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ON THE SOLUTION OF THE FUNCTIONAL EQUATION $f\circ g(z)=F(z)$, V

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In our previous paper we discussed the transcendental unsolvability of the functional equation $f\circ g(z)=F(z)$. In this note we shall extend some results in [4] to a more general class of functions and make use of the same terminology "transcendental solvability". Our basic tool is an elegant theorem of Edrei-Fuchs [2].

THEOREM 1. *Let $f(z)$ be an entire function of the form $P(z)e^{M(z)}$ with a polynomial $P(z)$. Assume that there exist two constants a, b such that $|a|\neq|b|$, $ab\neq 0$ and that $f(z)=a$ and $f(z)=b$ have their solutions on p straight lines $l_1, \cdots, l_p$, almost all, any two of which are not parallel with each other. Then $f(z)$ reduces to a polynomial.*

Proof. By Edrei-Fuchs' theorem in [2] $f(z)$ must be of finite order and hence $M(z)$ must be a polynomial. Denote it by

$$\alpha_n z^n+\alpha_{n-1}z^{n-1}+\cdots+\alpha_1 z+\alpha_0, \qquad \alpha_n\neq 0.$$

By a suitable change of variable we have

$$M(z)=z^n+\alpha_{n-2}z^{n-2}+\cdots+\alpha_1 z+\alpha_0$$

with new α_j. Hence our problem reduces to solve the following equation

$$(A_m z^m+\cdots+A_0)\exp(z^n+\alpha_{n-2}z^{n-2}+\cdots+\alpha_0)=a.$$

We have asymptotically

$$z^n\left(1+O\left(\frac{1}{z^2}\right)\right)=\log\frac{a}{A_m e^{\alpha_0}}+2q\pi i.$$

Hence the given p straight lines $l_1, \cdots, l_p$ must be parallel to one of

$$\arg z=\pm\frac{\pi}{2n}+\frac{2s}{n}\pi, \qquad s=0, \cdots, n-1,$$

respectively. Assume that l_1 is parallel to a radius given by

$$Re^{\pi i/2n}.$$

Then l_1 can be represented as $x_0+R\exp(i\pi/2n)$ with a real x_0. Let

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$$X_0+iY=\log\frac{a}{A_m e^{\alpha_0}}+2q\pi i$$

with real numbers X_0, Y. Then

$$(x_0+Re^{\pi i/2n})^n\left(1+O\left(\frac{1}{R^2}\right)\right)=X_0+iY.$$

Taking the real part, we have

$$nR^{n-1}x_0\mathcal{R}e^{(n-1)\pi i/2n}\left(1+O\left(\frac{1}{R}\right)\right)=X_0$$

This implies that $x_0=0$ and hence $X_0=0$. Therefore

$$\log\left|\frac{a}{A_m e^{\alpha_0}}\right|=X_0=0,$$

which shows that

$$|a|=|A_m e^{\alpha_0}|.$$

The same holds for each l_j. By the same procedure we have

$$|b|=|A_m e^{\alpha_0}|.$$

This is a contradiction. Therefore $M(z)$ must be a constant.

This theorem suggests the following conjecture: Let $f(z)$ be an entire function. Assume that there is a sequence $\{a_n\}$ such that $a_n\to\infty$ as $n\to\infty$ and that almost all the roots of $f(z)=a_n$ lie on p straight lines $l_1, \cdots, l_p$, any two of which are not paralled with each other. Then $f(z)$ reduces to a polynomial of degree at most $2p$.

Edrei [1] proved this conjecture, when $p=1$. By Edrei-Fuchs' theorem in [2] we can say that $f(z)$ is of finite order.

