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THEORY OF CONFORMAL CONNECTIONS

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Introduction.

The main purpose of the present paper is to give a modern introduction to the theory of conformal connections. There were, historically, several approaches to this subject. Our approach here is based on the theory of G-structures. We shall now briefly explain our method.

For a manifold M[1] of dimension n, we construct the bundle $P^2(M)$ of frames of 2nd order contact. Its structure group will be denoted by $G^2(n)$. We define a certain subgroup $H^2(n)$ of $G^2(n)$ which is isomorphic with an isotropy subgroup of the conformal transformation group $K(n)$ acting on the Möbius space of dimension n. A conformal structure on a manifold M is a subbundle P of $P^2(M)$ with structure group $H^2(n)$.

A conformal connection for the given conformal structure P is a Cartan connection satisfying some extra conditions. It will be shown that we can associate with each conformal structure a naturally defined conformal connection, so-called normal conformal connection.

§1. Prolongations of a Lie algebra.

Let V be a real vector space of dimension n and $\mathfrak{g}$ a Lie algebra of endomorphisms of V. $\mathfrak{g}$ may be considered as a subspace of $V\otimes V^*=\mathrm{Hom}(V, V)=\mathfrak{gl}(V)$, where V^* denotes the dual space of V. *The first prolongation* $\mathfrak{g}^{(1)}$ of $\mathfrak{g}$ is defined to be $\mathfrak{g}^{(1)}=\mathfrak{g}\otimes V^*\cap V\otimes S^2(V^*)\subset V\otimes V^*\otimes V^*$, where $S^2(V^*)$ denotes the space of symmetric tensors of degree 2 over V^*. Since $\mathfrak{g}\otimes V^*=\mathrm{Hom}(V,\mathfrak{g})$, an element $T\in\mathfrak{g}\otimes V^*$ is in $\mathfrak{g}^{(1)}$ if and only if

$$T(u)\cdot v=T(v)\cdot u \qquad \text{for all} \quad u, v\in V.$$

Set $\mathfrak{g}^{(2)}=(\mathfrak{g}^{(1)})^{(1)}$ and, in general, $\mathfrak{g}^{(k+1)}=(\mathfrak{g}^{(k)})^{(1)}$. The space $\mathfrak{g}^{(k)}$ is called *the k-th prolongation* of $\mathfrak{g}$. Then

$$\mathfrak{g}^{(k)}=\mathfrak{g}\otimes\underbrace{V^*\otimes\cdots\otimes V^*}_{k\text{-times}}\cap V\otimes S^{k+1}(V^*).$$

We call that $\mathfrak{g}$ is of finite type if $\mathfrak{g}^{(k)}=0$ for some (and hence all larger) k. If

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1) Throughout this paper, we shall denote by M a connected manifold of dimension $\geqq 3$, unless otherwise stated.

$\mathfrak{g}^{(k)}\neq 0$ for all k then $\mathfrak{g}$ is said to be of infinite type.

Let (,) be a non-degenerate symmetric bilinear form on V (of arbitrary signature). Let $\mathfrak{o}(V)$ be the orthogonal algebra of (,), that is, $\mathfrak{o}(V)$ is the set of $A\in\mathfrak{gl}(V)$ such that

$$(Au, v)+(u, Av)=0 \qquad \text{for all} \quad u, v\in V.$$

PROPOSITION 1. $\mathfrak{o}(V)^{(1)}=0.$

Proof. For any $T\in\mathfrak{o}(V)^{(1)}$ and any $u, v, w\in V$ we have

$$\begin{aligned}(T(u)\cdot v, w)&=(T(v)\cdot u, w)=-(u, T(v)\cdot w)=-(u, T(w)\cdot v)\\ &=(T(w)\cdot u, v)=(T(u)\cdot w, v)=-(w, T(u)\cdot v)\\ &=-(T(u)\cdot v, w).\end{aligned}$$

Thus $(T(u)v, w)=0$. Since w is arbitrary and (,) is non-degenerate, $T(u)v=0$ for all $u, v\in V$. Hence $T(u)=0$ for all $u\in V$. This implies $T=0$. (Q.E.D.)

Let (,) be as before and let $\mathfrak{co}(V)$ denote its conformal algebra. That is, $\mathfrak{co}(V)$ is the set of $A\in\mathfrak{gl}(V)$ such that

$$(Au, v)+(u, Av)=\lambda\cdot(u, v) \qquad \text{for all} \quad u, v\in V,$$

where λ is some scalar depending on A.

