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ON ORISPHERICAL SUBGROUPS OF A SEMISIMPLE LIE GROUP

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1. Let $\mathfrak{g}$ be a real semisimple Lie algebra, and let $\mathcal{G}$ be a Lie group whose Lie algebra is $\mathfrak{g}$. We take a one-parameter subgroup $g(t)=\exp(tX)$, $X\in\mathfrak{g}$, and define a orispherical subgroup $\mathcal{Z}$ relative to $g(t)$ as follows.

DEFINITION 1. $\mathcal{Z}$ is the set of all $z\in\mathcal{G}$ for which

$$\lim_{t\to\infty} g(t)zg(t)^{-1}=e \quad \text{(neutral element of } \mathcal{G}).$$

Orispherical subgroups were introduced by Gelfand, Graev, and Pyatetskii-Shapiro, and played an important role in the theory of representations and automorphic functions; [1], [2]. The purpose of this note is to show that $\mathcal{Z}$ is a connected closed subgroup of $\mathcal{G}$.

2. First of all, $\mathcal{Z}$ is easily seen to be connected. Let $z\in\mathcal{Z}$, then $\exp(tX)\cdot z\cdot\exp(-tX)=z_t\in\mathcal{Z}$ by definition. Of course z_t is continuous in t, and $z_t\to e$ as $t\to\infty$. Denote by $\mathcal{Z}_0$ the connected component of e of $\mathcal{Z}$. Since $\mathcal{Z}_0$ is open in $\mathcal{Z}$, we have $z_{t_0}\in\mathcal{Z}_0$ for sufficiently large t_0. But then z_t $(0\leqq t\leqq t_0)$ connects z to z_{t_0}, hence $z\in\mathcal{Z}_0$. This proves connectedness of $\mathcal{Z}$.

3. It is a classical result that any $X\in\mathfrak{g}$ can be expressed by a unique sum $Y+N$, where $Y, N\in\mathfrak{g}$ satisfy the conditions: i) $[Y, N]=0$; ii) ad Y is semisimple and all of its eigen values are real; iii) ad N has only pure imaginary eigen values. Here ad means the adjoint representation of $\mathfrak{g}$.

Let Ad be the adjoint representation of $\mathcal{G}$ into the set of Aut$(\mathfrak{g})$ of all automorphisms of the Lie algebra $\mathfrak{g}$. We denote the image Ad $\mathcal{G}$ of $\mathcal{G}$ by Int$(\mathfrak{g})$ Then it is obvious that if $z\in\mathcal{Z}$, we have

$$\lim_{t\to\infty} \mathrm{Ad}(\exp(tX))\cdot\mathrm{Ad}\, z\cdot\mathrm{Ad}(\exp(-tX))=E \quad \text{(Identity)}.$$

In Int$(\mathfrak{g})$, we consider the subset $\boldsymbol{Z}$ of all $\zeta\in\mathrm{Int}(\mathfrak{g})$ for which

$$\lim_{t\to\infty} \mathrm{Ad}(\exp(tX))\cdot\zeta\cdot\mathrm{Ad}(\exp(-tX))=E\,.$$

Now we put ad $X=\xi$, ad $Y=\eta$, ad $N=\nu$, then

$$\xi=\eta+\nu, \qquad \eta\nu=\nu\eta,$$
$$\mathrm{Ad}(\exp(tX))=e^{t\xi}=e^{t\nu}e^{t\eta}$$

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where $e^{\alpha}=\sum_{n=0}^{\infty}\alpha^n/n!$. We denote by $a_1,\cdots,a_n$ the proper values of η, and assume $a_1=a_2=\cdots=a_{k_1}<a_{k_1+1}=\cdots=a_{k_1+k_2}<\cdots<a_{k_1+\cdots+k_{m-1}+1}=a_{k_1+k_2+\cdots+k_{m-1}+2}=\cdots=a_{k_1+k_2+\cdots+k_m}$. Since ν has pure imaginary proper values and commute with η, η and ν are expressed in a suitably chosen basis $\{U_1,\cdots,U_n\}$ in $\mathfrak{g}^C$ (the complexification of $\mathfrak{g}$) as the matrices of the form