LEMMA 1. *Let $f(z)$ be an entire function of the form $P(z)e^{M(z)}$ with a polynomial $P(z)$ and a non-constant entire function $M(z)$. If there are p straight lines $l_1, \cdots, l_p$, any two of which are not parallel with each other, such that almost all roots of $f(z)=a$, $a\neq 0$, lie on $l_1, \cdots, l_p$, then $P(z)$ reduces to a constant and $M(z)=\alpha(z-z_0)^n+\beta$ for some z_0 and some positive integer n.*

Proof. By the proof of theorem 1 we have

$$z^n+\alpha_{n-2}z^{n-2}+\cdots+\alpha_1 z+m\log z\left(1+O\left(\frac{1}{z}\right)\right)=(2q\pi+y_0)i,$$

assuming $A_m\neq 0$. Here put

$$z=Re^{\pi i/2n}.$$

Assuming $\alpha_{n-2}\neq 0$ and taking the real part of both sides,

$$R^{n-2}\mathcal{R}\alpha_{n-2}e^{(n-2)\pi i/2n}\left(1+O\left(\frac{1}{R}\right)\right)=0,$$

which implies that

$$\mathcal{R}\alpha_{n-2}e^{(n-2)\pi i/2n}=0.$$

Similarly we have

$$\mathcal{R}\alpha_{n-2}e^{-(n-2)\pi i/2n}=0.$$

Hence

$$\cos(\beta+(n-2)\pi/2n)=\cos(\beta-(n-2)\pi/2n)=0,$$

which is clearly untenable, unless $n=2$. Here β is an argument of α_{n-2}. Hence $\alpha_{n-2}=0$. The same holds for each α_j, $1\leqq j\leqq n-2$. Now we have

$$z^n+m\log z\left(1+O\left(\frac{1}{z}\right)\right)=(2q\pi+y_0)i.$$

Taking the real part, we have

$$m\log R\left(1+O\left(\frac{1}{R}\right)\right)=0,$$

which shows a contradiction. Hence $A_m=0$. The same holds for each A_j, $1\leqq j\leqq m$. Thus we have the desired result.

THEOREM 2. *Let $F(z)$ be a meromorphic function whose image covers the Riemann sphere. Assume that ∞ is a Picard exceptional value of F and almost all the roots of $F(z)=A$ lie on p straight lines $\{l_j\}$, any two of which are not parallel with each other. Then the functional equation $f\circ g(z)=F(z)$ is not transcendentally solvable.*

Proof. Evidently we have

$$f(w)=(w-w_1)^n f^*(w),\qquad g(z)=w_1+P(z)e^{M(z)}$$

with a polynomial P, entire functions $f^*(w)$ and $M(z)$ and a negative integer n. By the assumption there is at least one solution w_2 of $f(w)=A$. Further $g(z)=w_2$ has solutions lying on $\{l_j\}$ almost all. Since $g(z)$ is transcendental, $P(z)$ must be a constant by Lemma 1. Then $F(z)$ has the form

$$F(z)=f\circ g(z)=C^n e^{nM(z)}f^*\circ(w_1+Ce^{M(z)})$$

with a constant C. This shows that $F(z)$ is an entire function. This contradicts the assumption.

LEMMA 2. *Let $f(z)$ be an entire function of the form $P(z)e^{M(z)}$ with polynomials $P(z)$ and $M(z)$. If there are p straight lines $l_1,\cdots,l_p$ such that almost all roots of $f(z)=a$, $a\neq 0$, lie on $l_1,\cdots,l_p$, then $P(z)$ reduces to a constant unless $M(z)$ is a constant.*

Proof. By the proof of theorem 1 there are $2n$ directions along which almost all a-points of $f(z)$ lie and they must start from a suitable point z_0. Then by Lemma 1 $P(z)$ must be a constant.

In the sequel ρ_G means the order of G.

THEOREM 3. *Let $F(z)$ be a meromorphic function of finite order, whose image covers the Riemann sphere. Assume that ∞ is a Picard exceptional value of F and almost all the zeros of $F(z)$ lie on p straight lines $l_1, \cdots, l_p$. Then the functional equation $f\circ g(z)=F(z)$ is not transcendentally solvable.*

Proof. Firstly we have

$$f(w)=(w-w_1)^{-n}f^*(w) \qquad \text{and} \qquad g(z)=w_1+P(z)e^{M(z)}$$

with a polynomial P and two entire functions f^* and M and a positive integer n. Hence

$$F(z)=P(z)^{-n}e^{-nM(z)}f^*\circ(w_1+P(z)e^{M(z)}).$$

By the order finiteness of $F(z)$ we have that the order of

$$G(z)=e^{-nM(z)}f^*\circ(w_1+P(z)e^{M(z)})$$

is finite and further $f^*(w)$ is transcendental. It is easy to prove that

$$\rho_G=\rho_{f^*\circ g}.$$

Hence

$$\rho_G=\rho_F<\infty$$

implies that $g(z)$ is an entire function of finite order and $f^*(w)$ is an entire function of order zero. Hence $M(z)$ must be a polynomial. By Lemma 2 we have the constancy of $P(z)$, which again implies that $F(z)$ must be an entire function. This is clearly a contradiction.