PROPOSITION 2. *$\mathfrak{co}(V)^{(1)}$ is isomorphic with V^*.*

Proof. For any $T\in\mathfrak{co}(V)^{(1)}$ we have a linear form λ on V defined by

$$(T(u)v, w)+(v, T(u)w)=\lambda(u)\cdot(v, w).$$

Thus we have a linear mapping of $\mathfrak{co}(V)^{(1)}\to V^*$. A T lying in its kernel would lie in $\mathfrak{o}(V)^{(1)}$ and thus vanish by Proposition 1. Hence the mapping is injective. Let us show that it is also surjective. To this effect we observe that (,) induces an isomorphism of V onto V^*. Thus $u\in V$ is mapped onto $u^*\in V^*$ where $u^*(v)=(u, v)$ for every $v\in V$. If we replace (,) by ρ(,), then under the new isomorphism u gets sent into ρu^*. In particular, the isomorphism of $V\otimes V^*$ onto $V^*\otimes V$ induced by (,) is independent of the scalar ρ. Let us denote this isomorphism by ϕ. For any $u^*\in V^*$, let μ: $V^*\to V\otimes V^*\otimes V^*$ be defined by

$$\mu(u^*)(v)=v\otimes u^*-\phi(u^*\otimes v)+u^*(v)\cdot I,$$

where I is the identity in $\mathfrak{gl}(V)$. From

$$\mu(u^*)(v_1)v_2=u^*(v_2)\cdot v_1+u^*(v_1)\cdot v_2-(v_1, v_2)\cdot u,$$

we have

$$\mu(u^*)(v_1)v_2=\mu(u^*)(v_2)v_1.$$

Furthermore,

$$(\mu(u^*)(v_1)v_2, v_3)+(v_2, \mu(u^*)(v_1)v_3)=2u^*(v_1)\cdot(v_2, v_3).$$

These imply that $\mu(u^*)$ is an element of $\mathfrak{co}(V)^{(1)}$. Thus $\mathfrak{co}(V)^{(1)}$ is isomorphic with V^*. (Q.E.D.)

PROPOSITION 3. *If* $\dim V \geqq 3$, *then* $\mathfrak{co}(V)^{(2)}=0$.

Proof. For any $u, v, x, y \in V$ and for any $T \in \mathfrak{co}(V)^{(2)}$ we have

$$(T(u, v)x, y)+(x, T(u, v)y)=\lambda(u, v)\cdot(x, y),$$

where λ is a symmetric bilinear form on V depending on T. If λ vanishes, then T belong to $\mathfrak{o}(V)^{(2)}$ and hence must vanish. Since λ is symmetric, to prove that a given λ vanishes it suffices to show that $\lambda(u, u)$ vanishes identically. Let us choose u and v with $(u, v)=0$. Then

$$\begin{aligned}\lambda(u, u)\cdot(v, v)&=2(T(u, u)v, v)=2(T(u, v)u, v)=-2(u, T(u, v)v)\\&=-2(u, T(v, v)u)=-\lambda(v, v)\cdot(u, u).\end{aligned}$$

Thus for every pair of orthonormal vectors u and v we have

$$\lambda(u, u)=-\lambda(v, v).$$

If $\dim V \geqq 3$, for every orthonormal vectors u, v, w we have

$$\lambda(u, u)=-\lambda(v, v)=\lambda(w, w)=-\lambda(u, u).$$

Hence $\lambda(u, u)=0$. (Q.E.D.)

The explicit treatment will be given in §4.

§2. *G*-structures.

Let M be a manifold of dimension n. A linear frame u at a point $x \in M$ is an ordered basis $X_1, \cdots, X_n$ of the tangent space $T_x(M)$. Let $L(M)$ be the set of all linear frames u at all points of M and let π be the mapping of $L(M)$ onto M which maps a linear frame u at x into x.

The general linear group $GL(n, \mathrm{R})$ acts on $L(M)$ on the right as follows: If $a=(a_j^i) \in GL(n, \mathrm{R})$ and $u=(X_1, \cdots, X_n)$ is a linear frame at x, then ua is, by definition, the linear frame $(\Sigma a_1^j X_j, \cdots, \Sigma a_n^j X_j)$[2] at x.

