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ON THE BIEBERBACH CONJECTURE FOR THE SIXTH COEFFICIENT

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§ 0. **Introduction.** Let $f(z)$ be a normalized regular function univalent in the unit circle $|z|<1$

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n.$$

In 1916 Bieberbach [1] proved $|a_2|\leqq 2$ and stated his famous conjecture $|a_n|\leqq n$, with the equals sign holding only for the Koebe function $z/(1-z)^2$ and its rotations. Later Löwner [10] established the deeper inequality $|a_3|\leqq 3$ by his parametric method, which was also proved by various authors. Garabedian and Schiffer [3] showed that $|a_4|\leqq 4$ in a lengthy paper and Charzynski and Schiffer [2] found its elementray proof. Several authors claimed recently that they proved the local maximality for general a_n or a special a_n at the Koebe function. In our earlier papers [8], [12], [13] we proved the local maximality for the coefficient a_6 at the Koebe function. Our method in these papers suggests certain possibility of a further global study. In [11] we proved that if a_2 is real non-negative then $\Re a_6\leqq 6$.

In this paper we shall prove the global maximality for the a_6 at the Koebe function, that is, $\Re a_6\leqq 6$ for $|\arg a_2|\leqq\pi/5$, with equality holding only for the Koebe function. Consequently we have the decisive answer to the Bieberbach conjecture for the sixth coefficient.

THEOREM. $$|a_6|\leqq 6.$$

Equality occurs only for the function $z/(1-e^{i\theta}z)^2$, θ real.

Our proof of this theorem is on the non-elementary level, since we make use of a result due to Jenkins [6] (Lemma 4 in this paper). However we believe there would be an elementary proof. This is supported by various phenomena. It should be remarked that the essential part in this paper has been proved within the elementary level.

Our method in this paper would suggest some further possibility for $|a_8|\leqq 8$. To this end we should prepare a certain simpler proof of local maximality for a_8 than in [9], which would not be so difficult, and several useful lemmas, which give some effective estimations of several combinations of the coefficients of lower in-

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dices. However the true difficulty for $|a_8|\leqq 8$ stems from a tremendous amount of calculation we must perform.

Section 1 is devoted to several preparatory lemmas and general inequalities, with which we start out. Section 2 to 6 are concerned with the case $1\leqq\Re a_2\leqq 2$, $|\Im a_2/\Re a_2|^2\leqq 1/10$. The main part in this paper consists of two sections 4 and 5. Section 7 is concerned with the case $1\leqq\Re a_2$, $1/10\leqq|\Im a_2/\Re a_2|^2\leqq 5-2\sqrt{5}$ and Section 8 is devoted to the case $0\leqq\Re a_2\leqq 1$, $|\Im a_2/\Re a_2|^2\leqq 5-2\sqrt{5}$. Sections 2, 3, 6, 7 and 8 are rather trivial parts in principle.

§1. This section is devoted to a general consideration and to establish several lemmas, which will be used later on.

Let $G_\mu(w)$ be the μ^{th} Faber polynomial which is defined by

$$G_\mu(g(z))=z^\mu+\sum_{\nu=1}^{\infty}\frac{b_{\mu\nu}}{z^\nu},\qquad g(z)=f(1/z^2)^{-1/2}.$$

Then it is known that $\nu b_{\mu\nu}=\mu b_{\nu\mu}$. With these notations Golusin's inequality [4] has the form

$$\sum_{\nu=1}^{\infty}\nu\left|\sum_{\mu=1}^{m}x_\mu b_{\mu\nu}\right|^2\leqq\sum_{\nu=1}^{m}\nu|x_\nu|^2$$

and Grunsky's inequality [5] has the form

$$\left|\sum_{\mu,\nu=1}^{m}\nu b_{\mu\nu}x_\mu x_\nu\right|\leqq\sum_{\nu=1}^{m}\nu|x_\nu|^2.$$

Jenkins [7] pointed out that Grunsky's inequality is a direct consequence of Golusin's. Our principal leading idea in our previous papers [8], [9], [12] and in this paper comes from this Jenkins' remark. By a simple calculation we have

$$\begin{aligned}
2b_{11}&=-a_2,\\
2b_{13}&=-(a_3-3a_2^2/4),\\
2b_{15}&=-(a_4-3a_2a_3/2+5a_2^3/8),\\
2b_{17}&=-(a_5-3a_2a_4/2-3a_3^2/4+15a_3a_2^2/8-35a_2^4/64),\\
2b_{33}&=-3(a_4-2a_2a_3+13a_2^3/12),\\
2b_{35}&=-3(a_5-2a_2a_4-5a_3^2/4+29a_3a_2^2/8-85a_2^4/64),\\
2b_{55}&=-5(a_6-2a_2a_5-3a_3a_4+4a_2^2a_4+21a_2a_3^2/4-59a_3a_2^3/8+689a_2^5/320).
\end{aligned}$$

From now on we shall use the following notations:

$$\begin{aligned}
p+ix'=2-x+ix'&=a_2,\\
y+iy'&=-2b_{13},\\
\eta+i\eta'&=-2b_{15},\\
\xi+i\xi'&=-2b_{17},\\
k&=x'/p.
\end{aligned}$$

And it is evident that $0\leqq p\leqq 2$ and $k^2\leqq 5-2\sqrt{5}$ when $|\arg a_2|\leqq\pi/5$.

Now we shall give here several lemmas.

LEMMA 1.

$$7(\xi^2+\xi'^2)+5(\eta^2+\eta'^2)+3(y^2+y'^2)+x'^2\leqq 4-p^2=4x-x^2.$$

Proof. This is a simple consequence of the area theorem for $f(1/z^2)^{-1/2}$.

LEMMA 2.

$$\eta+\left(2\beta-\frac{p}{2}\right)y$$
$$\leqq(2-p)\beta^2+\frac{2-p}{12}(4+2p+p^2)-\frac{1}{2}x'y'+\frac{1}{4}px'^2.$$

Proof. By the Grunsky inequality

$$|b_{11}x_1^2+6b_{13}x_1x_3+3b_{33}x_3^2|\leqq|x_1|^2+3|x_3|^2.$$

Here we take $x_3=1/3$ and put $x_1=\beta$. Taking the real part

$$\eta-\frac{p}{2}y+\frac{1}{2}x'y'+\frac{1}{12}p^3-\frac{p}{4}x'^2+2\beta y+\beta^2p$$
$$\leqq\frac{2}{3}+2\beta^2,$$

which is just the desired result.

We need a better inequality than Lemma 2.

LEMMA 3.

$$\frac{1}{2}p^3\left\{\eta+\left(\beta-\frac{p}{2}\right)y\right\}$$
$$\leqq\frac{4}{3}-\frac{1}{48}p^6+(4-p^2)\beta^2-2\beta py-y^2-(y'+\beta x')^2-\frac{p^3}{4}\left(x'y'-\frac{p}{2}x'^2\right)$$
$$-3\left\{\eta'+\left(\beta-\frac{p}{2}\right)y'+\left(\frac{p^2}{4}-\frac{x'^2}{12}\right)x'-\frac{1}{2}x'y\right\}^2$$
$$-5\left\{\xi'+\left(\beta-\frac{p}{2}\right)\eta'+\frac{1}{4}(p^2-x'^2)y'-\left(y'-\frac{p}{2}x'\right)y-\frac{1}{2}x'\eta\right\}^2.$$

Proof. By the Golusin inequality we have

$$5\left|a_5-2a_2a_4-\frac{5}{4}a_3^2+\frac{29}{8}a_3a_2^2-\frac{85}{64}a_2^4+\left(a_4-\frac{3}{2}a_2a_3+\frac{5}{8}a_2^3\right)\beta\right|^2$$
$$+3\left|a_4-2a_2a_3+\frac{13}{12}a_2^3+\left(a_3-\frac{3}{4}a_2^2\right)\beta\right|^2+\left|a_3-\frac{3}{4}a_2^2+a_2\beta\right|^2$$
$$\leqq\frac{4}{3}+4|\beta|^2.$$

Then we have

$$5\left\{\xi'-\frac{1}{2}(p\eta'+x'\eta)-yy'+\frac{1}{4}(p^2-x'^2)y'+\frac{p}{2}x'y+\beta\eta'\right\}^2$$

$$+3\left\{\eta-\frac{1}{2}py+\frac{1}{2}x'y'+\frac{1}{12}p^3-\frac{p}{4}x'^2+\beta y\right\}^2$$

$$+3\left\{\eta'-\frac{p}{2}y'-\frac{1}{2}x'y+\frac{p^2}{4}x'-\frac{1}{12}x'^3+\beta y'\right\}^2$$

$$+(y+\beta p)^2+(y'+\beta x')^2\leqq\frac{4}{3}+4\beta^2, \qquad \beta \text{ real}.$$

This implies the desired result.

LEMMA 4.

$$|y|<1.005 \quad \textit{and} \quad |y'|<1.005.$$

Proof. Jenkins [6] proved that

$$(y^2+y'^2)^{1/2}\leqq 1+2e^{-6}.$$

Since $1+2e^{-6}<1.005$, we have the desired result.

We are now in a position to explain a general consideration and to establish two inequalities, from which we start. By Grunsky's inequality with $m=5$, $x_2=x_4=0$, $x_3=\beta/6$, $x_1=\delta$ we have

$$\Big|a_6-2a_2a_5-3a_3a_4+4a_2^2a_4+\frac{21}{4}a_2a_3^2-\frac{59}{8}a_3a_2^3+\frac{689}{320}a_2^5$$

$$+\left(a_5-2a_2a_4-\frac{5}{4}a_3^2+\frac{29}{8}a_3a_2^2-\frac{85}{64}a_2^4\right)\beta+2\left(a_4-\frac{3}{2}a_2a_3+\frac{5}{8}a_2^3\right)\delta$$

$$+\frac{1}{4}\left(a_4-2a_2a_3+\frac{13}{12}a_2^3\right)\beta^2+\left(a_3-\frac{3}{4}a_2^2\right)\beta\delta+a_2\delta^2\Big|$$

$$\leqq\frac{2}{5}+\frac{|\beta|^2}{6}+2|\delta|^2.$$

Now we put $\beta=2p$ and $\delta=5p^2/8+By$. Then by taking the real part and by rearranging the terms we have

$$\begin{aligned}\Re a_6\leqq{}&\frac{2}{5}+\frac{2}{3}p^2-\frac{1}{12}p^5+\frac{7}{40}p^5+\frac{25}{64}p^4(2-p)\\ &-\left(\frac{3}{2}p^3-\frac{7}{8}px'^2\right)x'^2-\left(\frac{29}{8}p^2-\frac{11}{8}x'^2\right)x'y'-\frac{7}{4}py'^2-\frac{7}{2}px'\eta'\\ &-3y'\eta'-2x'\xi'-\frac{3}{2}x'y'y-\frac{5}{4}x'^2\eta-\frac{29}{8}px'^2y+\frac{1}{8}p^3y\end{aligned}\tag{A}$$

$$+\frac{5}{4}Bp^2(2-p)y+\left(\frac{7}{4}p-2Bp+2B^2-B^2p\right)y^2+(3-2B)y\eta.$$

