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INVARIANT SUBFIELDS OF RATIONAL FUNCTION FIELDS

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Let K be the rational function field $k(X_1, X_2, \cdots, X_n)$ of variables $X_1, X_2, \cdots, X_n$ over a field k. Let M be the vector space $\sum_{i=1}^{n} k\cdot X_i$ over k. Let $\mathfrak{g}$ be a finite group operating on K, induced by a representation ρ of $\mathfrak{g}$ with representation space M. Let L be the subfield of K consisting of elements which are invariant under $\mathfrak{g}$. The problem to consider here is whether L is purely transcendental over k. This problem has been answered affirmatively in the following cases: (0) $\mathfrak{g}$ is the symmetric group permuting $X_1, X_2, \cdots, X_n$, (1) $\mathfrak{g}$ is abelian and k is the complex number field, (2) $\mathfrak{g}$ is a cyclic group of order n, ρ is its regular representation and k contains the primitive n-th roots of unity, provided that the characteristic of k does not divide n (cf. [5]) and (3) k is of characteristic $p>0$, $\mathfrak{g}$ is a p-group and ρ is its regular representation (cf. [2], [3] and [4]). In this note we shall give a principle, written in language of algebraic groups, which covers the three cases (1), (2) and (3), and which may be applied to other cases where $\mathfrak{g}$ is soluble.

A connected algebraic group G is called k-soluble if there exists a normal chain $G_0=G\supset G_1\supset G_2\supset\cdots\supset G_r=\{e\}$ such that G_i is defined over k and G_i/G_{i+1} is isomorphic to G_a or G_m over k, where G_a and G_m are the additive group of the universal domain Ω and the multiplicative group of non-zero elements of Ω. The following property of k-soluble algebraic groups is used here (cf. [6]): let G be a k-soluble algebraic group; let V be a homogeneous space with respect to G over k, then the function field $k(V)$ over k is purely transcendental over k.

From this we have

(P) Let G be a k-soluble algebraic group such that $k(G)=K$; let $\mathfrak{g}$ be a finite subgroup of G which is rational over k such that the invariant subfield of K by the left translations of $\mathfrak{g}$ is L, then L is purely transcendental over k.

In fact, there exists the quotient variety $G/\mathfrak{g}$, defined over k, which is a homogeneous space with respect to G over k.

Let us consider the case where $\mathfrak{g}$ is abelian.

LEMMA. *Let $\mathfrak{g}$ be a finite abelian subgroup of $GL(n, k)$ of exponent m. Then, if k contains the primitive m-th roots of unity, there exists $x\in GL(n, k)$ such that $x\cdot\mathfrak{g}\cdot x^{-1}$ is contained in the set of matrices of the form*

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$$\begin{pmatrix} N_1 & 0 & \cdots\cdots & 0 \\ 0 & N_2 & 0 \cdots & 0 \\ \vdots & & \ddots & \vdots \\ & & & 0 \\ 0 & \cdots\cdots & 0 & N_s \end{pmatrix}$$

where N_i is upper triangular matrix with only one eigenvalue, and further if the characteristic of k does not divide m, $\mathfrak{g}$ is a group of semisimple matrices and $x \cdot \mathfrak{g} \cdot x^{-1}$ is diagonal.

The first part follows, for example, from the proof of the Lemma 6. 4 of [1]. To prove the second part, take any element $g \in \mathfrak{g}$ and let $g = g_s g_u$ be the multiplicative Jordan decomposition of g; then the orders of g_s and g_u divide that of g. If the characteristic of k is 0, any non-identity unipotent matrix has the infinite order (cf. Prop. 8. 1 of [1]); hence g is semisimple; in the characteristic $p>0$ case, a matrix is unipotent if and only if its order is a power of p (cf. Prop. 8. 1 of [1]); hence g is semisimple.

Now we have a proposition which generalizes (1) and (2).

PROPOSITION 1. *Let $\mathfrak{g}$ be a finite abelian group of exponent m. If the characteristic of k does not divide m and if k contains the primitive m-th roots of unity, L is purely transcendental over k.*

In fact, we may suppose that ρ is faithful and that $\mathfrak{g}$ is a subgroup of $GL(M)$. By the Lemma we have a base $Y_1, Y_2, \cdots, Y_n$ of the vector space M such that $Y_i^{\sigma} = \chi_i(\sigma) Y_i$ for $\sigma \in \mathfrak{g}$. Take G=the group of diagonal matrices with coordinate functions $Y_1, Y_2, \cdots, Y_n$, then $k(G)=K$. We can consider that $\mathfrak{g}$ is the subgroup of G consisting of diagonal matrices $(\chi_1(\sigma), \chi_2(\sigma), \cdots, \chi_n(\sigma))$ for $\sigma \in \mathfrak{g}$. Since $Y_i^{\sigma}(g) = \chi_i(\sigma) Y_i(g) = Y_i(\sigma g)$ for $g \in G$, the proposition follows from (P).