In order to introduce a differentiable structure in $L(M)$, let $(x^1, \cdots, x^n)$ be a local coordinate system in a coordinate neighborhood U in M. Every frame u at $x \in U$ can be expressed uniquely in the form $u=(X_1, \cdots, X_n)$ with $X_i=\Sigma X_i^k(\partial/\partial x^k)$, where (X_i^k) is a non-singular matrix. This shows that $\pi^{-1}(U)$ is in one-to-one correspondence with $U \times GL(n, \mathrm{R})$. We can make $L(M)$ into a differentiable manifold by taking (x^i) and (X_i^k) as a local coordinate system in $\pi^{-1}(U)$. $L(M)$ is a

2) Indices $i, j, k, \cdots$ run over the range $1, 2, \cdots, n$ and to simplify notation we adopt the convention that all repeated indices under a summation sign are summed.

principal fibre bundle over M with structure group $GL(n, \mathrm{R})$. We call $L(M)$ the bundle of linear frames over M.

A linear frame u at x can also be defined as an isomorphism of R^n onto $T_x(M)$. The two definitions are related to each other as follows: let $e_1, \cdots, e_n$ be the natural basis for R^n. A linear frame $u=(X_1, \cdots, X_n)$ at x can be given as a linear mapping u: $\mathrm{R}^n \to T_x(M)$ such that $u(e_i)=X_i$. The action of $GL(n, \mathrm{R})$ on $L(M)$ can be accordingly interpreted as follows:

Consider $a=(a^i_j)\in GL(n, \mathrm{R})$ as a linear transformation of R^n which maps e_j into $\Sigma a^i_j e_i$. Then ua: $\mathrm{R}^n \to T_x(M)$ is the composite of the following two mappings:

$$\mathrm{R}^n \xrightarrow{a} \mathrm{R}^n \xrightarrow{u} T_x(M).$$

A *G-structure* on a differentiable manifold M is, by definition, a reduction of the structure group $GL(n, \mathrm{R})$ of the bundle of linear frames $L(M)$ to the subgroup G.

Let (,) be a non-degenerate symmetric bilinear form on R^n and let $O(n)$ be its orthogonal group. An $O(n)$-structure $O(M)$ on M is the same as a Riemannian metric g. In fact, given $O(M)$, set $g_x(X, Y)=(u^{-1}X, u^{-1}Y)$ for every $X, Y\in T_x(M)$ and $u\in O(M)$ with $\pi(u)=x$. From the definition of $O(n)$, $g_x(X, Y)$ is independent of u with $\pi(u)=x$. Conversely, given a Riemannian metric on M, we let $O(M)$ be the set of all orthonormal frames, that is, of all $u\in L(M)$ which are isometries of R^n onto $T_x(M)$.

Let (,) be as before and let $CO(n)$ be its conformal group, that is, set of all elements $a\in GL(n, \mathrm{R})$ such that

$$(au, av)=\lambda\cdot(u, v) \qquad \text{for all} \quad u, v\in \mathrm{R}^n,$$

where λ is a positive function depending on a. A $CO(n)$-structure $CO(M)$ on M is the same as a "*conformal structure*" on M. Two Riemannian metric g and $\bar{g}$ on M are said to be conformally related if there exists a positive function ρ on M such that $\bar{g}=\rho^2 g$. Let $\{g\}$ be a class of conformally related Riemannian metrics on M. For an element g of $\{g\}$, $CO(M)$ is defined as the set of all $u\in L(M)$ such that

$$g_x(X, Y)=\rho\cdot(u^{-1}X, u^{-1}Y) \qquad \text{for all} \quad X, Y\in T_x(M).$$

Clearly $CO(M)$ does not depend on the choice of $g\in\{g\}$. Hence the set of all classes of conformally related Riemannian metrics on M are in one-to-one correspondence with the set of all $CO(n)$-structures on M. This fact will be treated in § 8 from slightly different point of view.

§ 3. Jets and frames of higher order contact (Theory of Ehresmann-Kobayashi).

Let M be a manifold of dimension n and R^n be a real number space of dimension n. Let U and V be neighborhoods of the origin 0 in R^n. Two mappings f: $U\to M$ and g: $V\to M$ give rise to the same *r-jet* at 0 if they have the same partial derivatives up to order r at 0. The equivalence class of f, thus defined, is denoted by $j^r_0(f)$.

If f is a diffeomorphism of a neighborhood of 0 onto an open subset of M, then the r-jet $j_0^r(f)$ at 0 is called an *r-frame* at $x=f(0)$. The set of M will be denoted by $P^r(M)$.

