KŌDAI MATH. SEM. REP.
20 (1967), 118–124

AN ASPECT OF LOGISTIC LAW

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1. Introduction.

In social or economic phenomena, it is empirically known that certain variables such as human populations increase monotonously or *grow* with time, obeying a certain law, the logistic law of growth. Few basic considerations, however, have been made from the theoretical point of view. (See for example Feller [1]). For investigating the logistic law more firmly it seems to the authors that the growing variables should be taken into account in correlation with suitable parameters.

Suggested by what we have obtained while dealing with actual statistical data, we shall give, in this paper, introducing an important notion of *translatability*, some concrete bases for the use of logistic law, which indicate a direction to the basic interpretation of the growing phenomena.

2. The logistic curve.

A variable $p(t)$, measured in an appropriate unit, will be called to obey logistic law of growth or simply logistic law, if

$$p(t)=\frac{1}{1+\alpha e^{-\beta t}}$$

for arbitrary positive constants α and β, where t $(-\infty<t<\infty)$ denotes time.

Note here that the growing variable $p(t)$ does not always mean a relative frequency or a probability. Note also that $p''(t_0)=0, p(t_0)=1/2$ for $t=t_0$ such that $\alpha e^{-\beta t_0}=1$.

3. A theoretical model.

Suppose there exists a parameter m, $m>0$, which has at any specified time t a correlation with a variable p, $p>0$, so that the relation between m and p can be represented by a regression line:

$$(1) \qquad p=a(t)m, \qquad 0<a(t)<1,$$

where $a(t)$ is a non-constant differentiable time function which is the gradient of the line at time t, and satisfies that $a(t_1)<a(t_2)$ if $t_1<t_2$.

Received September 14, 1967.

On the other hand for m_0 fixed arbitrarily such that $m_0>0$, the m and p will be assumed to be related linearly (in the statistical sense) as follows:

$$(1-m_0)p=m-m_0. \tag{2}$$

The above model may be as in Figure 1.

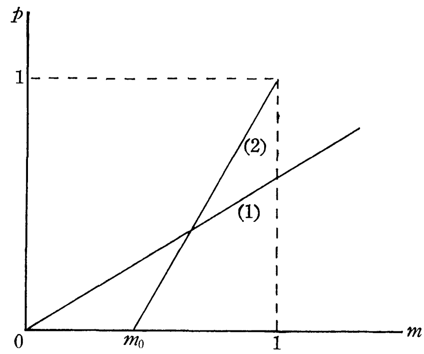

Figure 1.

4. Translatability assumption and a derivation of logistic curve.

If we define p as $p(t, m_0)$ after the elimination of m from equations (1) and (2) defined above, then we have

$$p(t, m_0)=\frac{m_0}{m_0-1+\frac{1}{a(t)}} \tag{3}$$

which may be considered as a time function with parameter m_0. Note that $p(t, m_1)>p(t, m_2)$ for any t if $m_1>m_2$.

DEFINITION. If, for any given $m_1>0$ and $m_2>0$, there exists a real number τ depending on m_1 and m_2 such that

$$p(t+\tau, m_2)=p(t, m_1) \qquad \text{for all} \quad t,$$

then $p(t, m_0)$ is termed *translatable with respect to* m_0.

THEOREM 1. *If $p(t, m_0)$ in* (3) *is translatable with respect to m_0, then it obeys logistic law. More specifically we have*

$$p(t, m_0)=\frac{1}{1+\frac{\alpha}{m_0}e^{-\beta t}}$$

for any $m_0>0$ *and for arbitrary constants* $\alpha>0$ *and* $\beta>0$.