$$\eta=\begin{pmatrix} \boxed{\begin{matrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_{k_1}\end{matrix}} & & & \mathbf{0} \\ & \boxed{} & & \\ & & \ddots & \\ \mathbf{0} & & & \boxed{\begin{matrix} a_{k_1+\cdots+k_{m-1}+1} & & 0 \\ & \ddots & \\ 0 & & a_{k_1+\cdots+k_m}\end{matrix}} \end{pmatrix},$$

$$\nu=\begin{pmatrix} \boxed{\begin{matrix} \sqrt{-1}\,b_1 & & * \\ & \ddots & \\ 0 & & \sqrt{-1}\,b_{k_1}\end{matrix}} & & & \mathbf{0} \\ & \boxed{\begin{matrix} \sqrt{-1}\,b_{k_1+1} & & * \\ & \ddots & \\ 0 & & \sqrt{-1}\,b_{k_1+k_2}\end{matrix}} & & \\ & & \ddots & \\ \mathbf{0} & & & \boxed{\begin{matrix} \sqrt{-1}\,b_{k_1+\cdots+k_{m-1}+1} & & * \\ & \ddots & \\ 0 & & \sqrt{-1}\,b_{k_1+\cdots+k_m}\end{matrix}} \end{pmatrix}.$$

We fix in the following the basis $U_1,\cdots,U_n$. Now we consider the subset $\boldsymbol{Z}'\subset\mathrm{GL}(\mathfrak{g}^C)$ (the set of all nonsingular linear transformations of $\mathfrak{g}^C$) of all ζ satisfying

$$\lim_{t\to\infty} e^{t\xi}\cdot\zeta\cdot e^{-t\xi}=E.$$

By simple computations of matrices, we easily obtain the following

LEMMA. *ζ belongs to $\boldsymbol{Z}'$ if and only if ζ has the form*

$$\begin{pmatrix} E_{k_1} & * & \cdots & * \\ 0 & E_{k_2} & \cdots & * \\ & & \ddots & \\ 0 & 0 & & E_{k_m}\end{pmatrix}$$

where there are zeros below the diagonal and E_{k_i} denotes the identity matrix of order k_i.

Moreover the followings are also trivial.

For a $\zeta \in GL(\mathfrak{g}^C)$, the two conditions

$$e^{t\xi}\cdot\zeta\cdot e^{-t\xi}\to E,$$

and

$$e^{t\eta}\cdot\zeta\cdot e^{-t\eta}\to E$$

are equivalent.

Since the set $\boldsymbol{Z}'$ is closed in $GL(\mathfrak{g}^C)$ by the above lemma and since the intersection of $\boldsymbol{Z}'$ with $\mathrm{Int}(\mathfrak{g})$ is clearly $\boldsymbol{Z}$, we have the following

PROPOSITION 1. *Let* $\mathfrak{g}$ *be a semisimple Lie algebra and let* $X \in \mathfrak{g}$. *Then the subgroup* $\boldsymbol{Z}$ *of* $\mathrm{Int}(\mathfrak{g})$, *consisting of all* $\zeta \in \mathrm{Int}(\mathfrak{g})$ *for which*

$$\lim_{t\to\infty} e^{t\,\mathrm{ad}\,X}\cdot\zeta\cdot e^{-t\,\mathrm{ad}\,X}=E$$

is a closed subgroup of $\mathrm{Int}(\mathfrak{g})$.

Moreover, X *is uniquely decomposed into a sum* $Y+N$ *as above. Then*

$$\left\{\zeta\in\mathrm{Int}(\mathfrak{g})\Big/\lim_{t\to\infty} e^{t\,\mathrm{ad}\,Y}\cdot\zeta\cdot e^{-t\,\mathrm{ad}\,Y}=E\right\}=\boldsymbol{Z}.$$

4. We consider the set $\mathcal{Z}_1$ of all $z\in\mathcal{G}$, for which

$$\lim_{t\to\infty}\exp(tX)\cdot z\cdot\exp(-tX)$$

exists and is equal to some element of the center $\mathcal{C}$ of $\mathcal{G}$.

