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A REMARK ON ULTRAHYPERELLIPTIC SURFACES

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1. Let R be an ultrahyperelliptic surface of finite order and with $P(R)=4$. Let p denote its order, then R is defined by

$$y^2=(e^H-\gamma)(e^H-\delta), \qquad \gamma\delta(\gamma-\delta)\neq 0$$

with $H(z)=\sum_{n=1}^{p} h_n z^n$, $h_p\neq 0$. [3]. Let S be another ultrahyperelliptic surface of non-zero finite order, that is,

$$y^2=g(z)\equiv z^s \prod_{n=1}^{\infty} E\left(\frac{z}{a_n}, q\right),$$

where $E(u, q)$ is the Weierstrass prime factor

$$E(u, q)=(1-u)\exp\sum_{j=1}^{q}\frac{u^j}{j}$$

and every a_n is a simple zero of $g(z)$ and $s=0$ or 1.

Hiromi and Mutō [1] proved the following result: Assume there exists a non-trivial analytic mapping φ from R into S. Then $p=n\cdot r$, where r is the order of $g(z)$ and n is an integer.

The aim of the present paper is to prove the following

THEOREM. *If S is an ultrahyperelliptic surface of non-zero finite order into which there is a non-trivial analytic mapping from an ultrahyperelliptic surface R of finite order and with $P(R)=4$, then the order of S is a half of an integer.*

2. Proof of theorem. For our purpose we need our previous result in [4], which asserts the existence of two functions $h(z)$ and $f(z)$ such that $f(z)$ is meromorphic in $|z|<\infty$ and $h(z)$ is a polynomial of degree n in the present situation [1] satisfying

$$f(z)^2(e^H-\gamma)(e^H-\delta)=g\circ h(z).$$

Here put $h(z)=\sum_{m=0}^{n} h_m^* z^m$.

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Assume $n>2p$. Then $h(z)-a_\mu=0$ has n roots $z_{\mu 1}, \cdots, z_{\mu n}$ such that

$$z_{\mu j}=\left(\frac{a_\mu-h_0^*}{h_n^*}\right)^{1/n} e^{2\pi i(j-1)/n}(1+\varepsilon^*), \qquad \lim_{\mu\to\infty}\varepsilon^*=0.$$

If μ satisfies $\mu\geqq\mu_0$, then every corresponding root of $h(z)-a_\mu=0$ is simple. Hence this is the case for $g\circ h(z)=0$. Thus this root must be a root of $(e^H-\gamma)(e^H-\delta)=0$. On the other hand $H=\log\gamma+2m\pi i$ (or $\log\delta+2m\pi i$) has p roots for a positive $m\geqq m_0$ and p roots for a negative $m\leqq -m_0$

$$\left(\frac{2m\pi i}{h_p}\right)^{1/p} e^{2\pi ik/p}(1+\varepsilon), \quad \lim_{m\to\pm\infty}\varepsilon=0, \qquad k=0, 1, \cdots, p-1.$$

These roots with a fixed k have the same direction asymptotically for $n\to\infty$ and for $n\to-\infty$, respectively. Hence $n>2p$ leads to a contradiction.

Next we need a number-theoretic lemma: Let $\{x_j\}$ be the set of points satisfying $x_{j+1}-x_j=2p/n$ and $x_1\in[0, 1]$. Let $\delta(j, x_1)$ be

$$\min\{x_j-[x_j], [x_j]+1-x_j\}.$$

Then

$$\Delta(n, p)\equiv \inf_{0\leqq x_1\leqq 1} \max_j \delta(j, x_1)\geqq\Delta>0,$$

if n is not a divisor of $2p$.

This is easy to prove. First of all we should remark that $\delta(j, x_1)$ is a continuous function of x_1 for every fixed j. Assume there exists a sequence $\{x_{1\nu}\}$ for which $\max_j\delta(j, x_{1\nu})\to 0$, $x_{1\nu}\to x_{1\infty}$ as $\nu\to\infty$. This implies that for every j $\delta(j, x_{1\nu})\to 0$ as $\nu\to\infty$. Thus $\delta(j, x_{1\infty})=0$ for all j. Put $j=0$, then $x_{1\infty}$ is equal to either 0 or 1. Next put $j=1$, then $x_{1\infty}+2p/n$ is equal to an integer. Hence $2p/n$ is an integer, which is a contradiction.

Assume that n is not a divisor of $2p$. Now consider π/p as the unit length and $2\pi/p-(\arg h_p)/p$ as the origin, that is, we shall consider the set of points $-(\arg h_p)/p+\pi/2p+k\pi/p$ as the set of integers. Now consider the set of $\{\arg z_{\mu j}\}$ as the set of $\{x_j\}$. Hence by the above fact for every $\mu\geqq\mu_0$ there is an index $j(\mu)$ such that the difference between all integer points $-(\arg h_p)/p+\pi/2p+k\pi/p$, $k=0, \pm 1, \cdots$ and $\arg z_{\mu j(\mu)}$ is greater than $\Delta\pi/p-\varepsilon'$, $\varepsilon'\to 0$ as $\mu\to\infty$. This shows that there is a sequence of simple zeros of $h(z)-a_\mu$ which does not belong to the set of simple zeros of $(e^H-\gamma)(e^H-\delta)$. This is untenable.

By the above result and by $p=n\cdot r$ we have that $2p/n=2r$ must be a positive integer. This is nothing but the content of our theorem.

3. Examples. Let S be the surface defined by

$$y^2=e^{zp/n}+e^{-zp/n},$$

where n is a divisor of $2p$ but not of p. This is really an ultrahyperelliptic surface. Let R be the surface defined by

$$y^2=e^{2z^p}+1.$$

Then $P(R)=4$. In this case we have a functional equation

$$f(z)^2(e^{2z^p}+1)=e^{h(z)^{p/n}}+e^{-h(z)^{p/n}}$$

with $f(z)=\exp(-z^p)$ and $h(z)=z^h$. This implies the existence of a non-trivial analytic mapping from R into S.

If p is an even integer, then there is another example of a non-trivial analytic mapping from R: $y^2=\exp(2z^p)-1$ into S: $y^2=(\exp(z^{p/n})-\exp(-z^{p/n}))/z^{p/n}$, where n is a divisor of $2p$ but not a divisor of p. Take $f(z)=z^{-p}\exp(-z^p)$, $h(z)=z^n$. These examples show that our result is best possible.

If the order of S is a half integer and not an integer, then $P(S)=2$. [2]. If the order of S is a positive integer, then $P(S)=4$ or $P(S)=2$ if there is a non-trivial analytic mapping from R with $P(R)=4$ and of finite order to S. [5]. In this situation there is an example of $P(S)=2$.

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