1$, $C_{j_1\cdots j_{n-1}1\cdots 1}\times C^*_{j_1\cdots j_n}=-1$, and for all others $C_{k_1\cdots k_{n'}}\times C^*_{j_1\cdots j_n}=0$, $A_j\times C^*_{j_1\cdots j_n}=0$ and $B_j\times C^*_{j_1\cdots j_n}=0$.

4. Space of differentials Γ_h. Let ω_1 and ω_2 be two real harmonic differentials on R. Then the *inner product* of ω_1 and ω_2 is defined by

$$(\omega_1, \omega_2)_R=\int_R \omega_1\omega_2^*,$$

where ω_2^* denotes the conjugate differential of ω_2 and $\omega_1\omega_2^*$ the exterior product of ω_1 and ω_2^*, and further the *norm* $\|\omega_1\|_R$ of ω_1 by $\|\omega_1\|_R^2=(\omega_1, \omega_1)_R$. Let Γ_h be the Hilbert space of the real harmonic differentials on R with finite norm. We define the some subspaces of Γ_h (cf. [3]).

Γ_{he}: the space of exact harmonic differentials.

Γ_{hse}: the space of semi-exact harmonic differentials.

$\omega\in\Gamma_{h0}$ if and only if for each $\varepsilon>0$ and each compact set E there exist a regular region R_0 and a harmonic differential ω_0 on R_0 such that $E\subset R_0$, $\omega_0=0$ along ∂R_0 and $\|\omega-\omega_0\|_{R_0}<\varepsilon$.

$\omega\in\Gamma_{hm}$ if and only if for each $\varepsilon>0$ and each compact set E there exist a canonical region R_0 and an exact harmonic differential ω_0 on R_0 such that $E\subset R_0$, $\omega_0=0$ along ∂R_0 and $\|\omega-\omega_0\|_{R_0}<\varepsilon$.

We can easily see that the inclusion relations

$$\Gamma_{hm}\subset\Gamma_{he}\subset\Gamma_{hse}, \qquad \Gamma_{hm}\subset\Gamma_{h0}\subset\Gamma_{hse}$$

hold. The subspaces formed by all conjugate differentials of differentials in Γ_{he}, Γ_{hse}, Γ_{hm} and Γ_{h0} are denoted by Γ^*_{he}, Γ^*_{hse}, Γ^*_{hm} and Γ^*_{h0} respectively.

The following orthogonal decompositions hold (cf. [3]):

$$\Gamma_h=\Gamma_{h0}\dot{+}\Gamma^*_{he}=\Gamma^*_{h0}\dot{+}\Gamma_{he}, \tag{1.1}$$

$$\Gamma_h=\Gamma_{hm}\dot{+}\Gamma^*_{hse}=\Gamma^*_{hm}\dot{+}\Gamma_{hse} \tag{1.2}$$

and

$$\Gamma_h=\Gamma_{hse}\cap\Gamma^*_{hse}\dot{+}\Gamma_{hm}\dot{+}\Gamma^*_{hm}.$$

Hence we have further the decomposition

$$\Gamma_h=\Gamma_{hm}\dot{+}\Gamma_{h0}\cap\Gamma^*_{hse}\dot{+}\Gamma^*_{he}=\Gamma^*_{hm}\dot{+}\Gamma^*_{h0}\cap\Gamma_{hse}\dot{+}\Gamma_{he}. \tag{1.3}$$

Here the space $\Gamma_{h0}\cap\Gamma^*_{hse}$ is identical to the closure of the space of the canonical harmonic differentials in Kusunoki's meaning whose conjugate differentials are semi-exact (cf. [6], [11]).

5. Space of differentials Λ_h. Partition the ideal boundary $\mathfrak{J}$ of R into two disjoint sets α and β such that α is closed on $\hat{R}$. The subspace $\Lambda_h=\Lambda_h(\alpha, \beta)$ of Γ_h is defined as the collection of all $\omega\in\Gamma_h$ which satisfy the following conditions for every non-compact regular region Ω such that $\partial\Omega$ consists of a Jordan curve and $\alpha(\Omega)=\phi$:

(i) For each $\varepsilon>0$ and each compact set $E\subset\Omega$ there exist a region $\Omega_0\subset\Omega$ and a harmonic differential ω_0 on $\bar{\Omega}_0$ such that $E\subset\Omega_0$, $\partial\Omega\subset\partial\Omega_0$, $\Omega_0\equiv\Omega\cap K$ for a certain regular region K, $\omega_0=0$ along $\partial\Omega_0-\partial\Omega$ and $\|\omega-\omega_0\|_{\Omega_0}<\varepsilon$;

(ii) ω^* is semi-exact on Ω.

It is shown that the present $\Lambda_h=\Lambda_h(\alpha, \beta)$ is identical to the space $\Lambda_h=\Lambda_h(\alpha, \beta, \phi)$ for the case of $\gamma=\phi$ defined in [10]. The proof is omitted.

The subspaces Λ_{h0} and Λ_{hm} of Λ_h are defined by $\Lambda_{h0}=\Lambda_h\cap\Gamma_{h0}$ and $\Lambda_{hm}=\Lambda_h\cap\Gamma_{hm}$. The spaces formed by all conjugate differentials of differentials in Λ_h, Λ_{h0} and Λ_{hm} are denoted by Λ_h^*, Λ_{h0}^* and Λ_{hm}^* respectively. Obviously, if $\beta=\phi$, then $\Lambda_h=\Lambda_h^*=\Gamma_h$, $\Lambda_{h0}=\Gamma_{h0}$, $\Lambda_{hm}=\Gamma_{hm}$, $\Lambda_{h0}^*=\Gamma_{h0}^*$ and $\Lambda_{hm}^*=\Gamma_{hm}^*$.