PROPOSITION 2. $\mathcal{Z}_1$ *is the complete inverse image* $\mathrm{Ad}^{-1}\boldsymbol{Z}$ *of* $\boldsymbol{Z}$.

Proof. From the definition of $\mathcal{Z}_1$, we have $\mathcal{Z}_1\supset\mathcal{C}$. Hence we need to show $\mathrm{Ad}\,\mathcal{Z}_1=\boldsymbol{Z}$. It is obvious that $\mathrm{Ad}\,\mathcal{Z}_1\subset\boldsymbol{Z}$. On the other hand, let $\zeta\in\boldsymbol{Z}$, then

$$e^{t\,\mathrm{ad}\,X}\cdot\zeta\cdot e^{-t\,\mathrm{ad}\,X}\to E \qquad (t\to\infty).$$

Take a z such that $\mathrm{Ad}\,z=\zeta$. Let U be a connected neighbourhood of e in $\mathcal{G}$ for which $U^{-1}=U$, $U^2\cap\mathcal{C}=e$. Then $\mathrm{Ad}\,U=\{\mathrm{Ad}\,g/g\in U\}$ is a neighbourhood of E in $\mathrm{Int}(\mathfrak{g})$. Hence we have

$$e^{t\,\mathrm{ad}\,X}\cdot\mathrm{Ad}\,z\cdot e^{-t\,\mathrm{ad}\,X}\in\mathrm{Ad}\,U$$

for all t such that $t\geqq T$, where T is sufficiently large. This means $\exp(tX)\cdot z\cdot\exp(-tX)\in U\cdot\mathcal{C}$ if $t\geqq T$. But $\{\exp(tX)\cdot z\cdot\exp(-tX)/t\geqq T\}$ is obviously a connected set. Hence there is only a $c\in\mathcal{C}$ for which

$$\exp(tX)\cdot z\cdot\exp(-tX)\in U\cdot c \qquad \text{if} \quad t\geqq T.$$

Of course, c is independent of U. This means

$$\lim_{t\to\infty} \exp(tX)\cdot z\cdot \exp(-tX)=c.$$

Hence $z\in\mathcal{Z}_1$ i.e. $\boldsymbol{Z}\subset \mathrm{Ad}\,\mathcal{Z}_1$.

Now combining prop. 1 and prop. 2, we get

PROPOSITION 3. *$\mathcal{Z}_1$ is a closed subgroup of $\mathcal{G}$.*

5. By the above prop. 3, we see that $\mathcal{Z}_1$ is a closed Lie subgroup of $\mathcal{G}$. Let $\mathcal{Z}_0$ be the connected component of $\mathcal{Z}_1$, and let $\mathfrak{z}$ be the Lie algebra of $\mathcal{Z}_0$ (or $\mathcal{Z}_1$). We assert that $\mathcal{Z}_0$ coincides with $\mathcal{Z}$.

Let $Z\in\mathfrak{z}$, then $\exp(sZ)\in\mathcal{Z}_0$ for any real s. Hence we have

$$\lim_{t\to\infty} \exp(tX)\exp(sZ)\exp(-tX)=c_s\in\mathcal{C}.$$

But since $\mathcal{C}$ is a countable set, there can be only a countable number of s_i $(i=1, 2, \cdots)$, for which c_{s_i} are mutually different.

On the other hand, it is obvious that $c_s\cdot c_t=c_{s+t}$. Hence, if we put

$$J_0=\{s/-\infty<s<\infty, c_s=e\}$$

then J_0 is an additive group, and we have

$$(*)\qquad (-\infty, \infty)=\bigcup_{i=1}^{\infty}(J_0+s_i)$$

where $J_0+s_i=\{s+s_i/s\in J_0\}$.