Proof. By translatability assumption there exists a number $\tau(m_0, \mu)>0$ for $\mu>0$ such that

$$p(t+\tau(m_0, \mu), m_0)=p(t, m_0+\mu).$$

Hence we have

$$\frac{m_0}{m_0-1+\dfrac{1}{a(t+\tau(m_0, \mu))}}=\frac{m_0+\mu}{m_0+\mu-1+\dfrac{1}{a(t)}}.$$

Therefore

$$(4) \qquad m_0\cdot\frac{\tau(m_0, \mu)}{\mu}\left\{\frac{\dfrac{1}{a(t+\tau(m_0, \mu))}-\dfrac{1}{a(t)}}{\tau(m_0, \mu)}\right\}+\frac{1}{a(t+\tau(m_0, \mu))}-1=0.$$

We easily see that

$$(5) \qquad \lim_{\mu\to 0}\frac{\tau(m_0, \mu)}{\mu}=A(m_0)>0$$

for some finite positive constant $A(m_0)$ depending on m_0. If we let $\mu\to 0$, then $\tau(m_0, \mu)\to 0$, hence by (4)

$$m_0A(m_0)b'(t)+b(t)-1=0,$$

where we put $b(t)=1/a(t)$.

The solution $b(t)$ will be readily obtained as

$$b(t)=\alpha e^{-t/m_0A(m_0)}+1$$

for arbitrary positive constant α.

Hence we have

$$(6) \qquad a(t)=\frac{1}{1+\alpha e^{-t/m_0A(m_0)}},$$

$$(7) \qquad p(t, m_0)=\frac{1}{1+\dfrac{\alpha}{m_0}e^{-t/m_0A(m_0)}}.$$

We next show that $m_0A(m_0)$=constant when considered as a function of m_0, otherwise by (6) $a(t)$ becomes to depend on parameter m_0. For this purpose fix $\tilde{p}$ arbitrarily such that $0<\tilde{p}<1$, then by translatability assumption we have

$$\lim_{\mu\to 0}\frac{t_2-t_1}{\mu}=\lim_{\mu\to 0}\frac{\tau(m_0, \mu)}{\mu}=A(m_0),$$

where $p(t_2, m_0)=\tilde{p}$ and $p(t_1, m_0+\mu)=\tilde{p}$, both of which may be transformed by (7) into

$$t_2=-m_0A(m_0)\log\left\{\frac{(1-\tilde{p})m_0}{\alpha\tilde{p}}\right\},$$

$$t_1=-(m_0+\mu)A(m_0+\mu)\log\left\{\frac{(1-\tilde{p})(m_0+\mu)}{\alpha\tilde{p}}\right\}.$$

Hence we easily see

$$\lim_{\mu\to 0}\frac{t_2-t_1}{\mu}=\left[\frac{d}{dz}\left\{zA(z)\log\left(\frac{1-\tilde{p}}{\alpha\tilde{p}}z\right)\right\}\right]_{z=m_0}=A(m_0),$$

which can be simplified to obtain

$$m_0A'(m_0)+A(m_0)=0,$$

therefore

$$\frac{d}{dm_0}\{m_0A(m_0)\}=0,$$

hence

$$m_0A(m_0)=\gamma \quad \text{(constant)}.$$

In conclusion

(8) $$a(t)=\frac{1}{1+\alpha e^{-\beta t}},$$

(9) $$p(t, m_0)=\frac{1}{1+\dfrac{\alpha}{m_0}e^{-\beta t}}$$

for positive constants α and $\beta=1/\gamma$.

5. Translatability property.

About (9) it is easily verified that for any $m_1>0$ and $m_2>0$

$$p(t+\tau, m_2)=p(t, m_1), \qquad \tau=\gamma\log\left(\frac{m_1}{m_2}\right).$$

Fix $m_0=m_0^*$ and τ. Define $m_0(\nu)$ as $\nu\tau=\gamma\log(m_0(\nu)/m_0^*)$, then $m_0(\nu)=(e^{\tau/\gamma})^{\nu}\cdot m_0^*$, $\nu=\cdots, -1, 0, 1, \cdots$. The sequence $\{m_0(\nu)\}$ is the sequence of m_0's which give a unit time delay τ for the logistic curves

The translatability property has convenient advantages. For instance, in predicting $p(t, m_0)$ for future time t, we sometimes encounter with an estimation of two constants α and γ, when the available data are only for short time interval. In this situation, data for p for various m_0 may be translated properly along the time axis to obtain the more precise and widely ranged (in time) statistical curve for $p(t, m_0)$, therefore the more information about α and γ.