By (1. 1), (1. 2) and (1. 3) we obtain immediately the following orthogonal decompositions:

$$\Lambda_h=\Lambda_{h0}\dot{+}\Lambda_h\cap\Gamma_{he}^*,\qquad \Lambda_h^*=\Lambda_{h0}^*\dot{+}\Lambda_h^*\cap\Gamma_{he}, \tag{1. 4}$$

$$\Lambda_h=\Lambda_{hm}\dot{+}\Lambda_h\cap\Gamma_{hse}^*,\qquad \Lambda_h^*=\Lambda_{hm}^*\dot{+}\Lambda_h^*\cap\Gamma_{hse} \tag{1. 5}$$

and

$$\Lambda_h=\Lambda_{hm}\dot{+}\Lambda_{h0}\cap\Gamma_{hse}^*\dot{+}\Lambda_h\cap\Gamma_{he}^*.$$

Here we can easily see that

$$\Lambda_{h0}\cap\Gamma_{hse}^*=\Lambda_h(\phi, \alpha\cup\beta)=\Gamma_{h0}\cap\Gamma_{hse}^*.$$

Thus we have that

$$\begin{aligned}\Lambda_h&=\Lambda_{hm}\dot{+}\Gamma_{h0}\cap\Gamma_{hse}^*\dot{+}\Lambda_h\cap\Gamma_{he}^*,\\ \Lambda_h^*&=\Lambda_{hm}^*\dot{+}\Gamma_{h0}^*\cap\Gamma_{hse}\dot{+}\Lambda_h^*\cap\Gamma_{he}.\end{aligned} \tag{1. 6}$$

6. Elementary differentials of Γ_h. Let Ω be a non-compact regular region. A function u being harmonic on Ω and continuous on the relative closure $\bar{\Omega}$ is called a *normalized function* on Ω, if for each $\varepsilon>0$ and each compact set $E\subset\Omega$ there exist a region $\Omega_0\subset\Omega$ and a harmonic function u_0 on Ω_0 such that $E\subset\Omega_0$, $\partial\Omega\subset\partial\Omega_0$, $\Omega_0\equiv\Omega\cap K$ for a certain canonical region K,

$$u_0=\begin{cases}u & \text{on } \partial\Omega,\\ 0 & \text{on } \partial\Omega_0-\partial\Omega\end{cases}$$

and $\|du-du_0\|_{\Omega_0}<\varepsilon$ (cf. [12], [6]). For the sake of conformity a harmonic function u on a relatively compact subregion Ω of R is always assumed to be a normalized function on Ω. The differential du of a normalized function u is called a *normalized differential*.

Let C be a dividing cycle in R which divides R into two subregions R_C and $\tilde{R}_C$ so determined that $C=\partial R_C=-\partial\tilde{R}_C$. A harmonic function u_C on R is called a *harmonic measure associated to* C if u_C and $1-u_C$ are normalized functions on R_C and $\tilde{R}_C$ respectively. u_C is uniquely determined up to the case where both harmonic measures on R_C and $\tilde{R}_C$ with respect to their ideal boundaries vanish. In the latter case we define $u_C\equiv 0$. Except for the latter case we have that

$$u_C+\tilde{u}_C=1,$$

where $\tilde{u}_C$ denotes the harmonic measure associated to $-C$. By σ_C and $\tilde{\sigma}_C$ we denote du_C and $d\tilde{u}_C$. Then $\sigma_C=-\tilde{\sigma}_C$.

Let $\{A_j, B_j\}_{j=1}^{g}$ $(g\leqq\infty)$ be a canonical homology basis of R modulo $\mathfrak{J}$. The elementary differential σ_{A_j} (σ_{B_j}) is defined as an element of Γ_h which satisfies the conditions:

(i) σ_{A_j} (σ_{B_j}) is a normalized differential on $R-A_j$ $(R-B_j)$;

(ii)
$$\int_{A_k}\sigma_{A_j}=0, \qquad \int_{B_k}\sigma_{A_j}=\delta_{jk}$$
$$\left(\int_{A_k}\sigma_{B_j}=-\delta_{jk}, \qquad \int_{B_k}\sigma_{B_j}=0\right) \qquad (j, k=1, \cdots, g).$$

We can easily see that each of the differentials σ_{A_j} (σ_{B_j}) is uniquely determined (cf. [12], [6]).

7. Elementary differentials of Λ_h. Let α and β be the partition of the ideal boundary $\mathfrak{J}$ of R defined in **2.** Let Ω be a non-compact regular region. A function u being harmonic on Ω and continuous on $\bar{\Omega}$ is called a *normalized function on* Ω *with respect to the space* Λ_h, if for each $\varepsilon>0$ and each compact set $E\subset\Omega$ there exist a region $\Omega_0\subset\Omega$ and a harmonic function u_0 on Ω_0 such that $E\subset\Omega_0$, $\partial\Omega\subset\partial\Omega_0$, $\Omega_0\equiv\Omega\cap K$ for a certain canonical region K, $\|du-du_0\|_{\Omega_0}<\varepsilon$ and

$$u_0=\begin{cases} u & \text{on } \partial\Omega, \\ 0 & \text{on } C\in\alpha_K \quad (C\subset\Omega), \\ \text{const.} & \text{on } C\in\beta_K \quad (C\subset\Omega) \end{cases}$$

with the constant so chosen that $\int_C du_0^*=0$ for each $C\in\beta_K$ $(C\subset\Omega)$, where α_K and β_K denote those defined in **2.** The present normalized function u with respect to the space Λ_h is obviously identical to the normalized function for the case of $\gamma=\phi$ defined in §1. 4 of [10].

Let C be a dividing cycle in R which divides R into two subregions R_C and $\tilde{R}_C$ so determined that $C=\partial R_C=-\partial\tilde{R}_C$. A harmonic function u_C on R is called a *generalized harmonic measure* (*with respect to* Λ_h) *associated to* C if u_C and $1-u_C$ are normalized functions on R_C and $\tilde{R}_C$ respectively with respect to Λ_h. The existence and the uniqueness of u_C refer to [10]. By σ_C we denote du_C.