Let $G^r(n)$ be the set of r-frames $j_0^r(g)$ at $0\in R^n$, where g is a diffeomorphism from a neighborhood of $0\in R^n$ onto a neighborhood of $0\in R^n$. The $G^r(n)$ is a group with multiplication defined by the composition of jets, that is, $j_0^r(g)\cdot j_0^r(g')=j_0^r(g\circ g')$. The group $G^r(n)$ acts on $P^r(M)$ on the right by $j_0^r(f)\cdot j_0^r(g)=j_0^r(f\circ g)$ for $j_0^r(f)\in P^r(M)$ and $j_0^r(g)\in G^r(n)$. Then $P^r(M)$ is a principal fibre bundle over M with group $G^r(n)$. $P^1(M)$ is nothing but the bundle of linear frames $L(M)$ with structure group $G^1(n)$ $=GL(n, R)$.

From now on we shall be mainly interested in $P^2(M)$ and $P^1(M)$.

We shall now define a 1-form on $P^2(M)$ with values in $R^n+\mathfrak{gl}(n, R)$, where $\mathfrak{gl}(n, R)$ denotes the Lie algebra of $GL(n, R)$. Let X be a vector tangent to $P^2(M)$ at $u=j_0^2(f)$. Denote by X' the image of X under the natural projection $P^2(M)\to P^1(M)$, it is a vector tangent to $P^1(M)$ at $u'=j_0^1(f)$. Since f is a diffeomorphism of a neighborhood of the origin $0\in R^n$ onto a neighborhood of $f(0)\in M$, it induces a diffeomorphism of a neighborhood of $e=j_0^1(id.)\in P^1(R^n)$ onto a neighborhood of $j_0^1(f)\in P^1(M)$. The latter induces on isomorphism of the tangent space $R^n+\mathfrak{gl}(n, R)$ of $P^1(R^n)$ at e onto the tangent space of $P^1(M)$ at $u'=j_0^1(f)$; this isomorphism will be denoted by $\tilde{u}$.

The *canonical form* θ on $P^2(M)$ is defined by

$$\theta(X)=\tilde{u}^{-1}(X').$$

Since $\tilde{u}$ depends only on $u=j_0^2(f)$, $\theta(X)$ is well defined. The 1-form θ takes its values in $R^n+\mathfrak{gl}(n, R)$.

We define an action of $G^2(n)$ on $R^n+\mathfrak{gl}(n, R)$ which will be denoted by *ad*. Let $j_0^2(g)\in G^2(n)$ and $j_0^1(f)\in P^1(R^n)$. The mapping of a neighborhood of $e\in P^1(R^n)$ onto a neighborhood of $e\in P^1(R^n)$ defined by

$$j_0^1(f)\to j_0^1(g\circ f\circ g^{-1})$$

induces a linear isomorphism of the tangent space $R^n+\mathfrak{gl}(n, R)$ of $P^1(R^n)$ at e onto itself. This linear isomorphism depends only on $j_0^2(g)$ and will be denoted by $ad(j_0^2(g))$.

Since $G^2(n)$ acts on $P^2(M)$ on the right, every element A of the Lie algebra $\mathfrak{g}^2(n)$ of $G^2(n)$ induces a vector field A^* on $P^2(M)$, which will be called the *fundamental vector field* corresponding to A.

PROPOSITION 4. *Let θ be the canonical form on $P^2(M)$. Then*

(i) $$\theta(A^*)=A' \quad \text{for} \quad A\in\mathfrak{g}^2(n)$$

where $A'\in\mathfrak{gl}(n, R)$ is the image of A under the natural homomorphism

$$\mathfrak{g}^2(n)\to\mathfrak{g}^1(n)=\mathfrak{gl}(n, R)$$

(ii) $$R_a^*\theta=ad(a^{-1})\theta \quad \text{for} \quad a\in G^2(n)$$

where R_a denotes the action of $a\in G^2(n)$ on $P^2(M)$.

PROPOSITION 5. *Let M and M' be manifolds of the same dimension n and let θ and θ' be the canonical forms on $P^2(M)$ and $P^2(M')$ respectively. Let f: $M\to M'$ be a diffeomorphism and denote by the same letter f the induced bundle isomorphism $P^2(M)\to P^2(M')$. Then*

$$f^*\theta'=\theta.$$

Conversely, if F: $P^2(M)\to P^2(M')$ is a bundle isomorphism such that

$$F^*\theta'=\theta,$$

then F is induced by a diffeomorphism f of the base manifolds.