THEOREM 4. *Let $F(z)$ be a meromorphic function whose image covers the Riemann sphere. Assume that ∞ is a Picard exceptional value of F and almost all the zeros of $F(z)$ lie on p straight lines and further the order of $N(r;\, 0, F)$ is finite. Then the functional equation $f\circ g(z)=F(z)$ is not transcendentally solvable.*

Proof. By a similar consideration as in theorem 3 we have

$$F(z)=P(z)^{-n}e^{-nM(z)}f^*\circ(w_1+P(z)e^{M(z)}).$$

If $f^*(w)=0$ has at least two roots w_2, w_3, we have

$$N(r;\, 0, f^*\circ g)\geqq m(r, Pe^M)-O(\log rm\,(r, Pe^M))$$

by the second fundamental theorem. If $f^*(w)=0$ has only one root w_2, we have

$$f^*(w)=(w-w_2)^s e^{L(w)}$$

and hence

$$N(r;\ 0, f^*\circ g)\sim sm(r, Pe^M).$$

In both cases we have

$$\rho_{N(r;0,F)}\geqq \rho_{e^M},$$

which implies the order finiteness of $g(z)=w_1+P(z)e^{M(z)}$. As in theorem 3 we have the desired result.

In the sequel we make use of the notation $\hat{\rho}_f$ as the hyper-order of f.

THEOREM 5. *Let $F(z)$ be a meromorphic function satisfying $\hat{\rho}_{F'}<p$. Assume that 0 is a Picard exceptional value of F' and almost all the poles of F' lie on p straight lines $l_1, \cdots, l_p$, any two of which are not parallel with each other and each of which carries an infinite number of poles of F'. Further assume that the image of F' covers the Riemann sphere. Then the functional equation $f\circ g(z)=F(z)$ is not transcendentally solvable.*

Proof. Consider the derived functional equation $f'\circ g(z)\cdot g'(z)=F'(z)$. Evidently $f'(w)=(w-w_1)^n/f^*(w)$ and $g(z)=w_1+P(z)e^{M(z)}$ with two entire functions f^*, M, a polynomial P and a positive integer n. If $f^*(w)$ has an infinite number of zeros $\{w_k^*\}$, almost all the solutions of all the equations $g(z)=w_k^*$, $k=1, 2, \cdots$, lie on the given p straight lines. Then theorem 1 implies that $g(z)$ must be a polynomial. Therefore $f^*(w)$ has only a finite number of zeros. Hence $f^*(w)=Q(w)e^{L(w)}$ with a polynomial Q and an entire function L. This implies that the lower order $\lambda_{f'}$ of f^* is not less than 1. By Lemma 1 we further have that $M(z)=\alpha(z-z_0)^n+\beta$ and $P(z)$ is a constant. Here n must be p by the assumption and by the proof of theorem 1 and Lemma 1. Hence $\rho_g=p$. Now by our earlier result in [3] we have

$$\hat{\rho}_{F'}\geqq \rho_g=p,$$

which contradicts $\hat{\rho}_{F'}<p$.

THEOREM 6. *Let $F(z)$ be a meromorphic function satisfying $\hat{\rho}_{F'}<0$. Assume that 0 is a Picard exceptional value of F' and almost all the poles of F' lie on p straight lines. Further assume that the image of F' covers the Riemann sphere. Then the functional equation $f\circ g(z)=F(z)$ is not transcendentally solvable.*

Proof. Evidently we have $f'(w)=(w-w_1)^n/f^*(w)$ and $g(z)=w_1+P(z)e^{M(z)}$ $=Q(z)e^{N(z)}$ with entire functions f^*, M, N, polynomials P, Q and a positive integer n.