Next we put $\beta=2p$ and $\delta=9p^2/16+By$. Then by taking the real part and by rearranging the terms we have

$$\begin{aligned}\Re a_6\leqq &\frac{2}{5}+\frac{2}{3}p^2-\frac{1}{12}p^5+\frac{7}{40}p^5+\frac{81}{256}p^4(2-p)\\ &-\left(\frac{3}{2}p^3-\frac{7}{8}px'^2\right)x'^2-\left(\frac{29}{8}p^2-\frac{11}{8}x'^2\right)x'y'-\frac{7}{4}py'^2-\frac{7}{2}px'\eta'-3y'\eta'\\ &-2x'\xi'-\frac{3}{2}x'y'y-\frac{5}{4}x'^2\eta-\frac{29}{8}px'^2y+\frac{9}{8}Bp^2(2-p)y\\ &+\left(\frac{7}{4}p-2Bp+2B^2-B^2p\right)y^2+(3-2B)y\eta+\frac{p^2}{8}(\eta+2py).\end{aligned}\tag{B}$$

§ **2.** In this section we shall be concerned with the case $y\leqq 0, p\geqq 1, k^2\leqq 1/10$. In this case we start from (A) with $B=3/2$. Then we have

$$\Re a_6\leqq R(p)-X-\frac{5}{4}x'^2\eta-\frac{19}{8}px'^2y+\frac{1}{2}(9-7p)y^2+\frac{1}{8}(p^3-10px'^2)y$$
$$+\frac{1}{8}(15p^2x-12x'y')y,$$

$$R(p)=\frac{2}{5}+\frac{2}{3}p^2-\frac{1}{12}p^5+\frac{7}{40}p^5+\frac{25}{64}p^4(2-p),$$

$$X=\left(\frac{3}{2}p^3-\frac{7}{8}px'^2\right)x'^2+\left(\frac{29}{8}p^2-\frac{11}{8}x'^2\right)x'y'+\frac{7}{4}py'^2+\frac{7}{2}px'\eta'$$
$$+3y'\eta'+2x'\xi'.$$

Since $k^2=x'^2/p^2\leqq 1/10$, $y\leqq 0$,

$$\frac{1}{8}(p^3-10px'^2)y\leqq 0.$$

Further by the area theorem

$$\frac{1}{8}(15p^2x-12x'y')y\leqq\frac{1}{8}y\left(\frac{15}{4}p^2x^2+\frac{15}{4}p^2x'^2-12x'y'+\frac{45}{4}p^2y'^2\right)$$
$$\leqq 0$$

for $p\geqq(64/75)^{1/4}$. Hence we may omit them. Further we have, applying Lemma 1,

$$R(p)=6-\frac{15}{4}x-\frac{x^2}{960}(4320-5480x+2120x^2-287x^3)$$
$$\leqq 6-\frac{x^2}{960}P(x)-\frac{15}{16}(x'^2+3y'^2+5\eta'^2+7\xi'^2+3y^2+5\eta^2)$$

with $P(x)=5220-5480x+2120x^2-287x^3$. Hence we have

$$\Re a_6 \leqq 6-\frac{x^2}{960}P(x)-X-\frac{15}{16}(x'^2+3y'^2+5\eta'^2+7\xi'^2)-\frac{5}{4}x'^2\eta-\frac{19}{8}px'^2y$$
$$+\left(\frac{9}{2}-\frac{7}{2}p-\frac{45}{16}\right)y^2-\frac{75}{16}\eta^2.$$

Here we make use of the following trivial inequalities:

$$-\frac{5}{4}x'^2\eta \leqq \frac{5}{8\alpha}x'^4+\frac{5}{8}\alpha\eta^2, \qquad \alpha>0,$$
$$-\frac{19}{8}px'^2y \leqq \frac{19}{16\beta}px'^4+\frac{19}{16}\beta p y^2, \qquad \beta>0.$$

If we put $\alpha=7.5$ and $\beta=1.5$, then the coefficients of η^2 and y^2 are zero and non-positive for $p \geqq 27/27.5$, respectively. Since $19px'^2/1.5 \leqq 12.67px'^2$ and $10x'^2/7.5 \leqq 0.49$ for $p^2+x'^2 \leqq 4$ and $x'^2/p^2 \leqq 1/10$, we have

$$\Re a_6 \leqq 6-\frac{x^2}{960}P(x)-Q,$$
$$16Q=(24p^3-26.67px'^2+14.5)x'^2+2(29p^2-11x'^2)x'y'+(28p+45)y'^2$$
$$+56px'\eta'+48y'\eta'+75\eta'^2+32x'\xi'+105\xi'^2.$$

Now consider the quadratic form Q and its associated symmetric matrix

$$\begin{pmatrix} 14.5+24p^3-26.67px'^2 & 29p^2-11x'^2 & 28p & 16 \\ 29p^2-11x'^2 & 45+28p & 24 & 0 \\ 28p & 24 & 75 & 0 \\ 16 & 0 & 0 & 105 \end{pmatrix}.$$

By a simple estimation for its principal diagonal minor determinants we can prove that Q is positive definite for $p \geqq 1$ and $k^2 \leqq 1/10$. This implies that

$$\Re a_6 \leqq 6-\frac{x^2}{960}P(x).$$

Since

$$P(x) \equiv 5220-5480x+2120x^2-287x^3>0$$

for $0 \leqq x \leqq 1$, we have

$$\Re a_6 \leqq 6$$

for $p \geqq 1$, $k^2 \leqq 1/10$ in this case. Equality occurs only for $x=0$, that is, for the Koebe function $z/(1-z)^2$.

§ **3.** This section is devoted to discuss the case $y \geqq 0$, $\eta+2py \leqq 0$, $1 \leqq p \leqq 2$, $k^2 \leqq 1/10$. In this case we start from (B), in which we put $B=3/2$. Now we remark that

$$(p^2-10x'^2)(\eta+2py)\leqq 0.$$

Hence we have

$$\Re a_6\leqq 6+S(x)-X-\frac{3}{2}x'y'y-\frac{9}{8}px'^2y+\frac{27}{16}p^2xy+\frac{1}{2}(9-7p)y^2,$$

$$S(x)=-10x+8x^2-\frac{11}{3}x^3+\frac{11}{12}x^4-\frac{11}{120}x^5+\frac{81}{256}xp^4$$

and X is the same expression as in § 2. Since

$$S(x)+\frac{27}{32}p^2\alpha x^2$$

$$=-4.9375x-\frac{x^2}{32}\left\{68-108\alpha-\left(\frac{377}{3}-108\alpha\right)x+\left(\frac{155}{3}-27\alpha\right)x^2-\left(\frac{181}{40}+\frac{8}{3}\right)x^3\right\}$$

$$\leqq-\frac{4.9375}{4}(x'^2+3y'^2+5\eta'^2+7\xi'^2+3y^2+5\eta^2)$$

$$-\frac{x^2}{32}\left\{107.5-108\alpha-\left(\frac{377}{3}-108\alpha\right)x+\left(\frac{155}{3}-27\alpha\right)x^2-\left(\frac{181}{40}+\frac{8}{3}\right)x^3\right\},$$

we have, with $\alpha=3/4$,

$$\Re a_6\leqq 6-x^2P(x)/96-Q(\beta)+Y(\beta),$$

$$P(x)=79.5-134x+94.25x^2-21.575x^3,$$

$$Q(\beta)=X+\frac{4.9375}{4}(x'^2+3y'^2+5\eta'^2+7\xi'^2)-\frac{3}{4}\beta x'^2y'^2,$$

$$Y(\beta)=\left(\frac{9}{2}-\frac{7}{2}p+\frac{9}{8}p^2-\frac{4.9375}{4}\cdot 3\right)y^2+\frac{3}{4\beta}y^2-\frac{4.9375}{4}\cdot 5\eta^2.$$

We consider the coefficient of y^2 in $Y(\beta)$, which is

$$\left(6.375+\frac{6}{\beta}-28p+9p^2\right)\Big/8.$$

Here we put $\beta=1/2$. In this case we have

$$12.375-28p+9p^2<0$$

for $1\leqq p\leqq 2$. Hence $Y(1/2)$ is non-positive for $1\leqq p\leqq 2$. Next we have

$$16Q(1/2)=(24p^3-14px'^2+19.75)x'^2+2(29p^2-11x'^2)x'y'$$
$$+(28p+59.25-6x'^2)y'^2+56px'\eta'+48y'\eta'+98.75\eta'^2+32x'\xi'+138.25\xi'^2.$$

The coefficient of y'^2 may be replaced by $28p+57$, since $6x'^2<2.2$ in this case. Consider the symmetric matrix associated with this modified quadratic form Q^* of Q

$$\begin{pmatrix} 19.75+24p^3-14px'^2 & 29p^2-11x'^2 & 28p & 16 \\ 29p^2-11x'^2 & 57+28p & 24 & 0 \\ 28p & 24 & 98.75 & 0 \\ 16 & 0 & 0 & 138.25 \end{pmatrix}.$$

It is easy to prove that all of its principal diagonal minor determinants are positive for $1\leqq p\leqq 2$, $k^2\leqq 1/10$. Hence we have

$$\Re a_6\leqq 6-\frac{x^2}{96}P(x).$$

Since $P(x)>0$ for $0\leqq x\leqq 1$, we have $\Re a_6\leqq 6$ for $1\leqq p\leqq 2$, $k^2\leqq 1/10$. Equality occurs only for the Koebe function $z/(1-z)^2$.

§ 4. In this section we shall be concerned with the case $y\geqq 0$, $\eta\geqq 0$, $1.2\leqq p\leqq 2$, $k^2\leqq 1/10$. Firstly we have, putting $\beta=p$, in Lemma 2,

$$\frac{3}{2}py\leqq \eta+\frac{3}{2}py$$

$$\leqq p^2x+\frac{x}{12}(4+2p+p^2)-\frac{1}{2}x'y'+\frac{1}{4}px'^2.$$

Hence

$$\frac{1}{8}p^3y\leqq \frac{p^2}{12}\left\{p^2x+\frac{x}{12}(4+2p+p^2)-\frac{1}{2}x'y'+\frac{1}{4}px'^2\right\}.$$

In this case we start from (A) with $B=3/2$. Then we have

$$\Re a_6\leqq R(p)-X+\frac{15}{8}p^2xy-\frac{3}{2}x'y'y-\frac{5}{4}x'^2\eta-\frac{29}{8}px'^2y+\frac{1}{8}p^3y+\frac{9-7p}{2}y^2,$$

where $R(p)$ and X are the same as in Section 2. By applying the above inequality we have

$$\Re a_6\leqq L+\frac{15}{8}p^2xy-X+\frac{p^3}{48}x'^2-\frac{p^2}{24}x'y'-\frac{3}{2}x'y'y-\frac{5}{4}x'^2\eta-\frac{29}{8}px'^2y+\frac{9-7p}{2}y^2,$$

$$L=R(p)+\frac{p^4}{12}x+\frac{p^2x}{144}(4+2p+p^2).$$

By a simple calculation we have

$$L=6-\frac{25}{12}x-\frac{23}{3}x^2+\frac{575}{72}x^3-\frac{53}{18}x^4+\left(\frac{1}{6}-\frac{7}{40}+\frac{25}{64}+\frac{1}{144}\right)x^5.$$