We shall now express the canonical form of $P^2(M)$ in terms of the local coordinate system of $P^2(M)$ which arises in a natural way from a local coordinate system of M. For this purpose it suffice to consider the case $M=\mathrm{R}^n$. Let $e_1, \cdots, e_n$ be the natural basis for R^n and $(x^1, \cdots, x^n)$ the natural coordinate system in R^n. Each frame $u=j_0^2(f)$ of R^n has a unique polynomial representation of the form

$$f(x)=\Sigma\left(u^i+\Sigma u_j^i x^j+\frac{1}{2}\Sigma u_{jk}^i x^j x^k\right)e_i$$

where $x=\Sigma x^i e_i$ and $u_{jk}^i=u_{kj}^i$. We take (u^i, u_j^i, u_{jk}^i) as the natural coordinate system in $P^2(\mathrm{R}^n)$. Restricting u_j^i and u_{jk}^i to $G^2(n)$ we obtain the natural coordinate system in $G^2(n)$, which will be denoted by (s_j^i, s_{jk}^i). For $u=j_0^2(f)\in P^2(M)$ with

$$f(x)=\Sigma\left(u^i+\Sigma u_j^i x^j+\frac{1}{2}\Sigma u_{jk}^i x^j x^k\right)e_i$$

and $s=j_0^2(g)\in G^2(n)$ with

$$g(x)=\Sigma\left(\Sigma s_j^i x^j+\frac{1}{2}\Sigma s_{jk}^i x^j x^k\right)e_i$$

we have $u\cdot s=j_0^2(f\circ g)$ with

$$\begin{aligned}(f\circ g)(x)=\Sigma\Big\{&u^i+\Sigma u_j^i\left(\Sigma s_l^j x^l+\frac{1}{2}\Sigma s_{lk}^j x^l x^k\right)\\&+\frac{1}{2}\Sigma u_{jk}^i\left(\Sigma s_l^j x^l+\frac{1}{2}\Sigma s_{la}^j x^l x^a\right)\left(\Sigma s_m^k x^m+\frac{1}{2}\Sigma s_{mb}^k x^m x^b\right)\Big\}e_i\\=\Sigma\Big\{&u^i+\Sigma u_j^i s_l^j x^l+\frac{1}{2}\Sigma(u_j^i s_{lk}^j+u_{jm}^i s_l^j s_k^m)x^l x^k+\cdots\Big\}e_i.\end{aligned}$$

Hence the action of $G^2(n)$ on $P^2(\mathrm{R}^n)$ is given by

$$(u^i, u_j^i, u_{jk}^i)(s_j^i, s_{jk}^i)=(u^i, \Sigma u_l^i s_j^l, \Sigma u_l^i s_{jk}^l+\Sigma u_{lm}^i s_j^l s_k^m).$$

In particular, the multiplication in $G^2(n)$ is given by

$$(\bar{s}_j^i, \bar{s}_{jk}^i)(s_j^i, s_{jk}^i)=(\Sigma\bar{s}_l^i s_j^l, \Sigma\bar{s}_l^i s_{jk}^l+\Sigma\bar{s}_{lm}^i s_j^l s_k^m).$$

Similarly we can introduce a coordinate system (u^i, u^i_j) in $P^1(\mathrm{R}^n)$ and a coordinate system (s^i_j) in $G^1(n)$ so that the natural homomorphisms $P^2(\mathrm{R}^n)\to P^1(\mathrm{R}^n)$ and $G^2(n)\to G^1(n)$ are given by $(u^i, u^i_j, u^i_{jk})\to(u^i, u^i_j)$ and $(s^i_j, s^i_{jk})\to(s^i_j)$ respectively.

Let $\{E_i, E^j_i\}$ be the basis for $\mathrm{R}^n+\mathfrak{gl}(n, \mathrm{R})$ defined by $E_i=(\partial/\partial u^i)_e$, $E^j_i=(\partial/\partial u^i_j)_e$. We set

$$\theta=\Sigma\theta^i E_i+\Sigma\theta^i_j E^j_i.$$

From the definition of the canonical form θ, we obtain by a straightforward calculation the following formulae (cf. [4]);

$$\theta^i=\Sigma v^i_k du^k,$$
$$\theta^i_j=\Sigma v^i_k du^k_j-\Sigma v^i_k u^k_{hj} v^h_l du^l,$$

where (v^i_j) denotes the inverse matrix of (u^i_j). From these formulae we have

PROPOSITION 6. *Let $\theta=(\theta^i, \theta^i_j)$ be the canonical form on $P^2(M)$. Then*

$$d\theta^i=-\Sigma\theta^i_k\wedge\theta^k.$$

§ 4. Möbius spaces and Möbius groups.