We assume, firstly, that $f^*(w)=0$ has an infinite number of roots. By its representations

$$(P'+PM')e^M=Qe^N,$$

which implies that

$$P'+PM'=Qe^{H}$$

for an entire function H. Firstly we shall consider the case that H is not a constant. In this case

$$M=\int^{z}\frac{Qe^{H}-P'}{P}dz+C, \qquad M+H+D=N$$

with constants C and D. Hence

$$F'=\frac{P^{n}e^{(n+1)M+H+D}Q}{f^{*}\circ(w_{1}+Pe^{M})}.$$

By Pólya's method

$$M_{f^{*}\circ g}(r)\geqq M_{f^{*}}\circ\left(dM_{g}\left(\frac{r}{1}\right)\right)\geqq d^{K}M_{g}\left(\frac{r}{2}\right)^{K}$$

for a constant d, $0<d<1$, and for every positive constant K, and hence

$$\hat{\lambda}_{f^{*}\circ g}\geqq\hat{\lambda}_{g},$$

where $\hat{\lambda}_{g}$ indicates the lower hyper-order of g. By its form and by Pólya's method we can easily prove that

$$\hat{\lambda}_{g}\geqq 1.$$

Further

$$T(r, F')=m(r, 1/F')+N(r;\ 0, F')+O(\log r)$$
$$=m(r, 1/F')+O(\log r)$$

and

$$m(r, f^{*}\circ g)\leqq m(r, 1/F')+(n+1)m(r, g)+m(r, e^{H})+O(\log r)$$
$$\leqq m(r, 1/F')+(n+2)m(r, g)+O(\log r).$$

Hence

$$m(r, f^{*}\circ g)-(n+2)m(r, g)\leqq T(r, F')+O(\log r).$$

Let w_{j}^{*}, $j=1, 2, \cdots$, be an infinite number of zeros of $f^{*}(w)$. By the second fundamental theorem

$$m(r, f^{*}\circ g)\geqq N(r;\ 0, f^{*}\circ g)\geqq\sum_{1}^{K+1}N(r;\ w_{j}, g)$$
$$\geqq Km(r, g)-O(\log rm(r, g))$$

with a negligible exceptional set of r and for an arbitrary large K. Hence

$$m(r, f^{*}\circ g)-(n+2)m(r, g)\geqq K'm(r, g)$$

with a negligible exceptional set of r. Hence

$$\hat{\rho}_{F'}\geqq\hat{\lambda}_g\geqq 1.$$

This contradicts the assumption. Therefore H reduces to a constant. Thus M must be a polynomial. In this case theorem 1 does work without any assumption on the situation of p straight lines. Then we can easily conclude that $g(z)$ is a polynomial. This is clearly untenable.

Now we shall consider the case that $f^*(w)=0$ has only a finite number of roots. In this case we have

$$f^*(w)=R(w)e^{L(w)}$$

with a polynomial R and an entire function L and hence

$$F'=\frac{P^n e^{nM} Q e^N}{R\circ(w_1+Pe^M)\cdot e^{L\circ(w_1+Pe^M)}}.$$

Let s be the degree of R. Then

$$T(r, F')\geqq N(r;\ \infty, F')=sm(r, Pe^M)+O(\log r)=sm(r, g)+O(\log r).$$

This implies that

$$1>\hat{\rho}_{F'}\geqq\hat{\rho}_g.$$

Next we want to prove that for an arbitrary positive K there is a sequence $\{r_n\}$ ($r_n\to\infty$ as $n\to\infty$) of r such

$$Ae^{m(r/4,g)}>Km(r, g)$$

through $\{r_n\}$. If not, for $r\geqq r_0$ there is a constant K_0 such that

$$Ae^{m(r/4,g)}\leqq K_0 m(r, g).$$

This implies that

$$\infty=\varliminf_{r\to\infty}\frac{m(r/4, g)+\log A}{\log r}\leqq\varliminf_{r\to\infty}\frac{\log m(r, g)+\log K_0}{\log r}$$

and hence

$$\infty=\varliminf_{r\to\infty}\frac{\log[m(r/4, g)+\log A]}{\log r}$$

$$\leqq\varliminf_{r\to\infty}\frac{\log[\log m(r, g)+\log K_0]}{\log r}=\hat{\lambda}_g\leqq\hat{\rho}_g.$$

This contradicts $\hat{\rho}_g<1$.