Hence for $\alpha>0$

$$L+\frac{15}{8}p^2xy\leqq L+\frac{15}{16}p^2\alpha x^2+\frac{15}{16\alpha}p^2y^2$$

$$=6-\frac{25}{12}x-\frac{92-45\alpha}{12}x^2+\frac{575-270\alpha}{72}x^3-\frac{424-135\alpha}{144}x^4+Mx^5+\frac{15}{16\alpha}p^2y^2,$$

where M is $1/6-7/40+25/64+1/144$. By putting $\alpha=2$ and by applying Lemma 1 we have

$$L+\frac{15}{8}p^2xy$$

$$\leqq 6-S(x)-\frac{25}{48}(x'^2+3y'^2+5\eta'^2+7\xi'^2+3y^2+5\eta^2)+\frac{15}{32}p^2y^2,$$

$$S(x)=\frac{33}{48}x^2-\frac{35}{72}x^3+\frac{77}{72}x^4-Mx^5.$$

Since

$$\frac{29}{8}px'^2+\frac{3}{2}x'y'+\frac{15}{116}y'^2\geqq 0$$

for $p\geqq 1.2$,

$$\left(-\frac{3}{2}x'y'-\frac{29}{8}px'^2\right)y\leqq\frac{15}{116}y'^2y$$

$$\leqq\frac{5}{29}xy\leqq\frac{5}{58\beta}x^2+\frac{5\beta}{58}y^2.$$

Thus we have

$$\Re a_6\leqq 6-x^2P(x,\beta)-Q+y^2\left(\frac{9}{2}-\frac{7}{2}p+\frac{15}{32}p^2-\frac{25}{16}+\frac{5\beta}{58}\right)-\frac{5}{4}x'^2\eta-\frac{125}{48}\eta^2,$$

$$x^2P(x,\beta)=S(x)-\frac{5}{58\beta}x^2,$$

$$Q=X-\frac{p^3}{48}x'^2+\frac{p^2}{24}x'y'+\frac{25}{48}(x'^2+3y'^2+5\eta'^2+7\xi'^2).$$

We consider the coefficient of y^2, which is

$$\frac{1}{29.32}(2726+80\beta-3248p+435p^2).$$

Here we put $\beta=5$. Then the above expression is negative for $1.2\leqq p\leqq 2$. Since $\eta\geqq 0$, we may omit $-(5/4)x'^2\eta-(125/48)\eta^2$. Hence we have

$$\Re a_6\leqq 6-P(x)x^2-Q, \qquad P(x)=P(x,5).$$

We consider the symmetric matrix associated with 48 Q

$$\begin{pmatrix} 25+71p^3-42px'^2 & 88p^2-33x'^2 & 84p & 48 \\ 88p^2-33x'^2 & 75+84p & 72 & 0 \\ 84p & 72 & 125 & 0 \\ 48 & 0 & 0 & 175 \end{pmatrix}.$$

Its principal diagonal minor determinants are

$$175,$$

$$175\cdot 125,$$

$$10500p+4251,$$

$$8679561+21745500p-92610000p^2+134628375p^3-38937500p^4$$
$$+(-100658250p+49875000p^2)x'^2-23821875x'^4\ (\equiv \varDelta).$$

We shall prove the positivity of these minor determinants. To this end, we may consider $\varDelta$ only. For this $\varDelta$ we have

$$\varDelta \geqq \phi(p) \equiv 8679561+21745500p-92610000p^2+124562550p^3-34188218.75p^4$$

for $k^2 \leqq 1/10$. We have $\phi(1)>0$ and $\phi(2)>0$. And it is easy to prove that $\phi'(p)>0$ for $1\leqq p\leqq 2$ and hence $\phi(p)$ is monotone increasing there. Thus $\phi(p)>0$ for $1\leqq p \leqq 2$. This shows that $\varDelta>0$ for $1\leqq p\leqq 2$, $k^2\leqq 1/10$. Hence Q is positive definite there. Further it is easy to prove that

$$29\cdot 72\, P(x)=1399.5-1015x+2233x^2-812.725x^3$$

is positive for $0\leqq x\leqq 1$. Thus we have the desired result:

$$\Re a_6 \leqq 6$$

for $1.2\leqq p\leqq 2$, $k^2\leqq 1/10$. Equality holds only for $x=0$.

§5. In this section we shall be concerned with the case $y\geqq 0$, $\eta\leqq 0$, $\eta+2py\geqq 0$, $1.6\leqq p\leqq 2$, $k^2\leqq 1/10$. This case is the most difficult one in principle and in practical sense. We need laborious calculations here. We need Lemma 3 in the following form:

$$\begin{aligned}
&\frac{p^3}{2}(\eta+2py)\\
&\leqq \frac{4}{3}-\frac{p^6}{48}+\frac{25}{16}p^2(4-p^2)-\frac{4-p^2}{8}5p^2y-y^2-\left(y'+\frac{5}{4}px'\right)^2\\
&\quad -3\left\{\eta'+\frac{3}{4}py'+\frac{1}{12}(3p^2-x'^2)x'-\frac{1}{2}x'y\right\}^2-\frac{1}{4}p^3x'y'+\frac{1}{8}p^4x'^2\\
&\quad -5\left\{\xi'+\frac{3}{4}p\eta'+\frac{1}{4}(p^2-x'^2)y'-\left(y'-\frac{p}{2}x'\right)y-\frac{1}{2}x'\eta\right\}^2.
\end{aligned}$$

This is easily obtained from Lemma 3 by putting $\beta=5p/4$ and by rearranging the terms.

Now we start from (B) and make use of the above inequality. Further we put $B=1$. We, then, have

$$\Re a_6 \leqq S(p)-X-\frac{3}{2}x'y'y-\frac{5}{4}x'^2\eta-\frac{29}{8}px'^2y+\frac{9}{8}p^2xy+\frac{8-5p}{4}y^2+y\eta$$

$$+\frac{1}{4p}\left\{\frac{4}{3}-\frac{1}{48}p^6+\frac{25}{16}p^2(4-p^2)-\frac{5}{8}p^2(4-p^2)y-y^2\right\}-\frac{p^2}{16}x'y'+\frac{p^3}{32}x'^2$$

$$-\frac{1}{4p}\left(y'+\frac{5}{4}px'\right)^2-\frac{3}{4p}\left\{\eta'+\frac{3}{4}py'+\frac{3p^2-x'^2}{12}x'-\frac{1}{2}x'y\right\}^2$$

$$-\frac{5}{4p}\left\{\xi'+\frac{3}{4}p\eta'+\frac{p^2-x'^2}{4}y'-\left(y'-\frac{p}{2}x'\right)y-\frac{1}{2}x'\eta\right\}^2,$$

$$S(p)=\frac{2}{5}+\frac{2}{3}p^2-\frac{p^5}{12}+\frac{7}{40}p^5+\frac{81}{256}p^4(2-p),$$

with the same X as in Section 2. By rearranging the terms we have

$$\Re a_6\leqq 6+\frac{1}{192p}(-504x-1608x^2+2676x^3-1509x^4+408.3x^5-44.15x^6)-X$$

$$-\frac{1}{4p}\left(y'+\frac{5}{4}px'\right)^2-\frac{3}{4p}\left\{\eta'+\frac{3}{4}py'+\frac{1}{12}(3p^2-x'^2)x'\right\}^2-\frac{1}{16}p^2x'y'$$

$$+\frac{1}{32}p^3x'^2-\frac{5}{4p}\left\{\xi'+\frac{3}{4}p\eta'+\frac{1}{4}(p^2-x'^2)y'\right\}^2+\frac{8-5p}{4}y^2-\frac{1}{4p}y^2$$

$$+y\eta+\frac{9}{8}p^2xy+\frac{10}{16p}\{y'\xi'+1.5py'\eta'+(p^2-x'^2)y'^2\}y-yQ^*$$

$$+\frac{5}{4p}x'\left\{\xi'+\frac{3}{4}p\eta'+\frac{1}{4}(p^2-x'^2)y'-px'\right\}\eta,$$

where

$$Q^*=\left(\frac{29}{8}p-\frac{3p^2-x'^2}{16p}\right)x'^2+\left\{\frac{15}{16}+\frac{5}{16}(p^2-x'^2)\right\}x'y'+\left(\frac{15}{16}p-\frac{3}{4p}\right)x'\eta'$$

$$-\frac{15}{16}y'\eta'+\frac{5}{4}x'\xi'-\frac{30}{16p}y'\xi'+\frac{5p}{32}(4-p^2).$$

Now we shall prove the non-negativity of Q^* for $1.6\leqq p\leqq 2$, $k^2\leqq 1/10$. By Lemma 1

$$32pQ^*\geqq 5p^2(x'^2+3y'^2+5\eta'^2+7\xi'^2)+(110p^2+2x'^2)x'^2+2p(15+5p^2-5x'^2)x'y'$$
$$+(15p^2-12)2x'\eta'-30py'\eta'+40px'\xi'-60y'\xi'.$$

Consider the symmetric matrix associated with the last quadratic form

$$\begin{pmatrix} 115p^2+2x'^2 & 15p+5p^3-5px'^2 & 15p^2-12 & 20p \\ 15p+5p^3-5px'^2 & 15p^2 & -15p & -30 \\ 15p^2-12 & -15p & 25p^2 & 0 \\ 20p & -30 & 0 & 35p^2 \end{pmatrix},$$

whose principal diagonal minor determinants are

$$115p^2+2x'^2,$$

$$-225p^2+1575p^4-25p^6+(180p^2+50p^4)x'^2-25p^2x'^4,$$

$$3240p^2-31050p^4+33750p^6-625p^8+(-2250p^2+6750p^4+1250p^6)x'^2-625p^4x'^4,$$

$$129600-108000p^2-2901600p^4-1386750p^6+1181250p^8-21875p^{10}$$

$$+(-45000p^2+71250p^4+236250p^6+43750p^8)x'^2-21875p^6x'^4.$$

All of them are positive for $1.6 \leqq p \leqq 2$, $k^2 \leqq 1/10$. This implies the non-negativity of Q^* there. Since $y \geqq 0$, $-yQ^* \leqq 0$ there. Hence we may omit $-yQ^*$. By using the trivial inequality

$$2d\,xy \leqq |d|\alpha x^2+\frac{|d|}{\alpha}y^2, \qquad \alpha>0,$$

we have

$$U \equiv \frac{1}{192p}(-504x-1608x^2+2676x^3-1509x^4+408.3x^5-44.15x^6)+\frac{9}{8}p^2xy$$

$$+\frac{10}{16p}\{\xi'+1.5p\eta'+(p^2-x'^2)y'\}y'y+\left(\frac{8-5p}{4}-\frac{1}{4p}\right)y^2+y\eta$$

$$\leqq \frac{1}{192p}\{-504x-(1608-864\alpha)x^2+(2676-1296\alpha)x^3-(1509-648\alpha)x^4$$

$$+(408.3-108\alpha)x^5-44.15x^6\}$$

$$+\left(2-\frac{5}{4}p-\frac{1}{4p}+\frac{9}{16\alpha}p^2+\frac{10}{32p\beta}\right)y^2+y\eta+\frac{10\beta}{32p}y'^2\{\xi'+1.5p\eta'+(p^2-x'^2)y'\}^2.$$

Here we put $\alpha=2$, $\beta=0.36$. Then by Lemma 1

$$U \leqq \frac{1}{192p}\{-4Ax+(A-6)x^2+84x^3-213x^4+192.3x^5-44.15x^6\}$$

$$-\frac{126-A}{192p}(x'^2+3y'^2+5\eta'^2+7\xi'^2)+y\eta-\frac{126-A}{192p}5\eta^2$$

$$+\frac{1}{32p}\left(64p-40p^2-8+9p^3+\frac{10}{0.36}-\frac{126-A}{6}\cdot 3\right)y^2$$

$$+\frac{3.6}{32p}\{\xi'+1.5p\eta'+(p^2-x'^2)y'\}^2\cdot\frac{4-p^2-x'^2}{3}.$$

Take $A=3$. Then the coefficient of y^2 is

$$\frac{1}{64p}\left(-139+\frac{20}{0.36}+128p-80p^2+18p^3\right) \equiv \frac{1}{64p}\psi(p).$$

Here $\psi(p)$ is monotone increasing for $0 \leqq p \leqq 2$ and $\psi(2)=-31/9$. Hence $\psi(p)<0$ there. Thus we may omit the term $\psi(p)y^2/64p$. Since $y\eta \leqq 0$, we may omit this term.