Let $\{A_j, B_j\}_{j=1}^{g}\ (g<\infty)$ be a canonical homology basis of R modulo $\mathfrak{F}$. The elementary differential σ_{A_j} (σ_{B_j}) with respect to the space Λ_h is defined as an element of Λ_h which satisfies the conditions:

(a) σ_{A_j} (σ_{B_j}) is a normalized differential on $R-A_j$ $(R-B_j)$ with respect to the space Λ_h;

(b)
$$\int_{A_k}\sigma_{A_j}=0, \qquad \int_{B_k}\sigma_{A_j}=\delta_{jk}$$
$$\left(\int_{A_k}\sigma_{B_j}=-\delta_{jk}, \qquad \int_{B_k}\sigma_{B_j}=0\right) \qquad (j, k=1, \cdots, g).$$

The differential σ_{A_j} (σ_{B_j}) uniquely exists (cf. [10]).

Especially the elementary differential σ_{A_j} (σ_{B_j}) with respect to $\Lambda_h(R; \phi, \alpha\cup\beta)=\Gamma_{h0}\cap\Gamma_{hse}^{*}$ is denoted by τ_{A_j} (τ_{B_j}).

8. Integral formulas.

LEMMA 1. 2. (Cf. Theorem 1. 3 of [10].) *Let $\omega\in\Lambda_h^{*}$ and u be a normalized function on a non-compact regular region Ω with respect to Λ_h. Then the equation*

$$(\omega, du^{*})_{\Omega}=\int_{\partial\Omega}u\,\omega$$

holds.

COROLLARY 1. 1. (Cf. Corollary 2. 3 of [10].) *Let $\omega\in\Lambda_h^{*}$ and σ_C be the differential of the generalized harmonic measure with respect to Λ_h defined in* **7**. *Then the equation*

$$(\omega, \sigma_C^{*})_R=\int_C\omega$$

holds.

COROLLARY 1. 2. (Cf. Corollary 2. 4 of [10].) *Let $\omega\in\Lambda_h^{*}$, and σ_{A_j} and σ_{B_j} be the elementary differentials with respect to Λ_h defined in* **7**. *Then the equations*

$$(\omega, \sigma_{A_j}^{*})_R=\int_{A_j}\omega, \qquad (\omega, \sigma_{B_j}^{*})_R=\int_{B_j}\omega$$

hold.

COROLLARY 1. 3. *Let $\omega\in\Gamma_{h0}^{*}\cap\Gamma_{hse}$, and τ_{A_j} and τ_{B_j} be the elementary differentials with respect to $\Gamma_{h0}\cap\Gamma_{hse}^{*}$ defined in* **7**. *Then the equations*

$$(\omega, \tau_{A_j}^{*})_R=\int_{A_j}\omega, \qquad (\omega, \tau_{B_j}^{*})_R=\int_{B_j}\omega$$

hold.

9. Theorem. Let $\{R_n\}_{n=1}^{\infty}$ be a canonical exhaustion of R. Let $\{A_j, B_j\}_{j=1}^{g}$ $(g\leqq\infty)$ be a canonical homology basis modulo $\mathfrak{J}$ associated to $\{R_n\}$. Let $\{C_j\}_{j=1}^{N}$ $(N\leqq\infty)$ be a canonical homology basis of dividing cycles modulo β associated to $\{R_n\}$. Let σ_{A_j} and σ_{B_j} $(j=1, \cdots, g)$ be the elementary differentials with respect to Λ_h and σ_{C_j} $(j=1, \cdots, N)$ be the differentials of the generalized harmonic measures with respect to Λ_h associated to C_j respectively. Let τ_{A_j} and τ_{B_j} $(j=1, \cdots, g)$ be the elementary differentials with respect to $\Gamma_{h0}\cap\Gamma_{hse}^{*}$. By making use of Corollaries 1. 1, 1. 2 and 1. 3 and the orthogonal decompositions (1. 4), (1. 5) and (1. 1) we can prove the following theorem (cf. Ch. V, Theorem 20 C of [3] for the proof and also [6]).

THEOREM 1. 1. (i) $\{\sigma_{A_j}, \sigma_{B_j}, \sigma_{C_k}\}$ $(j=1, \cdots, g;\ k=1, \cdots, N)$ *spans* Λ_{h0}; (ii) $\{\sigma_{C_j}\}$ $(j=1, \cdots, N)$ *spans* Λ_{hm}; (iii) $\{\tau_{A_j}, \tau_{B_j}\}$ $(j=1, \cdots, g)$ *spans* $\Gamma_{h0}\cap\Gamma_{hse}^{*}$.

10. Definition of relative extremal length. The various extensions of the concept of extremal length have already been introduced and investigated by Jenkins [5], Fuglede [4] and Ohtsuka [13]. In the present paper we shall make effective use of the relative extremal length defined in [10].

Let γ be a countable collection of locally rectifiable curves in R (cf. [13]) and $\mathfrak{C}$ be a family of such collections γ. For simplicity the elements of $\mathfrak{C}$ are called *curves*. Let ρ be a linear density of R (cf. [3]) and $\chi=\chi(\gamma)$ be a non-negative real valued function on $\mathfrak{C}$ such that for each $\gamma\in\mathfrak{C}$ a non-negative real valued $\chi(\gamma)$ is assigned. Let us define

$$L(\rho, \gamma, \chi(\gamma))=\begin{cases}\dfrac{1}{\chi(\gamma)}\displaystyle\int_{\gamma}\rho|dz| & \text{if } \chi(\gamma)\neq 0,\\ \infty & \text{if } \chi(\gamma)=0,\end{cases}$$

$$L(\rho, \mathfrak{C}, \chi)=\inf_{\gamma\in\mathfrak{C}} L(\rho, \gamma, \chi(\gamma))$$

and

$$A(\rho, R)=\iint_{R}\rho^2 dx\, dy,$$

where $z=x+iy$ denotes a local parameter. If there exists a non-void class P of ρ such that $L(\rho, \mathfrak{C}, \chi)$ and $A(\rho, R)$ are not simultaneously 0 or ∞, then we define the *extremal length of* $\mathfrak{C}$ *relative to* χ *or the relative extremal length of* $\mathfrak{C}$ *by*

$$\lambda(\mathfrak{C}, \chi)=\lambda_R(\mathfrak{C}, \chi)=\sup_{\rho\in P}\frac{L^2(\rho, \mathfrak{C}, \chi)}{A(\rho, R)}.$$