Now, $s\in J_0$ means

$$(\mathrm{a})\qquad \lim_{t\to\infty} \exp(tX)\exp(sZ)\exp(-tX)=e$$

and since the convergency of the left hand side is secured from the fact that Z belongs to $\mathfrak{z}$, the condition (a) is equivalent to

$$(\mathrm{b})\qquad \lim_{n\to\infty} \exp(nX)\exp(sZ)\exp(-nX)=e.$$

But if we take a neighbourhood U of e in $\mathcal{G}$, for which $U\cap\mathcal{C}=e$, then the condition (b) is equivalent to

$$\lim_{n\to\infty} \exp(nX)\exp(sZ)\exp(-nX)\in U$$

and this means

$$J_0=\bigcup_{n=1}^{\infty}\bigcap_{m\geqq n}\{s/\exp sZ\in\exp(-mX)\cdot U\cdot\exp(mX)\}.$$

Thus J_0 is a $G_{\delta\sigma}$-set and of course measurable.

By virtue of (*) measure of J_0 is positive, and since any measurable subgroup of positive measure of the additive group of all real numbers is the full group, we can see that $J_0=(-\infty, \infty)$, i.e. we have for all $s\in(-\infty, \infty)$

$$\lim_{t\to\infty} \exp(tX)\exp(sZ)\exp(-tX)=e.$$

This means $\exp(sZ)\in\mathcal{Z}$ for all s. Since $\mathcal{Z}_0$ is generated by $\exp Z$, $Z\in\mathfrak{z}$, we have $\mathcal{Z}_0\subset\mathcal{Z}$.

We have already seen that $\mathcal{Z}$ is connected. Hence $\mathcal{Z}\subset$ {the connected component of $\mathcal{Z}_1=\mathcal{Z}_0$}. Thus we have

PROPOSITION 4. *Let* $\mathfrak{g}$ *be a real semisimple Lie algebra, and let* $\mathcal{G}$ *be a connected Lie group whose Lie algebra is* $\mathfrak{g}$. *Let* $X\in\mathfrak{g}$ *and let* $\mathcal{Z}_1$ *be the set of all elements* z *of* $\mathcal{G}$ *for which*

$$\lim_{t\to\infty} \exp(tX)\cdot z\cdot\exp(-tX)$$

exists and contained in the center $\mathcal{C}$ *of* $\mathcal{G}$. *Let* $\mathcal{Z}$ *be the orispherical subgroup relative to X. Then the connected component of* $\mathcal{Z}_1$ *is* $\mathcal{Z}$. $\mathcal{Z}_1$ *and* $\mathcal{Z}$ *are closed subgroups of* $\mathcal{G}$.

6. For the Lie algebra $\mathfrak{z}$ of $\mathcal{Z}$, we have the following

PROPOSITION 5.

$$\mathfrak{z}=\left\{Z\in\mathfrak{g}\Big/\lim_{t\to\infty} e^{t\,\mathrm{ad}\,X}Z=0\right\}.$$

Proof. Denote the right hand side by $\mathfrak{z}'$. If $Z\in\mathfrak{z}'$ we have

$$\exp(e^{t\,\mathrm{ad}\,X}(sZ))\to e \qquad (t\to\infty)$$

for all s. But the left hand side is equal to

$$\exp(s\,\mathrm{Ad}\,(\exp(tX))Z)=\exp(tX)\exp(sZ)\exp(-tX).$$

Hence $\exp(sZ)\in\mathcal{Z}$ for all s and this implies $Z\in\mathfrak{z}$.

Conversely if $Z\in\mathfrak{z}$, then $\exp sZ\in\mathcal{Z}$. Hence $\mathrm{Ad}\exp(sZ)=e^{s\,\mathrm{ad}\,Z}\in\mathrm{Ad}\,\mathcal{Z}=\boldsymbol{Z}$. This implies $e^{s\,\mathrm{ad}\,Z}$ has the form mentioned in the above lemma. $\boldsymbol{Z}$ being nilpotent, we have $\mathrm{ad}\,Z=\log e^{\mathrm{ad}\,Z}$. Then $E+\mathrm{ad}\,Z$ has the same form as above, and this implies

$$e^{t\,\mathrm{ad}\,X}\cdot\mathrm{ad}\,Z\cdot e^{-t\,\mathrm{ad}\,X}\to 0 \qquad (t\to\infty).$$

The left hand side is $\mathrm{ad}\,(e^{t\,\mathrm{ad}\,X}Z)$. Since ad is an isomorphism, we have $e^{t\,\mathrm{ad}\,X}Z\to 0$. Hence $Z\in\mathfrak{z}'$.

References

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