We again make use of the trivial inequality and we have

$$U+\frac{5}{4p}x'\left\{\xi'+\frac{3}{4}p\eta'+\frac{p^2-x'^2}{4}y'-px'\right\}\eta$$

$$\leqq \frac{-1}{192p}xP^*(x)-\frac{41}{64p}(x'^2+3y'^2+5\eta'^2+7\xi'^2)+\frac{5}{8p\varepsilon}\eta^2-\frac{205}{64p}\eta^2$$

$$+\frac{1.2}{32p}\{\xi'+1.5p\eta'+(p^2-x'^2)y'\}^2(4-p^2-x'^2)+\frac{5\varepsilon}{8p}x'^2\left(\xi'+\frac{3}{4}p\eta'+\frac{p^2-x'^2}{4}y'-px'\right)^2.$$

If we put $\varepsilon=1/5$, then the coefficient of η^2 is

$$(-205+200)/64p=-5/64p<0.$$

Hence we may omit the term of η^2. Thus we have for $1.6\leqq p\leqq 2$, $k^2\leqq 1/10$

$$\Re a_6\leqq 6-\frac{x}{192p}P^*(x)-Q,$$

$$P^*(x)=12+3x-84x^2+213x^3-192.3x^4+44.15x^5,$$

$$Q=X+\frac{p^2}{16}x'y'-\frac{p^3}{32}x'^2+\frac{41}{64p}(x'^2+3y'^2+5\eta'^2+7\xi'^2)+\frac{1}{4p}\left(y'+\frac{5}{4}px'\right)^2$$

$$+\frac{3}{4p}\left\{\xi'+\frac{3}{4}py'+\frac{3p^2-x'^2}{12}x'\right\}^2+\frac{5}{4p}\left\{\xi'+\frac{3}{4}p\eta'+\frac{p^2-x'^2}{4}y'\right\}^2$$

$$-\frac{x'^2}{8p}\left(\xi'+\frac{3}{4}p\eta'+\frac{p^2-x'^2}{4}y'-px'\right)^2-\frac{1.2}{32p}(4-p^2-x'^2)\{\xi'+1.5p\eta'+(p^2-x'^2)y'\}^2.$$

By rearranging the terms, we have

$$Q=Q_1+x'^2Q_2+(1.9/64p)x'^6y'^2,$$

$$\begin{aligned}64pQ_1=&(41+25p^2+97p^4-72p^2x'^2)x'^2+2(20p+127p^3-40px'^2)x'y'\\ &+(139+139p^2-4.6p^4+2.4p^6-2p^4x'^2)y'^2+2(124p^2+5p^2x'^2)x'\eta'\\ &+2(132p+0.6p^3+3.6p^5)y'\eta'+(253+23.4p^2+5.4p^4-10p^2x'^2)\eta'^2\\ &+2(64p+8px'^2)x'\xi'+2(10.4p^2+2.4p^4-16x'^2)y'\xi'\\ &+2(45.6p+3.6p^3-2.4px'^2)\eta'\xi'+(357.4+2.4p^2-20x'^2)\xi'^2,\end{aligned}$$

$$\begin{aligned}64pQ_2=&(6p^2+x'^2/3)x'^2+2(-7p+2p^3-2px'^2)x'y'+\{9.2p^2-0.9p^4-(4.6+1.4p^2)x'^2\}y'^2\\ &+2(-4+p^2)x'\eta'+2(-0.6p-1.5p^3-2.1px'^2)y'\eta'+10.9p^2\eta'^2\\ &+2(5.6-2p^2-0.4'^2)y'\xi'+14.4\xi'^2.\end{aligned}$$

Consider the symmetric matrix associated with the quadratic form Q_2

(A_{ij}), $\quad A_{ij}=A_{ji}$,

$A_{11}=14.4$, $\quad A_{12}=0$, $\quad A_{13}=5.6-2p^2-0.4x'^2$, $\quad A_{14}=0$, $\quad A_{22}=10.9p^2$,

$A_{23}=-0.6p-1.5p^3-2.1px'^2$, $\quad A_{24}=-4+p^2$, $\quad A_{33}=9.2p^2-0.9p^4-(4.6+1.4p^2)x'^2$,

$$A_{34}=-7p+2p^3-2px'^2, \qquad A_{44}=6p^2+x'^2/3.$$

We may replace A_{44} by $6p^2+0.3x'^2$. Then we consider the principal diagonal minor determinants of this modified matrix, which are

$$14.4,$$

$$156.96p^2,$$

$$-347.008p^2+1662.272p^4-217.264p^6-(709.472p^2+327.904p^4)x'^2-65.248p^2x'^4,$$

$$501.76-3212.8p^2-9181.728p^4+14691.392p^6-2000.864p^8$$
$$+(988.16-2081.7024p^2-7669.1904p^4-789.7232p^6)x'^2$$
$$+(2.56-697.9616p^2-996.5792p^4)x'^4-19.5744p^2x'^6.$$

All of them are positive for $\sqrt{2}\leqq p\leqq 2$, $k^2\leqq 1/10$. Therefore Q_2 is positive definite there. Next we shall consider the symmetric matrix associated with the quadratic form Q_1

$$(a_{ij}), \qquad a_{ij}=a_{ji},$$

$$a_{11}=41+25p^2+97p^4-72p^2x'^2, \qquad a_{12}=20p+127p^3-40px'^2,$$

$$a_{22}=139+139p^2-4.6p^4+2.4p^6-2p^4x'^2, \qquad a_{13}=124p^2+5p^2x'^2,$$

$$a_{23}=132p+0.6p^3+3.6p^5, \qquad a_{33}=253+23.4p^2+5.4p^4-10p^2x'^2,$$

$$a_{14}=64p+8px'^2, \qquad a_{24}=10.4p^2+2.4p^4-16x'^2,$$

$$a_{34}=45.6p+3.6p^3-2.4px'^2, \qquad a_{44}=357.4+2.4p^2-20x'^2.$$

Since $1.6\leqq p\leqq 2$, $k^2\leqq 1/10$, we may replace a_{22} by $139+139p^2+p^4$. Then the corresponding quadratic form $Q_1{}^*$ satisfies $Q_1{}^*\leqq Q_1$. We calculate the principal diagonal minor determinants of the modified symmetric matrix. They are

$$\Delta_1\equiv 41+25p^2+97p^4-72p^2x'^2,$$

$$\Delta_2\equiv 5699+8774p^2+11919p^4-2621p^6+97p^8+(-8408p^2+152p^4-72p^6)x'^2-1600p^2x'^4,$$

$$\Delta_3\equiv 1441847+1638794.6p^2+1326954.8p^4-46713.96p^6+9638.88p^8+8638.92p^{10}$$
$$-219.24p^{12}-1257.12p^{14}$$
$$+(-2184214p^2-446903.2p^4-178399.6p^6+58330.72p^8+3524.24p^{10}+933.12p^{12})x'^2$$
$$+(-404800p^2-9635p^4-13875p^6-745p^8)x'^4+16000p^4x'^6,$$

$$\Delta_4\equiv 515316117.8+433271318.2p^2+468014409.52p^4-24376778.68p^6+10739180.16p^8$$
$$+711091.8p^{10}-47007p^{12}-385935.84p^{14}$$
$$+(-28836940-862985522.48p^2-191544069.6p^4-52626175.472p^6$$
$$+20671058.016p^8+515732.304p^{10}+318665.952p^{12}+26706.816p^{14})x'^2$$
$$+(-2655488-85293519.04p^2+3931167.8p^4+1476842.84p^6-1826377.336p^8$$

$$-166342.432p^{10}-19541.376p^{12})x'^4$$

$$+(15454976p^2+5273667.68p^4+1290509.92p^6+61366.56p^8)x'^6-575744p^4x'^8.$$

The positivity of the first two expressions for $1.6\leqq p\leqq 2$, $k^2\leqq 1/10$ is very easy to verify. Now we put

$$\Delta_3=\Phi_1(p)+\Phi_2(p)x'^2+\Phi_3(p)x'^4+\Phi_4(p)x'^6.$$

Then $\Phi_2(p)<0$, $\Phi_3(p)<0$ and $\Phi_4(p)>0$ for $0\leqq p\leqq 2$. Hence

$$\Delta_3\geqq\Phi_1(p)+p^2\Phi_2(p)/10+p^4\Phi_3(p)/100\equiv\Phi(p)$$

for $k^2\leqq 1/10$. By a simple calculation we have

$$\Phi(p)=1441847+1638794.6p^2+1108533.4p^4-95452.28p^6-8297.43p^8$$

$$+14333.242p^{10}+125.734p^{12}-1163.808p^{14}>0$$

for $0\leqq p\leqq 2$. Thus we have $\Delta_3>0$ for $0\leqq p\leqq 2$, $k^2\leqq 1/10$.

Next we put

$$\Delta_4=\varphi_1(p)+\varphi_2(p)x'^2+\varphi_3(p)x'^4+\varphi_4(p)x'^6+\varphi_5(p)x'^8.$$

Then $\varphi_4(p)+\varphi_5(p)x'^2>0$ for $1\leqq p\leqq 2$, $k^2\leqq 1$ and $\varphi_2(p)<0$, $\varphi_3(p)<0$ for $0\leqq p\leqq 2$. By $k^2\leqq 1/10$ we have

$$\Delta_4\geqq\varphi(p)\equiv\varphi_1(p)+p^2\varphi_2(p)/10+p^4\varphi_3(p)/100.$$

$\varphi(p)$ is the following polynomial of p

$$515316117.8+430387624.2p^2+381689302.392p^4-44384120.8304p^6+5515874.2908p^8$$

$$+2792966.03p^{10}-13697.54296p^{12}-355732.66912p^{14}+2475.26784p^{16}.$$

This is positive for $0\leqq p\leqq 2$. Hence $\Delta_4>0$ for $1\leqq p\leqq 2$, $k^2\leqq 1/10$. This implies the positive definiteness of Q_1^* and hence of Q_1 there. Thus we have

$$\Re a_6\leqq 6-\frac{x}{192p}P^*(x)$$

there.

Now consider the polynomial $P^*(x)$. It is easy to prove that there is an x_0 satisfying $P^{*\prime}(x_0)=0$ and $0<x_0<0.1$ and that $P^{*\prime}(x)>0$ for $0\leqq x<x_0$ and $P^{*\prime}(x)<0$ does hold for $x_0<x\leqq 0.4$. Hence $P^*(x)$ is monotone increasing for $0<x<x_0$ and decreasing for $x_0<x\leqq 0.4$. Since $P^*(0)=12>0$ and $P^*(0.4)>0$, $P^*(x)>0$ for $0\leqq x\leqq 0.4$.

Thus we have the desired result:

$$\Re a_6\leqq 6$$

for $0\leqq x\leqq 0.4$, $k^2\leqq 1/10$, with equality holding only for $x=0$, that is, for the Koebe function $z/(1-z)^2$.

A remark on our method should be mentioned here. Several tests we performed tell us that only the truncated Grunsky inequality does not give any enough informations even for the local maximality and hence for the global maximality.

In this point of view Jenkins' remark in [7] is very important and gives an important suggestion. Following this suggestion we made use of Golusin's inequality together with Grunsky's.

§6. In this section we are concerned with the remaining cases of the case $1\leqq p\leqq 1.6$, $k^2\leqq 1/10$.