It is permitted that $\lambda(\mathfrak{C}, \chi)=0$ or ∞. This definition is included in Fuglede's more general one (cf. [4] and also p. 92 of [13]). The following *L-normalization* of the relative extremal length will be useful: Let P_L be the subclass of P such that $\int_\gamma \rho|dz| \geqq \chi(\gamma)$ for any $\gamma \in \mathfrak{C}$. If $P_L \neq \phi$, then $\lambda(\mathfrak{C}, \chi) = 1/\inf_{\rho \in P_L} A(\rho, R)$. If $P \neq \phi$ and $P_L = \phi$, then $\lambda(\mathfrak{C}, \chi) = 0$.

11. Definitions and elementary properties. Now we shall define the terminologies related to the relative extremal length and enumerate its elementary properties. The proofs are omitted for those which are similar to the case of the ordinary extremal length (cf. [13]).

LEMMA 1. 3. *$\lambda(\mathfrak{C}, \chi)$ is conformally invariant.*

LEMMA 1. 4. *$\lambda(\mathfrak{C}, \chi)$ depends on R only through the requirement that the curves of $\mathfrak{C}$ are contained in R. Thus, if every $\gamma \in \mathfrak{C}$ is contained in $R'(\subset R)$, then $\lambda_R(\mathfrak{C}, \chi) = \lambda_{R'}(\mathfrak{C}, \chi)$.*

DEFINITION 1. 1. Assume that $\lambda(\mathfrak{C}, \chi) \neq 0$. If the relative extremal length problem is taken in the L-normalization, any $\rho \in P_L$ is called an *admissible metric.* If there exists $\rho_0 \in P_L$ for which $1/\lambda(\mathfrak{C}, \chi) = A(\rho_0, R)$, ρ_0 is called an *extremal metric.*

LEMMA 1. 5. *If ρ_1 and ρ_2 are extremal metrics for a common relative extremal length problem, then*

$$\rho_1(z) = \rho_2(z)$$

at most up to a set of measure zero on R.

LEMMA 1. 6. *If $\mathfrak{C}' \subset \mathfrak{C}$, then $\lambda(\mathfrak{C}', \chi) \geqq \lambda(\mathfrak{C}, \chi)$.*

LEMMA 1. 7. *If $\chi(\gamma) \geqq \chi_1(\gamma)$ for any $\gamma \in \mathfrak{C}$, then $\lambda(\mathfrak{C}, \chi) \leqq \lambda(\mathfrak{C}, \chi_1)$.*

LEMMA 1. 8. *Let Ω_n $(n=1, \cdots, \nu;\ \nu \leqq \infty)$ be disjoint open sets in R, $\mathfrak{C}_n$ $(n=1, \cdots, \nu)$ be families of curves in Ω_n respectively, and $\mathfrak{C}$ be a family of curves in R. If every $\gamma_n \in \mathfrak{C}_n$ $(n=1, \cdots, \nu)$ contains a $\gamma \in \mathfrak{C}$ ($\gamma = \phi$ is permitted) and $\chi(\gamma_n) \leqq \chi(\gamma)$ ($\chi(\gamma)=0$ for $\gamma = \phi$), then*

$$\frac{1}{\lambda(\mathfrak{C}, \chi)} \geqq \sum_{n=1}^{\nu} \frac{1}{\lambda(\mathfrak{C}_n, \chi)}.$$

LEMMA 1. 9. *Let $\mathfrak{C}$ and $\mathfrak{C}_n$ $(n=1, \cdots, \nu;\ \nu \leqq \infty)$ be families of curves in R such that $\mathfrak{C} \subset \bigcup_{n=1}^{\nu} \mathfrak{C}_n$. Then*

$$\frac{1}{\lambda(\mathfrak{C}, \chi)} \leqq \sum_{n=1}^{\nu} \frac{1}{\lambda(\mathfrak{C}_n, \chi)}.$$

By Lemmas 1. 8 and 1. 9, we have the following corollaries.

COROLLARY 1. 4. *Let Ω_n $(n=1, \cdots, \nu;\ \nu \leqq \infty)$ be disjoint open sets in R and $\mathfrak{C}_n$ $(n=1, \cdots, \nu)$ be families of curves in Ω_n respectively. Then*

$$\frac{1}{\lambda(\bigcup_{n=1}^{\nu}\mathfrak{C}_n, \chi)}=\sum_{n=1}^{\nu}\frac{1}{\lambda(\mathfrak{C}_n, \chi)}.$$

COROLLARY 1. 5. *If* $\lambda(\mathfrak{C}', \chi)=\infty$, *then* $\lambda(\mathfrak{C}\cup\mathfrak{C}', \chi)=\lambda(\mathfrak{C}, \chi)$.

COROLLARY 1. 6. *Let* $\mathfrak{C}_0=\{\gamma \mid \chi(\gamma)=0,\ \gamma\in\mathfrak{C}\}$. *Then* $\lambda(\mathfrak{C}, \chi)=\lambda(\mathfrak{C}-\mathfrak{C}_0', \chi)$ *for any* $\mathfrak{C}_0'\subset\mathfrak{C}_0$.

By the definition, if $\chi(\gamma)\equiv 1$, then $\lambda(\mathfrak{C}, \chi)$ is the same as the ordinary extremal length $\lambda(\mathfrak{C})$; $\lambda(\mathfrak{C}, 1)\equiv\lambda(\mathfrak{C})$.

LEMMA 1. 10. *Let* $\chi_0=\inf_{\gamma\in\mathfrak{C}}\chi(\gamma)$ *and* $\chi_1=\sup_{\gamma\in\mathfrak{C}}\chi(\gamma)$. *Then the inequality*

$$\chi_0^2\,\lambda(\mathfrak{C}, \chi)\leqq\lambda(\mathfrak{C})\leqq\chi_1^2\,\lambda(\mathfrak{C}, \chi)$$

holds provided each side has meaning.