Let us consider the case where ρ is the regular representation of $\mathfrak{g}$. Let $\Omega[\mathfrak{g}]$ be the group ring of $\mathfrak{g}$ over Ω. Then the unit group of $\Omega[\mathfrak{g}]$ has a structure of a connected algebraic group G defined over the prime field Z_p of k such that $k(G)=K$ and $\mathfrak{g}$ can be imbedded in G by $\sigma \to 1 \cdot \sigma$. Then, the notation being as above, by (P) we have

PROPOSITION 2. *If the algebraic group G is k-soluble, L is transcedental over k.*

When $\mathfrak{g}$ is a p-group, the following Lemma gives the structure of the algebraic group G.

LEMMA. *If k is of characteristic $p>0$ and $\mathfrak{g}$ is a p-group, G is a connected nilpotent algebraic group defined over Z_p and has the direct decomposition $G = G_s \times G_u$ over Z_p, where G_s is central and isomorphic to G_m over Z_p and G_u is the unipotent part of G.*

Let N be the radical of the algebra $\Omega[\mathfrak{g}]$; let s be a positive integer such that $N^s=\{0\}$. Let $U_i=\{a\in\Omega[\mathfrak{g}]\mid a\equiv e \bmod N^i\}$. Then we have

(i) $$(G, G)\subset U_1,$$

(ii) $$(G, U_i)\subset U_{i+1}.$$

In fact, for any $a=\sum_{\sigma\in\mathfrak{g}}a_\sigma\cdot\sigma$, let $\mathrm{tr}(a)=\sum_{\sigma\in\mathfrak{g}}a_\sigma$, then tr is a rational homomorphism of G onto G_m defined over Z_p; we have $a=\mathrm{tr}(a)e+\sum_{\sigma\in\mathfrak{g}}a_\sigma(\sigma-e)=\mathrm{tr}(a)e+r(a)\equiv\mathrm{tr}(a)e$, mod N and $a^{-1}=\mathrm{tr}(a)^{-1}(e+\mathrm{tr}(a)^{-1}r(a))^{-1}\equiv\mathrm{tr}(a)^{-1}e$, mod N, where $r(a)\in N$; hence, for $a, b\in G$, $aba^{-1}b^{-1}\equiv\mathrm{tr}(a)\mathrm{tr}(b)\mathrm{tr}(a)^{-1}\mathrm{tr}(b)^{-1}e=e$, mod N; thus we have (i). To show (ii), take $a\in G$ and $b\in U_i$; then $a=\mathrm{tr}(a)e+r(a)$ and $b=e+r(b)$, where $r(a)\in N$ and $r(b)\in N^i$; then $aba^{-1}b^{-1}=e+(ab-ba)a^{-1}b^{-1}=e+(r(a)r(b)-r(b)r(a))a^{-1}b^{-1}\equiv e$, mod N^{i+1}; thus we have (ii). Since $U_s=\{e\}$, we have that G is nilpotent. Each element $a\in G$ has a unique expression $a=\mathrm{tr}(a)e\cdot(e+\mathrm{tr}(a)^{-1}r(a))$. It is easily seen that the semisimple part G_s and the unipotent part G_u of G are defined over Z_p and that we have the Lemma.

Since any connected algebraic group of unipotent matrices defined over a perfect field k is k-soluble, we have the following Corollary of the Proposition 2 which is nothing but (3).

COROLLARY. *If k is of characteristic $p>0$ and if ρ is the regular representation of a p-group $\mathfrak{g}$, L is purely transcendental over k.*

BIBLIOGRAPHY

[1] BOREL, A., Groupes linéaires algébriques. Ann. of Math. **64** (1956), 20–82.

[2] KUNIYOSHI, H., On a problem of Chevalley. Nagoya Math. Journ. 8 (1955), 65–67.

[3] KUNIYOSHI, H., On purely transcendency of certain field. Tôhoku Math. Journ. **6** (1955), 101–108.

[4] KUNIYOSHI, H., Certain subfields of rational function fields. Proc. International Symp. on Algebraic Number Theory. (1955), 241–243.

[5] MASUDA, K., On a problem of Chevalley. Nagoya Math. Journ. 8 (1955), 59–63.

[6] ROSENLICHT, M., Some rationality questions on algebraic groups. Annali di Mat. **43** (1957), 25–50.

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