Case 1. $1\leqq p\leqq 1.2$, $k^2\leqq 1/10$. In this case we may assume $y\geqq 0$. We start from (A) with $B=1/4$. We, then, have

$$\Re a_6\leqq M(p)-X+\frac{25}{64}p^4(2-p)+\frac{5}{16}p^2(2-p)y+\frac{1}{8}p^3y+\frac{2+19p}{16}y^2+2.5y\eta$$

$$-\frac{3}{2}x'y'y-\frac{5}{4}x'^2\eta-\frac{29}{8}px'^2y,$$

where X is the same one as in Section 2 and

$$M(p)=\frac{2}{5}+\frac{2}{3}p^2+\frac{11}{120}p^5.$$

By applying Lemma 1 to $6-M(p)$, we have

$$\Re a_6\leqq 6-(6-M(p))\frac{p^2}{4}+\frac{25}{64}p^4(2-p)+\frac{5}{16}p^2(2-p)y+\frac{1}{8}p^3y+\frac{2+19p}{16}y^2$$

$$+2.5y\eta-\frac{6-M(p)}{4}(3y^2+5\eta^2)-X-\frac{6-M(p)}{4}(x'^2+3y'^2+5\eta'^2+7\xi'^2)$$

$$-\frac{3}{2}x'y'y-\frac{5}{4}x'^2\eta-\frac{29}{8}px'^2y.$$

Evidently we have $6-M(p)\geqq 4.411904$ for $1\leqq p\leqq 1.2$. Since $|y|<1.005$, we have

$$-(6-M(p))\frac{p^2}{4}+\frac{25}{64}p^4(2-p)+\frac{5}{16}p^2(2-p)y+\frac{1}{8}p^3y$$

$$\leqq\frac{p^2}{64}(-30.390464-12.06p+50p^2-25p^3)\equiv\frac{p^2}{64}\varphi(p).$$

Put $\varphi(p)=\varphi_1(p)-0.06p$. It is very easy to prove $\varphi_1(p)\leqq\varphi_1(1.2)=-15.990464$ for $1\leqq p\leqq 1.2$. Thus we have $\varphi(p)\leqq\varphi_1(1.2)-0.06=-16.050464$. On the other hand for $1\leqq p\leqq 1.2$

$$\left(-\frac{29}{8}px'^2-\frac{3}{2}x'y'\right)y\leqq\frac{9}{58}y'^2y\leqq\frac{3}{58}(4-p^2)1.005.$$

By these estimations we have

$$\frac{p^2}{64}\varphi(p)+\frac{3}{58}(4-p^2)1.005\leqq-\frac{16.050464}{64}+\frac{9.045}{58}<0$$

and hence we may omit the corresponding terms

$$-(6-M(p))\frac{p^2}{4}+\frac{25}{64}p^4(2-p)+\frac{5}{16}p^2(2-p)y+\frac{1}{8}p^3y-\frac{29}{8}px'^2y-\frac{3}{2}x'y'y.$$

Therefore for $1\leqq p\leqq 1.2$

$$\Re a_6<6-L(\alpha)-Q(\alpha),$$

$$L(\alpha)=\left\{\frac{3}{4}(6-M(p))-\frac{2+19p}{16}\right\}y^2-\frac{5}{2}y\eta+\left\{\frac{5}{4}(6-M(p))-\frac{5\alpha}{8}\right\}\eta^2,$$

$$Q(\alpha)=X+\frac{4.411904}{4}(x'^2+3y'^2+5\eta'^2+7\xi'^2)-\frac{5}{8\alpha}x'^4$$

with $\alpha>0$.

Now we consider the quadratic form $L(\alpha)$ of y, η, which satisfies

$$16L(\alpha)\geqq 28.142848y^2-40y\eta+(88.23808-10\alpha)\eta^2$$

in $1\leqq p\leqq 1.2$. Here we put $\alpha=7$. Then the right hand side form is non-negative. Hence $L(7)\geqq 0$ there. Hence we have

$$\Re a_6<6-Q, \qquad Q=Q(7).$$

Here we have

$$16Q=\left(24p^3-14px'^2+17.647616-\frac{10}{7}x'^2\right)x'^2+2(29p^2-11x'^2)x'y'$$
$$+(28p+52.942848)y'^2+56px'\eta'+48y'\eta'+88.23808\eta'^2$$
$$+32x'\xi'+123.533312\xi'^2.$$

Since $1\leqq p\leqq 1.2$, $x'^2\leqq p^2/10$, we have

$$\frac{10}{7}x'^2\leqq\frac{10}{7}\cdot\frac{p^2}{10}\leqq\frac{1.44}{7}\leqq 0.205715.$$

Hence for $1\leqq p\leqq 1.2$, $k^2\leqq 1/10$

$$16Q\geqq Q^*,$$

$$Q^*=(24p^3+17.4-14px'^2)x'^2+2(29p^2-11x'^2)x'y'+(28p+52.9)y'^2$$
$$+56px'\eta'+48y'\eta'+88\eta'^2+32x'\xi'+123\xi'^2.$$

Q^* is positive definite for $1\leqq p\leqq 1.2$, $k^2\leqq 1/10$. This is easily shown by taking the symmetric matrix associated with Q^* and by calculating its principal diagonal minor determinants. Consequently Q is also positive definite there. Hence we have

$$\Re a_6<6$$

there. This is the desired result.

Case 2. $1.2\leqq p\leqq 1.5$, $k^2\leqq 1/10$. In this case we may assume $y\geqq 0$, $\eta\leqq 0$. We start from (A) with $B=0$. Then we have

$$\Re a_6 \leqq 6-(6-M(p))+\frac{25}{64}p^4(2-p)+\frac{1}{8}p^3y+\frac{7}{4}py^2+3y\eta$$

$$-X-\frac{3}{2}x'y'y-\frac{5}{4}x'^2\eta-\frac{29}{8}px'^2y,$$

where X and $M(p)$ are the same as in the case 1. As in the above case

$$(6-M(p))\frac{p^2}{4}-\frac{25}{64}p^4(2-p)-\frac{p^3}{8}1.005$$

$$=\frac{p^2}{960}(1344-120.6p-910p^2+375p^3-22p^5)\equiv\frac{p^2}{960}\psi(p)$$

and

$$\psi(p)\geqq\psi(1.5)=\frac{6853.2}{32}$$

for $1.2\leqq p\leqq 1.5$. Further

$$-\frac{3}{2}x'y'y-\frac{29}{8}px'^2y\leqq y\left(-\frac{29}{8}px'^2+\frac{3\alpha}{4}x'^2+\frac{3}{4\alpha}y'^2\right)$$

$$\leqq y\left(-\frac{29}{8}px'^2+\frac{3}{4}\alpha x'^2+\frac{4-p^2}{4\alpha}\right).$$

Here we put $\alpha=5.8$. Then this implies that

$$-\frac{3}{2}x'y'y-\frac{29}{8}px'^2y\leqq\frac{4-p^2}{23.2}y.$$

By making use of Lemma 4 and Lemma 1 we have

$$\frac{p^2}{960}\psi(p)-\frac{4-p^2}{23.2}y\geqq\frac{1.44}{960}\cdot\frac{6853.2}{32}-\frac{2.5728}{23.2}>0.21\geqq\frac{0.63}{4}y^2.$$

Thus we have

$$\Re a_6<6-\frac{0.63}{4}y^2+\frac{7}{4}py^2-\frac{3}{4}(6-M(p))y^2-\frac{5}{4}(6-M(p))\eta^2-Q(\varepsilon)+\frac{5\varepsilon}{8}\eta^2+3y\eta,$$

$$Q(\varepsilon)=X+\frac{6-M(p)}{4}(x'^2+3y'^2+5\eta'^2+7\xi'^2)-\frac{5}{8\varepsilon}x'^4.$$

Consider the coefficient of y^2, which is

$$-\frac{0.63}{4}+\frac{7}{4}p-\frac{3}{4}(6-M(p))=-\frac{1}{16}(69.72-28p-8p^2-1.1p^5).$$

This is negative for $1.2\leqq p\leqq 1.5$. Now we put $\varepsilon=6.5$. Then the coefficient of η^2 is negative. Since $y\eta\leqq 0$, we may omit the term $3y\eta$. Hence we have

$$\Re a_6<6-Q,\qquad Q=Q(6.5).$$

Since $6-M(p)\geqq 3.40390625$ and $20x'^2/13<0.4$ for $1.2\leqq p\leqq 1.5$, $k^2\leqq 1/10$, we have

$$16Q \geqq Q^*,$$

$$Q^* = (24p^3 + 13.2 - 14px'^2)x'^2 + 2(29p^2 - 11x'^2)x'y' + (28p + 40.8)y'^2$$
$$+ 56px'\eta' + 48y'\eta' + 68\eta'^2 + 32x'\xi' + 95.2\xi'^2.$$

Q^* is positive definite for $1.2 \leqq p \leqq 1.5$, $k^2 \leqq 1/10$. This is easily verified by taking its associated symmetric matrix and its principal diagonal minor determinants. This implies that $\Re a_6 < 6$ for $1.2 \leqq p \leqq 1.5$, $k^2 \leqq 1/10$.

Case 3. $1.5 \leqq p \leqq 1.6$, $k^2 \leqq 1/10$. In this case we may assume $y \geqq 0$, $\eta \leqq 0$. We start from (A) with $B = 1/4$. By the same argument as in Case 2 we have

$$\Re a_6 \leqq 6 + U + \frac{2+19p}{16} y^2 + 2.5y\eta - X - \frac{6-M(p)}{4}(x'^2 + 3y'^2 + 5\eta'^2 + 7\xi'^2) - \frac{3}{2}x'y'y$$
$$- \frac{5}{4}x'^2\eta - \frac{29}{8}px'^2y - \frac{6-M(p)}{4}(3y^2 + 5\eta^2),$$

where $M(p)$ and X are the same as in Case 2 and

$$U = -(6-M(p))\frac{p^2}{4} + \frac{25}{64}p^4(2-p) + \frac{5}{16}p^2xy + \frac{1}{8}p^3y.$$

U satisfies

$$U \leqq -(6-M(p))\frac{p^2}{4} + \frac{25}{64}p^4(2-p) + \frac{p^3}{8}1.005 + \frac{5p^2}{32}\alpha(2-p)^2 + \frac{5p^2}{32\alpha}y^2 \qquad (\alpha > 0)$$
$$= -\frac{p^2}{960}L(\alpha) + \frac{5p^2}{32\alpha}y^2,$$

$$L(\alpha) = 1344 - 600\alpha + (600\alpha - 120.6)p - (910 + 150\alpha)p^2 + 375p^3 - 22p^5.$$

Here we put $\alpha = 2$ and $L = L(2)$. Then L is a polynomial of p, whose minimum in $1.5 \leqq p \leqq 1.6$ is attained at $p = 1.6$ and is equal to 78.75328. On the other hand for $1.5 \leqq p \leqq 1.6$ we have

$$-\frac{29}{8}px'^2y - \frac{3}{2}x'y'y \leqq \frac{9}{58}y'^2y \leqq \frac{3}{58}(4-p^2)1.005 < 0.091.$$