By Pólya's method

$$m(r, f^*\circ g) \geqq \frac{1}{3}\log M_{f^*\circ g}\left(\frac{r}{2}\right) \geqq \frac{1}{3}\log M_{f^*\circ}\left(dM_g\left(\frac{r}{4}\right)\right) \qquad (0<d<1)$$

$$\geqq \frac{1}{3}dM_g\left(\frac{r}{4}\right) \geqq \frac{d}{3}e^{2\pi m(r/4, g)}$$

Hence we have

$$T(r, F')+O(\log r) \geqq m(r, f^*\circ g)-(n+2)m(r, g)$$

$$\geqq Bm(r, f^*\circ g) \qquad (0<B<1)$$

through $\{r_n\}$. In this case it is not matter whether H is a constant or not. This implies that $\hat{\rho}_{F'} \geqq \hat{\lambda}_{f^*\circ g}$. Since $\lambda_{f^*} \geqq 1$, we, further, have $\hat{\lambda}_{f^*\circ g} \geqq \lambda_g \geqq 1$. We now arrived at a contradiction.

In the sequel we use the notation

$$\hat{\rho}_F = \overline{\lim_{r\to\infty}}\frac{\log\log\log T(r, F)}{\log r}.$$

THEOREM 7. *Let $F'(z)$ be the derived function of a meromorphic function $F(z)$. Assume that ∞ is a Picard exceptional value of F', which has at least one pole, and almost all the zeros of F' lie on p straight lines, any two of which are not parallel with each other. Assume further that either $\hat{\rho}_{F'} < \rho_{N(r; 0, F')}$ or $0 < \rho_{N(r; 0, F')}$, $\hat{\rho}_{F'} < \infty$. Then the functional equation $f\circ g(z)=F(z)$ is not transcendentally solvable.*

Proof. Evidently we have $f'(w)=(w-w_1)^{-n}f^*(w)$, $g(z)=w_1+P(z)e^{M(z)}$ with a polynomial P, two entire functions f^*, M and a positive integer n. If $f^*(w)$ has at least one zero w_2, $g(z)=w_2$ has its roots on the given p straight lines almost all. Hence by Lemma 1 $P(z)$ must be a constant and then F' is reduced to an entire function, which is clearly a contradiction. Hence $f^*(w)$ has no zero. This implies that

$$f^*(w)=e^{L(w)}$$

and

$$F'(z)=P(z)^{-n}e^{-nM(z)}e^{L\circ(w_1+P(z)e^{M(z)})}(P'(z)+P(z)M'(z))e^{M(z)}.$$

In both cases we assumed that

$$0<\rho_{N(r; 0, F')}.$$

Hence

$$0<\rho_{N(r; 0, F')}=\rho_{N(r; 0, P'+PM')}$$

$$\leqq \rho_{M'}=\rho_M.$$

This implies that

$$\rho_{e^M}=\infty \qquad \text{and} \qquad \hat{\rho}_{e^M} \geqq \rho_M.$$

Therefore we have

$$\hat{\rho}_{F'} \geqq \rho_{e^M} = \infty$$

and

$$\hat{\rho}_{F'} \geqq \hat{\rho}_{e^M} \geqq \rho_M.$$

The latter inequalities imply an absurdity relation

$$\hat{\rho}_{F'} \geqq \rho_{N(r;0,F')}.$$

Thus we have the desired result.

REFERENCES

[1] EDREI, A., Meromorphic functions with three radially distributed values. Trans. Amer. Math. Soc. **78** (1955), 276–293.

[2] EDREI, A. AND W. H. J. FUCHS, On meromorphic functions with regions free of poles and zeros. Acta Math. **108** (1962), 113–145.

[3] OZAWA, M., On the solution of the functional equation $f\circ g(z)=F(z)$, III. Kōdai Math. Sem. Rep. **20** (1968), 257–263.

[4] OZAWA, M., On the solution of the functional equation $f\circ g(z)=F(z)$, IV. Kōdai Math. Sem. Rep. **20** (1968), 272–278.

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