LEMMA 1. 11. *If* $\chi(\gamma)\neq 0$ *for every* $\gamma\in\mathfrak{C}$, *then* $\lambda(\mathfrak{C}, \chi)$ *and* $\lambda(\mathfrak{C})$ *are simultaneously finite or infinite.*

Proof. Let $\mathfrak{C}_n=\{\gamma \mid 1/n\leqq\chi(\gamma)\leqq n, \gamma\in\mathfrak{C}\}$. Then $\mathfrak{C}=\bigcup_{n=1}^{\infty}\mathfrak{C}_n$ and by Lemma 1. 10

$$\frac{1}{n^2}\lambda(\mathfrak{C}_n, \chi)\leqq\lambda(\mathfrak{C}_n)\leqq n^2\,\lambda(\mathfrak{C}_n, \chi).$$

If $\lambda(\mathfrak{C}, \chi)=\infty$, then by Lemma 1. 6 $\lambda(\mathfrak{C}_n, \chi)=\infty$ and thus by the above inequality $\lambda(\mathfrak{C}_n)=\infty$ for all n. Hence by Lemma 1. 9 for $\chi\equiv 1$ we find that $\lambda(\mathfrak{C})=\infty$. The converse assertion is also similarly proved.

§ 2. Existence of $\omega\in\Lambda_{h0}^*$ with given periods.

1. Statement of relative extremal length problem. Let R be an open Riemann surface. Let α and β be the partition of the ideal boundary $\mathfrak{J}$ of R defined in § 1. 2. Let $\{R_n\}_{n=1}^{\infty}$ be a canonical exhaustion of R. Let $\{A_j, B_j\}_{j=1}^{g}$ ($g\leqq\infty$) be a canonical homology basis modulo $\mathfrak{J}$ associated to $\{R_n\}$. Let $\{C_j\}_{j=1}^{N}$ ($N\leqq\infty$) be a canonical homology basis of dividing cycles modulo β associated to $\{R_n\}$.

Let $\mathfrak{C}$ be a class of curves defined by

$$\mathfrak{C}=\{\gamma \mid \gamma \text{ is a Jordan curve in } R \text{ such that } \gamma\not\approx 0 \pmod{\beta}\}.$$

Let a_j, b_j ($j=1, \cdots, g$) and c_j ($j=1, \cdots, N$) be a system of real numbers. We define a function $\chi=\chi(\gamma)$ on $\mathfrak{C}$ by

$$\chi(\gamma)=\left|\sum_j(\gamma\times B_j)a_j+\sum_j(A_j\times\gamma)b_j+\sum_j(\gamma\times C_j^*)c_j\right|, \tag{2. 1}$$

where C_j^* denotes the conjugate relative cycle of C_j respectively. Our problem is to compute the relative extremal length $\lambda(\mathfrak{C}, \chi)$.

2. Existence of $\omega\in\Lambda_{h0}^*$ for R with finite character. For simplicity, by a *differential $\omega\in\Lambda_{h0}^*$ with periods a_j, b_j, c_j* we shall call the differential $\omega\in\Lambda_{h0}^*$ which satisfies the period condition:

$$(2.2)\qquad \begin{aligned} &\int_{A_j}\omega=a_j, \qquad \int_{B_j}\omega=b_j \qquad (j=1,\cdots,g),\\ &\int_{C_j}\omega=c_j \qquad (j=1,\cdots,N). \end{aligned}$$

We note that by Corollary 1.1 if there exists a differential $\omega\in\Lambda_{h0}^*$ with periods a_j, b_j, c_j, then $c_k\neq 0$ for some k implies that the differential of the generalized harmonic measure $\sigma_{C_k}\neq 0$.

Now we shall prove that if $g<\infty$, $N<\infty$ and furthermore the differentials of the generalized harmonic measures $\sigma_{C_j}\neq 0$ $(j=1,\cdots,N)$, then there exists always one and only one differential $\omega\in\Lambda_{h0}^*$ with periods a_j, b_j, c_j for an arbitrarily given system of real numbers a_j, b_j, c_j.

We shall use the simplified notations:

$$\begin{aligned} &L_{2j-1}=A_j, \qquad L_{2j}=B_j, \qquad L_{2g+k}=C_k,\\ &\sigma_{2j-1}=\sigma_{A_j}, \qquad \sigma_{2j}=\sigma_{B_j}, \qquad \sigma_{2g+k}=\sigma_{C_k},\\ &q_{2j-1}=a_j, \qquad q_{2j}=b_j, \qquad q_{2g+k}=c_k \qquad (j=1,\cdots,g;\, k=1,\cdots,N);\\ &p_{jk}=\int_{L_j}\sigma_k^* \qquad (j,k=1,\cdots,2g+N). \end{aligned}$$

Here σ_{A_j} and σ_{B_j} are the elementary differentials with respect to Λ_h defined in § 1. 7.

LEMMA 2. 1. *$p_{jk}=p_{kj}$, $\sum_{j,k} x_j x_k p_{jk}>0$ for any real numbers x_j $(j=1,\cdots,2g+N)$ not simultaneously zero and thus* det. $|p_{jk}|>0$.

The present lemma is proved by a standard method making use of Corollaries 1. 1 and 1. 2.

Since by Theorem 1. 1 the space Λ_{h0}^* is spanned by $\sigma_{A_j}^*$, $\sigma_{B_j}^*$ $(j=1,\cdots,g)$ and $\sigma_{C_j}^*$ $(j=1,\cdots,N)$, and $g<\infty$ and $N<\infty$, we find that

$$\omega=\sum_{j=1}^{2g+N} x_j\sigma_j^*$$

with some real coefficients x_j $(j=1,\cdots,2g+N)$. Then by the condition (2. 2) we obtain the system of equations:

$$(2.3)\qquad \sum_{k=1}^{2g+N} p_{jk}x_k=q_j \qquad (j=1,\cdots,2g+N).$$

By Lemma 2. 1 we know that (2. 3) has a unique solution $(x_1,\cdots,x_{2g+N})$, which assures the existence and the uniqueness of the desired ω.