Hence for $1.5 \leqq p \leqq 1.6$

$$-\frac{p^2}{960}L - \frac{29}{8}px'^2y - \frac{3}{2}x'y'y \leqq -\frac{2.25}{960}78.75328 + 0.091$$
$$< -\frac{89.83488}{960} \leqq -\frac{11.22936}{160}y^2$$

by Lemma 1. Hence summing up the coefficients of y^2 and multiplying 160 we have $-663.22936 + 190p + 92.5p^2 + 11p^5$, which is negative for $1.5 \leqq p \leqq 1.6$. By the trivial inequality

$$-\frac{5}{4}x'^2\eta \leqq \frac{5}{8}\varepsilon\eta^2+\frac{5}{8\varepsilon}x'^4$$

we have the non-positivity of the coefficient of η^2 by taking $\varepsilon=5.8$. Thus we have

$$\Re a_6<6-Q,$$

$$Q=X+\frac{6-M(p)}{4}(x'^2+3y'^2+5\eta'^2+7\xi'^2)-\frac{5}{46.4}x'^4.$$

By rearranging the terms and by remarking

$$6-M(p)\geqq 2.932138\dot{6}>2.932,$$

we have $16Q\geqq Q^*$,

$$Q^*=\left(24p^3+11.728-\left(14p+\frac{5}{2.9}\right)x'^2\right)x'^2+2(29p^2-11x'^2)x'y'+(28p+35.184)y'^2$$

$$+56px'\eta'+48y'\eta'+58.64\eta'^2+32x'\xi'+82.096\xi'^2.$$

We may replace the coefficient of x'^2 by $24p^3+11-14px'^2$, since $1.5\leqq p\leqq 1.6$, $k^2\leqq 1/10$. Now we adopt new variables $2\eta'$, $4\xi'$ instead of η', ξ'. Then the associated symmetric matrix is

$$\begin{pmatrix} 24p^3+11-14px'^2 & 29p^2-11x'^2 & 14p & 4 \\ 29p^2-11x'^2 & 28p+35.184 & 12 & 0 \\ 14p & 12 & 14.66 & 0 \\ 4 & 0 & 0 & 5.131 \end{pmatrix}.$$

Its principal diagonal minor determinants are

5.131,

$5.131\cdot 14.66$,

$5.131\,(410.48p+371.79744)$,

$$15035.86027104+16600.22168p-35383.704384p^2+67622.15995136p^3$$

$$-12712.25774p^4+(-45671.87330496p+18504.23316p^2)x'^2-9101.67566x'^4.$$

All of them are positive for $1\leqq p\leqq 1.6$, $k^2\leqq 1/10$. Hence Q is non-negative for $1.5\leqq p\leqq 1.6$, $k^2\leqq 1/10$. This implies the desired result:

$$\Re a_6<6$$

there.

§7. In this section we are concerned with the case $1\leqq p$, $1/10\leqq k^2\leqq 5-2\sqrt{5}$. We divide this case into several subcases: 1) $y\geqq 0$, 2) $y\leqq 0$, $1\leqq p\leqq 1.6$, $1/10\leqq k^2\leqq 1/4$; 3) $y\leqq 0$, $1.6\leqq p$, $1/10\leqq k^2\leqq 1/4$ or $1/4\leqq k^2\leqq 5-2\sqrt{5}$; 4) $y\leqq 0$, $1.5\leqq p\leqq 1.6$, $1/4\leqq k^2\leqq 5-2\sqrt{5}$; 5) $y\leqq 0$, $1.3\leqq p\leqq 1.5$, $1/4\leqq k^2\leqq 5-2\sqrt{5}$; 6) $y\leqq 0$, $1\leqq p\leqq 1.3$, $1/4\leqq k^2$

$\leqq 5-2\sqrt{5}$.

Case 1). In this case we start from (A) with $B=3/2$. Then

$$\Re a_6 \leqq S-X-\frac{5}{4}x'^2\eta,$$

$$S=R(p)+\frac{1}{8}p^3y+\frac{15}{8}p^2(2-p)y+\frac{9-7p}{2}y^2-\frac{3}{2}x'y'y-\frac{29}{8}px'^2y,$$

$$R(p)=\frac{2}{5}+\frac{2}{3}p^2-\frac{1}{12}p^5+\frac{7}{40}p^5+\frac{25}{64}p^4(2-p)$$

and X is the same one as in Section 6. On the other hand we have

$$-\frac{29}{8}px'^2y+\frac{1}{8}p^3y-\frac{3}{2}x'y'y=\frac{5}{4}p^3y\left(\frac{1}{10}-k^2\right)-\left(\frac{19}{8}px'^2+\frac{3}{2}x'y'\right)y$$

$$\leqq -\left(\frac{19}{8}px'^2+\frac{3}{2}x'y'\right)y \leqq \frac{1}{4}y'^2y \leqq \frac{1}{12}(4-p^2)y$$

$$\leqq \frac{\beta}{24}(2+p)x^2+\frac{1}{24\beta}(2+p)y^2$$

with $\beta>0$. Hence

$$S\leqq 6-\frac{15}{4}x-\frac{x^2}{960}(4320-5480x+2120x^2-287x^3)+\frac{\beta}{24}(4-x)x^2$$

$$+\frac{15\alpha}{16}p^2x^2+\left(\frac{15}{16\alpha}p^2-\frac{7p}{2}+\frac{9}{2}+\frac{2+p}{24\beta}\right)y^2$$

$$\leqq 6-\frac{x^2}{960}\{5220-3600\alpha-160\beta-(5480-3600\alpha-40\beta)x+(2120-900\alpha)x^2-287x^3\}$$

$$+\left(\frac{15}{16\alpha}p^2-\frac{7}{2}p+\frac{9}{2}+\frac{2+p}{24\beta}-\frac{45}{16}\right)y^2-\frac{75}{16}\eta^2-\frac{15}{16}(x'^2+3y'^2+5\eta'^2+7\xi'^2).$$

Here we put $\alpha=1.1$, $\beta=1$. Then we have

$$S\leqq 6-\frac{x^2}{960}(1100-1480x+1130x^2-287x^3)+\left(\frac{15}{17.6}p^2-\frac{7}{2}p+\frac{9}{2}+\frac{2+p}{24}-\frac{45}{16}\right)y^2$$

$$-\frac{75}{16}\eta^2-\frac{15}{16}(x'^2+3y'^2+5\eta'^2+7\xi'^2).$$

Consider the coefficient of y^2. We can easily prove that this coefficient is negative for $1\leqq p\leqq 2$. By putting $L(x)=1100-1480x+1130x^2-287x^3$ we have

$$\Re a_6\leqq 6-\frac{x^2}{960}L(x)-X-\frac{15}{16}(x'^2+3y'^2+5\eta'^2+7\xi'^2)+\frac{5\varepsilon}{8}x'^4-\left(\frac{75}{16}-\frac{5}{8\varepsilon}\right)\eta^2,$$

$$\varepsilon>0.$$

Here we put $\varepsilon=2/15$. Then the coefficient of η^2 is zero. Therefore we have

$$\Re a_6 \leqq 6-\frac{x^2}{960}L(x)-Q,$$

$$Q=X+\frac{15}{16}(x'^2+3y'^2+5\eta'^2+7\xi'^2)-\frac{1}{12}x'^4.$$

The symmetric matrix associated with Q is

$$\begin{pmatrix} 15+24p^3-(14p+4/3)x'^2 & 29p^2-11x'^2 & 28p & 16 \\ 29p^2-11x'^2 & 28p+45 & 24 & 0 \\ 28p & 24 & 75 & 0 \\ 16 & 0 & 0 & 105 \end{pmatrix}.$$

This is positive definite for $1\leqq p$ $(\leqq 2)$, $1/10\leqq k^2\leqq 5-2\sqrt{5}$. Since $L(x)>0$ for $0\leqq x\leqq 1$ and $x>0$ for $1\leqq p$, $1/10\leqq k^2\leqq 5-2\sqrt{5}$, we have

$$\Re a_6<6$$

there, which is nothing but the desired result.

Case 2). We start from (A) with $B=3/2$ in this case. Then we have

$$\Re a_6\leqq 6+Y-X-\frac{5}{4}x'^2\eta-\frac{29}{8}px'^2y-\frac{3}{2}x'y'y+\frac{15}{8}p^2xy+\frac{1}{8}p^3y$$
$$+\frac{9-7p}{2}y^2,$$

$$Y=-\frac{15}{4}x-\frac{x^2}{960}(4320-5480x+2120x^2-287x^3),$$

where X is the same as in the above case. Now we have

$$Y\leqq-\frac{x^2}{960}P(x)-\frac{15}{16}(x'^2+3y'^2+5\eta'^2+7\xi'^2)-\frac{45}{16}y^2-\frac{75}{16}\eta^2,$$

$$P(x)=5220-5480x+2120x^2-287x^3$$

and

$$(-y)\left(\frac{3}{2}x'y'-\frac{15}{8}p^2x\right)$$

$$\leqq(-y)\left(\frac{3}{2}x'y'-\frac{15}{32}p^2x'^2-\frac{45}{32}p^2y'^2-\frac{15}{32}p^2x^2\right)\leqq 0$$

for $p\geqq 1$. Thus we have

$$\Re a_6\leqq 6-\frac{x^2}{960}P(x)-X-\frac{15}{16}(x'^2+3y'^2+5\eta'^2+7\xi'^2)+\frac{5\alpha}{8}x'^4+\frac{29}{16}p\beta x'^4$$

$$+\frac{29}{16\beta}py^2+\left(-\frac{7p}{2}+\frac{9}{2}-\frac{45}{16}\right)y^2+\frac{5}{8\alpha}\eta^2-\frac{75}{16}\eta^2.$$

Here we put $\alpha=2/15$, $\beta=1$. Then the coefficient of y^2 is non-positive for $p\geqq 1$ and that of η^2 is zero. Hence we have

$$\Re a_6\leqq 6-\frac{x^2}{960}P(x)-Q,$$

$$Q=X+\frac{15}{16}(x'^2+3y'^2+5\eta'^2+7\xi'^2)-\frac{1}{12}x'^4-\frac{29}{16}px'^4$$

for $1\leqq p$. The symmetric matrix associated with Q is

$$\begin{pmatrix} 15+24p^3-43px'^2-\frac{4}{3}x'^2 & 29p^2-11x'^2 & 28p & 16 \\ 29p^2-11x'^2 & 45+28p & 24 & 0 \\ 28p & 24 & 75 & 0 \\ 16 & 0 & 0 & 105 \end{pmatrix}.$$

We may replace the (1, 1) element by $13.1+24p^3-43px'^2$, since $4x'^2/3\leqq 1.85$ in this case. Evidently its principal diagonal minor determinants are positive with the exception of the determinant $\varDelta$ of this modified matrix. For $\varDelta$ we have

$$\varDelta=3133480.5+2350950p-3704400p^2+8841000p^3-1330875p^4$$
$$+(-14189805-4457250p)px'^2-952875x'^4.$$

If $1/10\leqq k^2\leqq 1/4$, then

$$\varDelta\geqq 3133480.5+2350950p-3704400p^2+5293548.75p^3-2504742.1875p^4$$

and the right hand side is positive for $1\leqq p\leqq 1.6$. This implies the positive definiteness of Q and hence we may omit it. Since $P(x)>0$ for $0\leqq x\leqq 1$ and $x>0$ in this case, we have $\Re a_6<6$ in this case. This is the desired result.