Note the following. Since

$$A(m_0)=\lim_{\mu\to 0}\frac{\tau(m_0, \mu)}{\mu}=\left[\frac{\partial}{\partial\mu}\tau(m_0, \mu)\right]_{\mu=0},$$

this quantity indicates *translating velocity* of $p(t, m_0)$ at m_0, which has the form $A(m_0)=\gamma/m_0$ for positive constant γ.

6. Necessary and sufficient condition, and a simple method for estimating constants.

On the basis of our model discussed in the preceding sections, we shall show a necessary and sufficient condition for the growing variable $p(t)$ to obey logistic law, and also show using this condition a simple method for estimating constants involved in the logistic curve equation.

Generally the variable $p(t)$ obeying the logistic law has the following form:

$$p(t)=\frac{c}{1+\alpha e^{-\beta t}},$$

where c, α, and β are assumed to be unknown positive constants. In the sections preceded, as the constant c was assumed to be known, we put $c=1$ without loss of generality.

Suppose we have, like in actual situations, samples $p(t)$ for $t=0, 1, \cdots, T$ $(T\geqq 1)$ (e.g. t indicates month or year).

In a (m, p)-plane, draw the following lines, (where m is an abstract parameter) like in Figure 2:

(10) $$p=p(t)m \qquad (m>0),$$

(11) $$p=p(t+1), \qquad t=0, 1, \cdots, T.$$

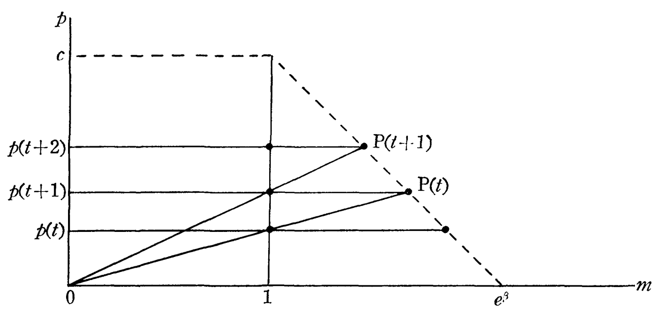

Figure 2

Let the points of intersections of (10) and (11) be denoted by P(t), then

$$\mathrm{P}(t)=\left(\frac{p(t+1)}{p(t)},\ p(t+1)\right), \qquad t=0, 1, \cdots, T.$$

We have

THEOREM 2. *For a positive growing variable, i.e. strictly monotonously increasing positive variable $p(t)$, a necessary and sufficient condition for $p(t)$ to obey logistic law, i.e. $p(t)=c/(1+\alpha e^{-\beta t})$ $(t=0, 1, \cdots, T)$, is that the point sequence* {P(t)} *is on a straight line: $p=-Am+B$ for some constants A, B and $A>0$.*

Proof. Necessity is easily verified, and indeed {P(t)} is on the line:

$$p=-\frac{c}{e^{\beta}-1}m+\frac{ce^{\beta}}{e^{\beta}-1}=\frac{c}{e^{\beta}-1}(e^{\beta}-m). \tag{12}$$

Now if {P(t)} is on a straight line $p=-Am+B$ with a negative gradient $-A$, then $B>0$, since $p(t)>0$, and $p(t+1)=-A[p(t+1)/p(t)]+B$. Hence, putting $1/p(t)=q(t)$, we have a difference equation

$$q(t+1)-\frac{A}{B}q(t)-\frac{1}{B}=0, \tag{13}$$

which has the general solution $q(t)=C(A/B)^{t}+1/(B-A)$ with an arbitrary constant C. Since $p(t+1)>p(t)>0$, we have $B-A>0$, or $1>A/B>0$. Furthermore, because of the positive and strictly increasing $p(t)$, we have $C>0$. Therefore $p(t)=c/(1+\alpha e^{-\beta t})$, where $c=B-A>0$, $\alpha=C(B-A)>0$, $\beta=-\log(A/B)>0$. q.e.d.