3. The case of compact bordered surfaces. Let R_0 be a compact bordered Riemann surface. Let $\{A_j, B_j\}_{j=1}^{g}$ be a canonical homology basis of R_0 modulo ∂R_0. Partition the collection of the boundary components of R_0 into two disjoint sets α and β. Let $C_0, \cdots, C_N$ denote the elements of α. Then $\{C_j\}_{j=1}^{N}$ forms a canonical homology basis of dividing cycles modulo β.

By the consequence of **2**, there always exists the unique differential $\omega \in \Lambda_{h0}^{*}$ with periods a_j, b_j, c_j, where Λ_{h0}^{*} is one for the present R_0.

We should note that the present ω can be also defined as the harmonic differential on R_0 which satisfies the conditions:

(i) $\omega^*=0$ along ∂R_0;

(ii) $$\int_{A_j} \omega = a_j, \qquad \int_{B_j} \omega = b_j \qquad (j=1, \cdots, g);$$

(iii) $$\int_{C_j} \omega = c_j \qquad (j=1, \cdots, N),$$

$$\int_{C} \omega = 0 \qquad \text{for each } C \in \beta.$$

Let $\mathfrak{C}$ and $\chi(\gamma)$ be the class of curves and the function respectively defined in **1** for the present R_0.

LEMMA 2. 2. $\lambda(\mathfrak{C}, \chi) = \|\omega\|_{R_0}^{-2}$.

Proof. If $\omega \equiv 0$, then the equality holds in the meaning: $\lambda(\mathfrak{C}, \chi) = \|\omega\|_{R_0}^{-2} = \infty$.

Assume that $\omega \not\equiv 0$. Since ω^* has only a finite number of critical points on $\bar{R}_0$, each level curve l of ω^* in R_0 is simple closed up to a finite number of components through such critical points. Let $\mathfrak{C}_l$ be the collection of such Jordan curves l. Obviously $\mathfrak{C}_l \subset \mathfrak{C}$, $\overline{\bigcup_{l \in \mathfrak{C}_l} l} = R_0$ and $l \cap l' = \phi$ for distinct $l, l' \in \mathfrak{C}_l$. We shall orient each level curve $l \in \mathfrak{C}_l$ so that ω is positive along it.

Except at a finite number of the critical points of ω, an integral of $\omega + i\omega^*$ can be used as a local parameter z. Let ρ be an admissible metric. Then by the Schwarz inequality

$$\left(\int_l \rho |dz|\right)^2 = \left(\int_l \rho\omega\right)^2 \leqq \int_l \rho^2 \omega \int_l \omega \qquad (l \in \mathfrak{C}_l).$$

Here we note that

$$\int_l \rho |dz| \geqq \chi(l), \qquad \int_l \omega = \chi(l).$$

Then

$$\int_l \omega \leqq \int_l \rho^2 \omega.$$

Integrating in ω^* for all variation over $\mathfrak{C}_l$, we obtain

$$||\omega||_{R_0}^2 \leqq A(\rho, R_0),$$

and thus $\lambda(\mathfrak{C}, \chi) \leqq ||\omega||_{R_0}^{-2}$. On the other hand, since ρ_0 defined by $\rho_0|dz| = |\omega + i\omega^*|$ is an admissible metric, we obtain

$$||\omega||_{R_0}^2 = A(\rho_0, R_0) \geqq \frac{1}{\lambda(\mathfrak{C}, \chi)}.$$

4. Construction of $\omega \in \Lambda_{h0}^*$ by exhaustion. Let R be a generic open Riemann surface. In the present number we shall assume that there exists the differential $\omega \in \Lambda_{h0}^*$ with given periods a_j, b_j, c_j, where g or N is not necessarily finite. By the orthogonal decomposition (1. 4) the differential ω is uniquely determined.

We shall preserve the notations in § 1. 1 and § 1. 2. Let $\Lambda_{h0}^{*n} = \Lambda_{h0}^{*n}(\alpha_n, \beta_n)$ $(n=1, 2, \cdots)$ denote the space Λ_{h0}^* defined for R_n respectively in place of R. Let ω_n $(n=1, 2, \cdots)$ be the differential of Λ_{h0}^{*n} respectively with periods a_j, b_j $(j=1, \cdots, \kappa_n)$, c_j $(j=1, \cdots, \iota_n)$. Now we shall prove that the sequence $\{\omega_n\}_{n=1}^\infty$ strongly converges to the differential ω.

Since $\omega_n - \omega_m$ for $n \geqq m$ is exact on R_m, by the orthogonal decomposition (1. 1) we find that

$$(\omega_m, \omega_n - \omega_m)_{R_m} = 0 \qquad (n \geqq m).$$

Hence

$$||\omega_n - \omega_m||_{R_m}^2 \leqq ||\omega_n||_{R_n}^2 - ||\omega_m||_{R_m}^2 \qquad (n \geqq m),$$

which implies that $||\omega_n||_{R_n}^2$ is monotone increasing with n. Similarly we see that

$$(\omega_n, \omega - \omega_n)_{R_n} = 0$$

and thus

$$||\omega - \omega_n||_{R_n}^2 \leqq ||\omega||_R^2 - ||\omega_n||_{R_n}^2.$$

Consequently, there exists the finite limit $\lim_{n\to\infty} ||\omega_n||_{R_n}^2 \leqq ||\omega||_R^2 < \infty$ and the sequence $\{\omega_n\}$ strongly converges to a differential $\omega' \in \Lambda_{h0}^*$. Obviously the differential ω' satisfies the period condition (2. 2). Hence by the uniqueness we have $\omega' \equiv \omega$. It is also easily shown that $\lim_{n\to\infty} ||\omega_n||_{R_n}^2 = ||\omega||_R^2$.