Case 3). We have $p<1.8$ when $1/4\leqq k^2\leqq 5-2\sqrt{5}$ and $p<1.91$ when $1/10\leqq k^2\leqq 1/4$. We start from (A) with $B=4$. Then

$$\Re a_6\leqq 6-\frac{15}{4}x-\frac{x^2}{960}(4320-5480x+2120x^2-287x^3)-X-\frac{5}{4}x'^2\eta-\frac{3}{2}x'y'y$$

$$-\frac{29}{8}px'^2y+\frac{1}{8}p^3y+5p^2(2-p)y-\frac{89p-128}{4}y^2-5y\eta,$$

where X is the same as in the above case. By Lemma 1 we have for $1.6\leqq p$

$$(-y)(2px'^2-5p^2(2-p))\leqq(-y)\left(2px'^2-\frac{5}{4}p^2x'^2\right)\leqq 0$$

and

$$(-y)\left(\frac{0.73}{2}x'y'-\frac{1}{8}p^3\right)\leqq -y\left(-\frac{p^3}{32}x'^2+\frac{0.73}{2}x'y'-\frac{3p^3}{32}y'^2\right)\leqq 0.$$

Hence by using Lemma 1 to $-15x/4$ we have

$$\Re a_6\leqq 6-\frac{x^2}{960}P(x)-X^*-\frac{5}{4}x'^2\eta-\frac{2.27}{2}x'y'y-\frac{13}{8}px'^2y$$

$$-\frac{1}{4}(89p-128)y^2-5y\eta-\frac{45}{16}y^2-\frac{75}{16}\eta^2,$$

$$P(x)=5220-5480x+2120x^2-287x^3,$$

$$X^*=X+\frac{15}{16}(x'^2+3y'^2+5\eta'^2+7\xi'^2).$$

We use the trivial inequality $2Auv\leqq|A|\alpha u^2+|A|v^2/\alpha$. Then consider the quadratic form of y and η, which is

$$-\frac{1}{16}[\{(356-13\delta)p-467-9.08\beta\}y^2+80y\eta+(75-10\alpha)\eta^2],\ \alpha>0,\ \beta>0,\ \delta>0.$$

Here we put $\alpha=2.5$, $\beta=2$, $\delta=2.5$. Then the above form is always negative for $p\geqq 1.6$ unless $y=\eta=0$. Thus we have

$$\Re a_6\leqq 6-\frac{x^2}{960}P(x)-Q,$$

$$Q=X^*-\frac{1}{4}x'^4-\frac{13}{40}px'^4-\frac{2.27}{8}x'^2y'^2.$$

We make the symmetric matrix associated with Q

$$\begin{pmatrix} 15+24p^3-19.2px'^2-4x'^2 & 29p^2-11x'^2 & 28p & 16 \\ 29p^2-11x'^2 & 45+28p-4.54x'^2 & 24 & 0 \\ 28p & 24 & 75 & 0 \\ 16 & 0 & 0 & 105 \end{pmatrix}.$$

Here we may replace the (1, 1) element by $15+24p^3-22px'^2$, since $4x'^2\leqq 2.5px'^2$ for $p\geqq 1.6$. Evidently we have the positivity of the principal diagonal minor determinants except the determinant of this modified matrix. For it we have

$$\Delta\equiv 3691881+2769900p-3704400p^2+8841000p^3-1330875p^4$$

$$+(-456994.5-8018010p+546982.8p^2-858060p^3)x'^2+(-952875+786555p)x'^4$$

$$>3691881+2769900p-3945693.096p^2+4607490.72p^3-1113265.1691p^4-453055.68p^5$$

for $1/4\leqq k^2\leqq 5-2\sqrt{5}<0.528$. The last polynomial is positive for $1.6\leqq p\leqq 1.8$.

When $1/10\leqq k^2\leqq 1/4$, we have

$$\Delta>3691881+2769900p-3818648.625p^2+6836497.5p^3-1203658.05p^4-206649.45p^5,$$

which is positive for $1.6 \leqq p < 1.91$.

In both cases Q is positive definite there. Further $P(x) > 0$ for $0 \leqq x \leqq 1$ and $x > 0$ in this case. Hence we have the desired result.

Case 4). We start from (A) with $B=4$ in this case. We have

$$-y\left(\frac{18.9}{8}px'^2 - 5p^2x\right) \leqq 0, -\frac{0.73}{2}x'y'y + \frac{1}{8}p^3y \leqq 0$$

for $1.5 \leqq p \leqq 1.6$ similarly. Thus we have

$$\Re a_6 \leqq 6 - \frac{x^2}{960}P(x) - Q(\alpha, \beta, \delta) - Y(\alpha, \beta, \delta),$$

$$P(x) = 5220 - 5480x + 2120x^2 - 287x^3,$$

$$Q(\alpha, \beta, \delta) = X + \frac{15}{16}(x'^2 + 3y'^2 + 5\eta'^2 + 7\xi'^2) - \frac{5}{8\alpha}x'^4 - \frac{2.27}{4\beta}x'^2y'^2 - \frac{10.1}{16\delta}px'^4,$$

$$16Y(\alpha, \beta, \delta) = \{4(89p - 128) - 9.08\beta - 10.1\delta p + 45\}y^2 + 80y\eta + (75 - 10\alpha)\eta^2,$$

$$\alpha > 0,\ \beta > 0,\ \delta > 0$$

with the same X as in the above case. Here we put $\alpha = 2.1$, $\beta = 1.5$, $\delta = 2$. Since $x^2P(x)$ is monotone increasing in $0.4 \leqq x \leqq 0.5$ and $P(0.4) = 3348.832$, we have

$$-\frac{x^2}{960}P(x) \leqq -\frac{0.16}{960}P(0.4) < -\frac{0.16}{960} \cdot \frac{3}{4} 3348.832y^2$$

by Lemma 1. Consider

$$Y(2.1, 1.5, 2) + \frac{0.16}{960} \cdot \frac{3}{4} 3348.832y^2$$

$$= \frac{1}{16}(6.697664 + 335.8p - 480.62)y^2 + 5y\eta + \frac{27}{8}\eta^2.$$

This is positive for $p \geqq 1.5$. Thus we have

$$\Re a_6 < 6 - Q, \qquad Q = Q\ (2.1,\ 1.5,\ 2).$$

We make the symmetric matrix associated with Q, which is

$$\begin{pmatrix} 15 + 24p^3 - 19.05px'^2 - \frac{10}{2.1}x'^2 & 29p^2 - 11x'^2 & 28p & 16 \\ 29p^2 - 11x'^2 & 45 + 28p - \frac{18.16}{3}x'^2 & 24 & 0 \\ 28p & 24 & 75 & 0 \\ 16 & 0 & 0 & 105 \end{pmatrix}.$$

We may replace the (1, 1) and (2, 2) elements by $15+24p^3-22.3px'^2$, $45+28p-6.1x'^2$, respectively, since $p\geqq 1.5$. Then this modified matrix is positive definite for $1.5\leqq p\leqq 1.6$, $1/4\leqq k^2\leqq 5-2\sqrt{5}<0.528$. This implies the positive definiteness of Q there and hence we arrived at the desired result.

Case 5). We start from (A) with $B=2$ in this case. Since

$$-y(12.6px'^2-20p^2x)\leqq 0,$$

$$-y\left(\frac{0.57}{2}x'y'-\frac{1}{8}p^3\right)\leqq -y\left(\frac{0.57}{2}\sqrt{5-2\sqrt{5}}\,p\cdot 1.005-\frac{p^3}{8}\right)\leqq 0$$

for $1.3\leqq p\leqq 1.5$, $1/3\leqq k^2\leqq 5-2\sqrt{5}$, we have

$$\Re a_6\leqq 6-\frac{x^2}{960}P(x)-Q(\alpha,\beta,\delta)-Y(\alpha,\beta,\delta),$$

$$Q(\alpha,\beta,\delta)=X+\frac{15}{16}(x'^2+3y'^2+5\eta'^2+7\xi'^2)-\frac{5}{8\alpha}x'^4-\frac{2.43}{4\beta}x'^2y'^2-\frac{16.4}{16\delta}px'^4,$$

$$16Y(\alpha,\beta,\delta)=(45+100p-128-9.72\beta-16.4p\delta)y^2+16y\eta+(75-10\alpha)\eta^2,$$

where α, β, δ are positive number and $P(x)$ and X are the same expressions as in Case 4, We put $\alpha=7.7$, $\beta=1$, $\delta=2$. Now $x^2P(x)$ is monotone increasing for $0.5\leqq x\leqq 0.7$. Hence we have by Lemma 1

$$-\frac{x^2}{960}P(x)\leqq -\frac{0.25}{960}2974.125$$

$$\leqq -\frac{2974.125}{960}\frac{1}{16}(p^2+3y^2+5\eta^2+x'^2+3y'^2+5\eta'^2+7\xi'^2).$$

Further again by Lemma 1 we have

$$-\frac{2974.125}{960}\frac{1}{16}p^2\leqq -\frac{2974}{960}\frac{1.69}{16}\leqq -\frac{2974}{960}\frac{1.69}{16}\frac{3}{4}y^2\leqq -\frac{3.9}{16}y^2$$

for $1.3\leqq p$. Now we have

$$Y(7.7,1,2)+\frac{2974.125}{960}\frac{1}{16}(3y^2+5\eta^2)+\frac{3.9}{16}y^2$$

$$\geqq \frac{1}{16}(7y^2+16y\eta+13\eta^2)\geqq 0.$$

Hence we have

$$\Re a_6<6-Q^*,\qquad Q^*=Q(7.7,1,2)+\frac{2974.125}{960}\frac{1}{16}(x'^2+3y'^2+5\eta'^2+7\xi'^2).$$

Now we consider $16Q^*$, which is just

$$\{15+24p^3-22.2px'^2-(10/7.7)x'^2\}x'^2+2(29p^2-11x'^2)x'y'+(45+28p-9.72x'^2)y'^2$$
$$+56px'\eta'+48y'\eta'+75\eta'^2+32x'\xi'+105\xi'^2+3.098046875(x'^2+3y'^2+5\eta'^2+7\xi'^2).$$

Since $0.8px'^2+10x'^2/7.7\leqq 3.098046875$ and $9.72x'^2\leqq 11.545173$ for $p\leqq 1.5$, $k^2\leqq 5-2\sqrt{5}$ <0.5279, we may replace $16Q^*$ by the following quadratic form Q_1:

$$Q_1=(15+24p^3-21.4px'^2)x'^2+2(29p^2-11x'^2)x'y'+(42+28p)y'^2+56px'\eta'$$
$$+48y'\eta'+80\eta'^2+32x'\xi'+126\xi'^2.$$

Consider the symmetric matrix associated with Q_1, introducing new variables $(x', y', 4\eta', 2\xi')$, which is

$$\begin{pmatrix} 15+24p^3-21.4px'^2 & 29p^2-11x'^2 & 7p & 8 \\ 29p^2-11x'^2 & 42+28p & 6 & 0 \\ 7p & 6 & 5 & 0 \\ 8 & 0 & 0 & 31.5 \end{pmatrix}$$

We may replace the (4, 4) element by 30. Denote this modified matrix by χ. The positivity of the principal diagonal minor determinants of χ is evident except the determinant of χ. This is

$$67164+54040p-61740p^2+157200p^3-25350p^4$$
$$+(-139428p+5820p^2)x'^2-18150x'^4,$$

which is positive for $1.3\leqq p\leqq 1.5$, $1/4\leqq k^2\leqq 5-2\sqrt{5}$. Hence we have the desired result: $\Re a_6<6$ for $1.3\leqq p\leqq 1.5$, $1/4\leqq k^2\leqq 5-2\sqrt{5}$.