By this theorem, the nearer -1 the correlation coefficient ρ between m and p at the points {P(t)} derived from samples $\{p(t)\}$, hence between $p(t+1)/p(t)$ and $p(t)$ for $t=0, 1, \cdots, T$, the more appropriate it may be to assume that $p(t)$ obeys logistic law.

In the equation (12) appeared in the proof of Theorem 2, put $m=1$, then $p=c$, and let $p\to 0$, then $m\to e^{\beta}$. Hence if the correlation coefficient ρ is near -1, the important constants c and β can be simply estimated on the basis of points P(0), P(1), $\cdots$, P(T). Note that the remaining constant α only translates the logistic curve along the time axis.

It seems that the following Corollary (obtained by an inspection of the equation (13)) gives a more powerful estimation of constants c and β.

COROLLARY. *For a positive growing variable $p(t)$, a necessary and sufficient condition for $p(t)$ to obey the logistic law, i.e. $p(t)=c/(1+\alpha e^{-\beta t})$ $(t=0, 1, \cdots, T)$ is that the point sequence*

$$\{\mathrm{Q}(t)\}\equiv\left\{\left(\frac{1}{p(t)},\ \frac{1}{p(t+1)}\right)\right\}$$

is on a straight line: $p=Rm+S$ *for some* $1>R>0$ *and* $S>0$.

By this Corollary c and β may be estimated by

$$c=\frac{1-R}{S}, \qquad \beta=-\log R,$$

since (13) gives $B=1/S$, $A=R/S$.

Any logistic curve can be completely represented by a triple (c, α, β) of three positive real numbers c, α, and β. We may call, therefore, (c, α, β) itself a logistic curve. The set of all possible logistic curves $\{(c, \alpha, \beta); c>0, \alpha>0, \beta>0\}$ is classified into equivalence classes by the *translatability equivalence relation* $\sim$, i.e. $(c, \alpha, \beta) \sim (c', \alpha', \beta')$ if and only if (c, α, β) is translatable to (c', α', β'). Hence $(c, \alpha, \beta) \sim (c', \alpha', \beta')$ if and only if $c=c'$ and $\beta=\beta'$. Therefore any equivalence class is represented by a pair (c, β) of two positive real numbers c and β.

By the Corollary above the relations $c=(1-R)/S$ and $\beta=-\log R$ give a one-to-one correspondence between the set of all equivalence classes $\{(c, \beta); c>0, \beta>0\}$ and the set $\{(R, S); 1>R>0, S>0\}$ of all possible straight lines of the form of $p=Rm+S$ $(1>R>0, S>0)$. This fact suggests how one can decide whether a given set of logistic curves is invariant under the translatability equivalence relation. And if the translatability relation holds among a set of logistic curves (e.g. those generated by letting parameter m_0 in the preceding sections take various values), the constants c and β can be efficiently determined because of the correspondence of this set to a single straight line (R, S).

ACKNOWLEDGMENTS. The authors are much indebted to Prof. K. Kunisawa for his constant encouragement and for his having given them a valuable opportunity for the research which bore this paper. They express their thanks especially to Messrs. A. Ukegawa and M. Shibasaki of the Business Efficiency Division of The Japan Monopoly Corporation for a financial aid and a set of raw statistical data. They also wish to acknowledge the understanding gained by several discussions with their colleagues, K. Kawamura, T. Ikefuji and N. Ueno.

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