5. Theorem. Let R be a generic open Riemann surface. Let a_j, b_j $(j=1, \cdots, g;$ $g \leqq \infty)$ and c_j $(j=1, \cdots, N; N \leqq \infty)$ be an arbitrarily given system of real numbers. Then we have the theorem.

THEOREM 2. 1. *There exists the unique differential $\omega \in \Lambda_{h0}^*$ which satisfies the period condition*:

$$\int_{A_j}\omega=a_j,\qquad \int_{B_j}\omega=b_j\qquad (j=1,\cdots,g),$$

$$\int_{C_j}\omega=c_j\qquad (j=1,\cdots,N),$$

if and only if $\lambda(\mathfrak{C},\chi)>0$ *for the relative extremal length* $\lambda(\mathfrak{C},\chi)$ *defined in* **1**.

Further provided any of these conditions is satisfied, the equality

$$\lambda(\mathfrak{C},\chi)=\|\omega\|_R^{-2}$$

holds. Here $\lambda(\mathfrak{C},\chi)=\infty$ *occurs if and only if* $a_j=b_j=c_j=0$ *for all* j *and thus* $\omega\equiv 0$.

The first half of the present theorem is essential only if at least one of g and N is infinity.

Proof. Necessity: If $\omega\equiv 0$, then $a_j=b_j=c_j=0$ and thus $\lambda(\mathfrak{C},\chi)=\|\omega\|_R^{-2}=\infty$.

Assume that $\omega\not\equiv 0$. Let $\{\omega_n\}_{n=1}^{\infty}$ be the sequence of differentials defined in **4**. Let $\mathfrak{C}_n$ $(n=1,2,\cdots)$ be the class of curves $\mathfrak{C}$ in **1** defined for R_n respectively in place of R. Then $\mathfrak{C}_n\subset\mathfrak{C}$ $(n=1,2,\cdots)$ and thus by Lemma 1. 6 $\lambda(\mathfrak{C}_n,\chi)\geqq\lambda(\mathfrak{C},\chi)$. Further by Lemma 2. 2

$$\lambda(\mathfrak{C}_n,\chi)=\|\omega_n\|_{R_n}^{-2}.$$

Since $\lim_{n\to\infty}\|\omega_n\|_{R_n}^2=\|\omega\|_R^2$, we find that

$$\|\omega\|_R^{-2}\geqq\lambda(\mathfrak{C},\chi).$$

On the other hand we see that

$$\left|\int_\gamma\omega\right|=\chi(\gamma)\qquad\text{for each }\gamma\in\mathfrak{C}$$

and thus ρ_0 defined by $\rho_0|dz|=|\omega+i\omega^*|$ is an admissible metric for $\lambda(\mathfrak{C},\chi)$. Hence

$$\|\omega\|_R^2=A(\rho_0,R)\geqq\frac{1}{\lambda(\mathfrak{C},\chi)}.$$

Consequently,

$$0<\lambda(\mathfrak{C},\chi)=\|\omega\|_R^{-2}<\infty.$$

Sufficiency: If $\lambda(\mathfrak{C},\chi)=\infty$, then $\lambda(\mathfrak{C}_n,\chi)=\infty$ for all n, thus by Lemma 2. 2 $a_j=b_j=c_j=0$ for all j and thus $\omega\equiv 0$.

Assume that $0<\lambda(\mathfrak{C},\chi)<\infty$. Then

$$\|\omega_n\|_{R_n}^2=1/\lambda(\mathfrak{C}_n,\chi)\leqq 1/\lambda(\mathfrak{C},\chi)<\infty\qquad\text{for all }n.$$

Thus by the consequence of **4** the finite limit

$$A=\lim_{n\to\infty}\|\omega_n\|_{R_n}^2 \qquad (0<A\leqq 1/\lambda(\mathfrak{C},\chi)<\infty)$$

exists and $\{\omega_n\}$ strongly converges to a differential $\omega\in\Lambda_{h0}^*$ such that $\|\omega\|_R^2=A$. Obviously the differential ω satisfies the period condition.

6. Existence of $\omega\in\Gamma_{h0}^*$ with given periods. Let R be an open Riemann surface, $\mathfrak{J}$ be the ideal boundary of R and $\{R_n\}_{n=1}^\infty$ be a canonical exhaustion of R. Let $\{A_j, B_j\}_{j=1}^g$ $(g\leqq\infty)$ be a canonical homology basis modulo $\mathfrak{J}$ associated to $\{R_n\}$ and $\{C_j\}_{j=1}^N$ $(N\leqq\infty)$ be a canonical homology basis of dividing cycles associated to $\{R_n\}$. Let a_j, b_j $(j=1,\cdots,g)$ and c_j $(j=1,\cdots,N)$ be an arbitrarily given system of real numbers. Let $\mathfrak{C}$ be a class of curves defined by

$$\mathfrak{C}=\{\gamma\mid\gamma \text{ is a Jordan curve in } R \text{ such that } \gamma\not\sim 0\}$$

and $\chi=\chi(\gamma)$ be the function on $\mathfrak{C}$ defined by (2. 1).

The following theorem is Theorem 2. 1 of the case of $\beta=\phi$.

THEOREM 2. 2. *There exists the unique differential $\omega\in\Gamma_{h0}^*$ which satisfies the condition*:

$$\int_{A_j}\omega=a_j, \qquad \int_{B_j}\omega=b_j \qquad (j=1,\cdots,g),$$

$$\int_{C_j}\omega=c_j \qquad (j=1,\cdots,N),$$

if and only if $\lambda(\mathfrak{C},\chi)>0$.

Further provided any of these conditions is satisfied, the equality

$$\lambda(\mathfrak{C},\chi)=\|\omega\|_R^{-2}$$

holds. Here $\lambda(\mathfrak{C},\chi)=\infty$ occurs if and only if $a_j=b_j=c_j=0$ for all j and thus $\omega\equiv 0$.