Case 6). We start from (A) with $B=3/2$ in this case. Since

$$-y(15.3px'^2-15p^2x)\leqq -yp^2(15.3\cdot 1.3\cdot 0.5279-10.5)<0$$

and

$$-y(0.96x'y'-p^3)\leqq -py(0.96|y'|\sqrt{5-2\sqrt{5}}-p^2)\leqq 0$$

for $1\leqq p\leqq 1.3$, $1/4\leqq k^2\leqq 5-2\sqrt{5}$ (<0.5279), we have

$$\Re a_6\leqq 6-\frac{x^2}{960}P(x)-X^*-Y-\frac{5}{4}x'^2\eta-\frac{2.76}{2}x'y'y-\frac{13.7}{8}px'^2y,$$

$$X^*=X+\frac{15}{16}(x'^2+3y'^2+5\eta'^2+7\xi'^2),$$

$$16Y=(56p-27)y^2+75\eta^2,$$

with the same X and $P(x)$ as in Case 5. Now we consider the minimum of $x^2P(x)$ in $0.7\leqq x\leqq 1$. Then $x^2P(x)\geqq 0.7^2P(0.7)=1138.93591$. Further we have

$$-px'^2y\leqq p^3(5-2\sqrt{5})|y|\leqq 1.005\,(5-2\sqrt{5})1.3^3.$$

Hence

$$-\frac{x^2}{960}P(x)-px'^2y<-\frac{19}{960}<0.$$

Now by the trivial inequality we have

$$\Re a_6 \leqq 6 - \frac{19}{960} - Q(\alpha, \beta, \delta) - Z(\alpha, \beta, \delta),$$

$$Q(\alpha, \beta, \delta) = X^* - \frac{5}{8\alpha} x'^4 - \frac{2.76}{4\beta} x'^2 y'^2 - \frac{5.7}{16\delta} p x'^4,$$

$$Z(\alpha, \beta, \delta) = Y - \frac{2.76}{4} \beta y^2 - \frac{5.7}{16} p\delta y^2 - \frac{5}{8} \alpha \eta^2.$$

Here we put $\alpha=7.5$, $\beta=1.5$, $\delta=2$. Then

$$16Z(7.5, 1.5, 2) = (44.6p - 43.56) y^2 \geqq 0$$

for $p \geqq 1$. Hence for $1 \leqq p \leqq 1.3$, $1/4 \leqq k^2 \leqq 5 - 2\sqrt{5}$

$$\Re a_6 < 6 - Q, \qquad Q = Q(7.5, 1.5, 2).$$

We now make the symmetric matrix associated with Q

$$\begin{pmatrix} 15+24p^3-16.85px'^2-\frac{4}{3}x'^2 & 29p^2-11x'^2 & 28p & 16 \\ 29p^2-11x'^2 & 45+28p-\frac{22.08}{3}x'^2 & 24 & 0 \\ 28p & 24 & 75 & 0 \\ 16 & 0 & 0 & 105 \end{pmatrix}.$$

Here we may replace the (1, 1) and (2, 2) elements by $15+24p^3-18.2px'^2$, $45+28p-7.36x'^2$, respectively. Then the modified matrix is positive definite for $1 \leqq p \leqq 1.3$, $1/4 \leqq k^2 \leqq 5-2\sqrt{5}$ and hence Q is positive definite there. Thus we have the desired result: $\Re a_6 < 6$ for $1 \leqq p \leqq 1.3$, $1/4 \leqq k^2 \leqq 5-2\sqrt{5}$.

Summing up the results in this section we have

$$\Re a_6 < 6$$

for $1 \leqq p$, $1/10 \leqq k^2 \leqq 5-2\sqrt{5}$.

§8. In this section we shall be concerned with the case $0 \leqq p \leqq 1$, $k^2 \leqq 5-2\sqrt{5}$. We divide this case into several subcases: 1) $y \geqq 0$, $k^2 \leqq 1/29$, 2) $y \geqq 0$, $k^2 \geqq 1/29$, 3) $y \leqq 0$, $k^2 \leqq 1/29$, 4) $y \leqq 0$, $1/29 \leqq k^2$.

Case 1). Since $|y| < 1.005$, we have, by starting from (A) with $B=0$,

$$\Re a_6 \leqq 6 - P(p) - X - \frac{3}{2} x'y'y - \frac{5}{4} x'^2 \eta + \frac{7}{4} p y^2 + 3y\eta,$$

$$P(p) = 6 - \frac{2}{5} - \frac{2}{3} p^2 - \frac{11}{120} p^5 - \frac{25}{64} p^4 (2-p) - \frac{1.005}{8} p^3,$$

with the same X as in the above case. We consider $P(p)$. By a simple calculation we have

$$P(p)=6-\frac{1}{960}\psi(p),$$

$$\psi(p)=384+640p^2+120.6p^3+750p^4-287p^5.$$

It is easy to prove that $\psi(p)$ is monotone increasing for $0\leqq p\leqq 1$. Hence $\psi(p)\leqq\psi(1)=1607.6$ for $0\leqq p\leqq 1$. Therefore

$$P(p)\geqq 6-\frac{1607.6}{960}=4.32+\frac{13}{2400}$$

$$>1.08\,(p^2+x'^2+3y^2+3y'^2+5\eta^2+5\eta'^2+7\xi'^2).$$

Hence we have

$$\Re a_6<6-1.08p^2-X^*+\frac{3}{4\alpha}x'^2y'^2+\frac{3\alpha}{4}y^2+\frac{5}{8\beta}x'^4+\frac{5\beta}{8}\eta^2$$

$$+\frac{7}{4}py^2+3y\eta-3.24y^2-5.4\eta^2,$$

$$X^*=X+1.08(x'^2+3y'^2+5\eta'^2+7\xi'^2),\qquad \alpha>0,\ \beta>0.$$

Here we put $\alpha=1.3$, $\beta=1.4$. Then

$$(9.06-7p)y^2-12y\eta+18.1\eta^2\geqq 0$$

for $0\leqq p\leqq 1$. Hence we have

$$\Re a_6<6-1.08p^2-Q,\qquad Q=X^*-\frac{3}{5.2}(x'y')^2-\frac{5}{11.2}x'^4.$$

We make the symmetric matrix associated with Q

$$\begin{pmatrix} 17.28+24p^3-14px'^2-(10/1.4)x'^2 & 29p^2-11x'^2 & 28p & 16 \\ 29p^2-11x'^2 & 51.84+28p-(12/1.3)x'^2 & 24 & 0 \\ 28p & 24 & 86.4 & 0 \\ 16 & 0 & 0 & 120.96 \end{pmatrix}.$$

It is easy to prove that this matrix is positive definite for $0\leqq p\leqq 1$, $k^2\leqq 5-2\sqrt{5}<0.528$. This implies the desired result.

Case 2). In this case we have

$$p^3y-29px'^2y=29p^3\left(\frac{1}{29}-k^2\right)y\leqq 0.$$

Hence we may omit them in this case. Hence we can apply the above estimation in Case 1) and then we have the desired result.

Case 3). In this case again we have $p^3y-29px'^2y\leqq 0$. Therefore we can apply the estimation in Case 1) and then we have the desired result.

Case 4). Firstly we assume that $0.5\leqq p\leqq 1$. We start from (A) with $B=3/2$. Since

$$-py(28.4x'^2-15px)=-p^2y(28.4k^2p-15x)\leqq 0$$

for $p\leqq 1$, $k^2\leqq 5-2\sqrt{5}<0.528$, we have

$$\Re a_6\leqq 6-P(p)-X-\frac{3}{2}x'y'y+\frac{9-7p}{2}y^2-\frac{5}{4}x'^2\eta+\frac{1}{8}p^3y-\frac{0.6}{8}px'^2y,$$

$$P(p)=6-\frac{2}{5}-\frac{2}{3}p^2-\frac{11}{120}p^5-\frac{25}{64}p^4(2-p),$$

with the same X as in the above case. Further

$$p^3y-0.6px'^2y\leqq -yp^3(0.6\cdot 0.528-1)<0$$

for $k^2\leqq 5-2\sqrt{5}<0.528$. Since $P(p)$ is monotone decreasing for $0\leqq p\leqq 1$,

$$P(p)\geqq P(1)=4.451041\dot{6}.$$

And further

$$|(3/2)x'y'y|\leqq (3/4)|x'|(y^2+y'^2)\leqq (1/4)(0.528)^{1/2}(4-p^2)\leqq 0.6816.$$

Hence we have

$$\begin{aligned}\Re a_6&<6-3.7694-X+\frac{5}{8\beta}x'^4+\frac{1}{2}(9-7p)y^2+\frac{5\beta}{8}\eta^2\\ &<6-\frac{3.76}{4}p^2-Q(\beta)+\frac{1}{2}(9-7p-5.64)y^2-\frac{1}{8}(37.6-5\beta)\eta^2,\end{aligned}$$

$$Q(\beta)=X+\frac{3.76}{4}(x'^2+3y'^2+5\eta'^2+7\xi'^2)-\frac{5}{8\beta}x'^4.$$

When $0.5\leqq p\leqq 1$, the coefficient of y^2 is negative and further the coefficient of η^2 is negative by putting $\beta=7.5$. Hence we have

$$\Re a_6<6-Q,\qquad Q=Q(7.5).$$

We consider the symmetric matrix associated with Q

$$\begin{pmatrix} 15.04+24p^3-14px'^2-(10/7.5)x'^2 & 29p^2-11x'^2 & 28p & 16\\ 29p^2-11x'^2 & 45.12+28p & 24 & 0\\ 28p & 24 & 75.2 & 0\\ 16 & 0 & 0 & 105.28\end{pmatrix}.$$

Again it is easy to prove the positive definiteness of this matrix and hence of Q. This implies the desired result.

When $0 \leqq p \leqq 0.5$, $k^2 \leqq 5-2\sqrt{5}$, we start from (A) with $B=0$. In this case we have

$$-\frac{29}{8}px'^2y \leqq 1.005 \cdot \frac{29}{8} \cdot 0.528 \cdot \frac{1}{8} < 0.24045,$$

$$\left| -\frac{3}{2}x'y'y \right| \leqq \frac{3}{4}|x'|(y^2+y'^2) \leqq \frac{4-p^2}{4}|k|p \leqq 0.528^{1/2} \cdot 0.5 < 0.3635.$$

Hence

$$\Re a_6 \leqq 6-P(p)-X+\frac{7}{4}py^2+3y\eta-\frac{5}{4}x'^2\eta+0.60395.$$

Since $P(p) \geqq P(0.5)$ for $0 \leqq p \leqq 0.5$ and $P(0.5) \geqq 5.3938$, we have

$$\Re a_6 < 6-4.78-X+\frac{7}{4}py^2+3y\eta-\frac{5}{4}x'^2$$

$$\leqq 6-Q(\alpha)-Y(\alpha)-\frac{4.78}{4}p^2,$$

$$Q(\alpha)=X+\frac{4.78}{4}(x'^2+3y'^2+5\eta'^2+7\xi'^2)-\frac{5}{8\alpha}x'^4,$$

$$4Y(\alpha)=(14.34-7p)y^2-12y\eta+(23.9-2.5\alpha)\eta^2.$$

Here we put $\alpha=8$. Then

$$4Y(8)=(14.34-7p)y^2-12y\eta+3.9\eta^2 \geqq 0$$

for $0 \leqq p \leqq 0.5$. Hence we have

$$\Re a_6 < 6-Q, \qquad Q=Q(8).$$

It is very easy to prove the positive definiteness of Q. Thus we have the desired result.

Thus we have completed our proof of $\Re a_6 \leqq 6$ for $|a_2| \leqq 2$, $k^2 \leqq 5-2\sqrt{5}$ with equality holding only for the Koebe function $z/(1-z)^2$. Thus we have the affirmative answer to the Bieberbach conjecture for the sixth coefficient.

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