7. Existence of $\omega\in\Gamma_{h0}^*\cap\Gamma_{hse}$ with given periods. Let a_j, b_j $(j=1,\cdots,g)$ be an arbitrarily given system of real numbers. Let $\mathfrak{C}$ be the class of curves defined by

$$\mathfrak{C}=\{\gamma\mid\gamma \text{ is a Jordan curve in } R \text{ such that } \gamma\not\sim 0 \pmod{\mathfrak{J}}\}$$

and $\chi=\chi(\tau)$ be the function on $\mathfrak{C}$ defined by

$$\chi(\gamma)=\left|\sum_j(\gamma\times B_j)a_j+\sum_j(A_j\times\gamma)b_j\right|.$$

The following theorem is Theorem 2. 1 of the case of $\beta=\mathfrak{J}$.

THEOREM 2. 3. *There exists the unique differential $\omega\in\Gamma_{h0}^*\cap\Gamma_{hse}$ which satisfies the condition*:

$$\int_{A_j} \omega = a_j, \qquad \int_{B_j} \omega = b_j \qquad (j=1, \cdots, g)$$

if and only if $\lambda(\mathfrak{C}, \chi)>0$.

Further provided any of these conditions is satisfied, the equality

$$\lambda(\mathfrak{C}, \chi)=\|\omega\|_R^{-2}$$

holds. Here $\lambda(\mathfrak{C}, \chi)=\infty$ *occurs if and only if* $a_j=b_j=0$ *for all* j *and thus* $\omega\equiv 0$.

§3. Existence of $\omega\in\Lambda_{hm}^*$ with given periods.

1. Statement of relative extremal length problem. Let R be an open Riemann surface. Let α and β be the partition of the ideal boundary $\mathfrak{J}$ of R defined in §1. 2. Let $\{R_n\}_{n=1}^{\infty}$ be a canonical exhaustion of R and $\{C_j\}_{j=1}^{N}$ $(N\leqq\infty)$ be a canonical homology basis of dividing cycles modulo β associated to $\{R_n\}$ defined in §1. 2.

Let $\mathfrak{C}$ be the class of curves in R defined by

$\mathfrak{C}=\{\gamma \mid \gamma$ is a dividing cycle consisting of a finite collection of Jordan curves which uniquely determines a partition of R into two disjoint sets R_γ and $\tilde{R}_\gamma$ with $\gamma=\partial R_\gamma=-\partial\tilde{R}_\gamma$ and is such that $\gamma\not\sim 0 \pmod{\beta}\}$.

Let c_j $(j=1, \cdots, N)$ be an arbitrarily given system of real numbers. Let $\chi=\chi(\gamma)$ be the function on $\mathfrak{C}$ defined by

$$\chi(\gamma)=\left|\sum_j (\gamma\times C_j^*)c_j\right|,$$

where C_j^* denotes the conjugate relative cycle of C_j respectively defined in §1. 3.

Our present problem is to compute the relative extremal length $\lambda(\mathfrak{C}, \chi)$.

2. Existence of $\omega\in\Lambda_{hm}^*$ for R with finite character. By a *differential* $\omega\in\Lambda_{hm}^*$ *with periods* c_j we call the differential $\omega\in\Lambda_{hm}^*$ which satisfies the period condition:

$$\int_{C_j} \omega = c_j \qquad (j=1, \cdots, N).$$

If there exists the differential $\omega\in\Lambda_{hm}^*$ with periods c_j, then by Corollary 1. 1 $c_k\neq 0$ for some k implies that the differential of the generalized harmonic measure $\sigma_{C_k}\not\equiv 0$ and by the orthogonal decomposition (1. 5) the differential ω is uniquely determined.

By the method similar to §2. 2 it is shown that if $N<\infty$ and $\sigma_{C_j}\not\equiv 0$ $(j=1, \cdots, N)$ then there always exists one and only one differential $\omega\in\Lambda_{hm}^*$ with given periods c_j $(j=1, \cdots, N)$.

3. The case of compact bordered Riemann surfaces. Let R_0 be a compact bordered Riemann surface. Partition the set of components of ∂R_0 into two disjoint sets α and β. Let C_j $(j=0, \cdots, N)$ be all elements of α. Then $\{C_j\}_{j=1}^{N}$ can be taken as a canonical homology basis of dividing cycles modulo β. Let ω be the harmonic differential on R_0 uniquely determined by the conditions:

(i) ω^* is exact and $\omega^*=0$ along ∂R_0;

(ii)
$$\int_{C_j}\omega=c_j \qquad (j=1, \cdots, N),$$
$$\int_{C}\omega=0 \qquad \text{for each } C\in\beta.$$

Let $\mathfrak{C}$ and χ be the class of curves and the function respectively defined in **1** for R_0 in place of R.

The following lemma is proved by the method similar to Lemma 2. 2.

LEMMA 3. 1. $$\lambda(\mathfrak{C}, \chi)=\|\omega\|_{R_0}^{-2}.$$

4. Theorem. Let R be a generic open Riemann surface. Let c_j $(j=1, \cdots, N;$ $N\leqq\infty)$ be an arbitrarily given system of real numbers. Then we have the theorem.

THEOREM 3. 1. *There exists the unique differential* $\omega\in\Lambda_{hm}^*$ *which satisfies the condition*:

$$\int_{C_j}\omega=c_j \qquad (j=1, \cdots, N)$$

if and only if $\lambda(\mathfrak{C}, \chi)>0$ *for the relative extremal length* $\lambda(\mathfrak{C}, \chi)$ *defined in* **1**.

Further provided any of these conditions is satisfied, then equality

$$\lambda(\mathfrak{C}, \chi)=\|\omega\|_R^{-2}$$

holds. Here $\lambda(\mathfrak{C}, \chi)=\infty$ *occurs if and only if* $c_j=0$ *for all* j *and thus* $\omega\equiv 0$.

The proof which is omitted is similar to the method of Theorem 2. 1 making use of Lemma 3. 1.

If $\beta=\phi$, then the present theorem shows the consequence with respect to the existence of the differential $\omega\in\Gamma_{hm}^*$ with given periods c_j $(j=1, \cdots, N)$.

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SCHOOL OF ENGINEERING,
OKAYAMA UNIVERSITY.