Let E^n be a Euclidean space of dimension n with coordinate system $(y^1, \cdots, y^n)$ and with metric $\varepsilon=(\varepsilon_{ij})$.

Let E^{n+2} be a Euclidean space of dimension $n+2$ with coordinate system $(y^0, y^1, \cdots, y^n, y^\infty)$, and with metric

$$\tilde{\varepsilon}=(\tilde{\varepsilon}_{\alpha\beta})=\begin{pmatrix} 0 & 0 & -1 \\ 0 & \varepsilon_{ij} & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$ [3)]

Let P_{n+1} be the real projective space of dimension $n+1$, constructed from E^{n+2}, with homogeneous coordinate system $(y^0, y^1, \cdots, y^n, y^\infty)$. Let $\Xi^n=\mathrm{E}^n\cup\{\infty\}$ be the one point compactification of E^n by a so-called "point at infinity".

A hypersphere S^{n-1} in E^n may be represented by the ratio of $n+2$ real numbers $a^0, a^1, \cdots, a^n, a^\infty$ as follows:

$$a^0\Sigma\varepsilon_{jk}y^j y^k-2\Sigma\varepsilon_{jk}a^j y^k+2a^\infty=0. \tag{1}$$

A point $(a^0, a^1, \cdots, a^n, a^\infty)$ in $\mathrm{E}^{n+2}-\{0\}$ can also be considered as a point in P^{n+1}.

If $a^0\neq 0$ and $\Sigma\varepsilon_{jk}a^j a^k-2a^0a^\infty\geqq 0$, the equation (1) gives a real hypersphere of radius $\{(\Sigma\varepsilon_{jk}a^j a^k-2a^0a^\infty)/a^0a^0\}^{1/2}$ and centered at $(a^1/a^0, \cdots, a^n/a^0)$. In particular, $\Sigma\varepsilon_{jk}a^j a^k-2a^0a^\infty=0$ is the condition for the equation (1) to represent a point sphere, that is, a single point $(a^1/a^0, \cdots, a^n/a^0)$.

Let $\mathfrak{S}$ denote the set of all point hyperspheres. If we let the special case

3) Indices $\alpha, \beta, \cdots$ run over the range $0, 1, 2, \cdots, n, \infty$.

$a^0=a^1=\cdots=a^n=0$ correspond to the point at infinity $\{\infty\}$ in Ξ^n, the elements of $\mathfrak{S}$ are in one-to-one correspondence with the points of Ξ^n.

Let Q be the quadric in P_{n+1} defined by the equation

$$\Sigma\varepsilon_{jk}y^j y^k-2y^0y^\infty=0.$$

Then the elements of $\mathfrak{S}$ are in one-to-one correspondence with the points of Q.

We set $x^i=y^i/y^0$ for $i=1,\cdots,n$ and we shall take $(x^1,\cdots,x^n)$ as a local coordinate system of Ξ^n in the neighborhood defined by $y^0\neq 0$. Then Ξ^n is homeomorphic with Q. We call Ξ^n the *Möbius space* of dimension n.

An element of the projective transformation group $PL(n+1,\mathrm{R})$ of P_{n+1} which leaves Q invariant induces a transformation of Ξ^n.

Let $\tilde{O}(n+2)$ denote the set of all elements $s=(s^\alpha_\beta)$ of $GL(n+2,\mathrm{R})$ which leave the metric $\tilde{\varepsilon}$ invariant, that is, $\Sigma\tilde{\varepsilon}_{\lambda\mu}s^\lambda_\alpha s^\mu_\beta=\tilde{\varepsilon}_{\alpha\beta}$, and denote by $\tilde{Q}$ the cone in E^{n+2} defined by the equation $\Sigma\tilde{\varepsilon}_{\alpha\beta}y^\alpha y^\beta=0$. Then $\tilde{O}(n+2)$ acts transitively on $\tilde{Q}$ and every element of $\tilde{O}(n+2)$ leaves $\tilde{Q}$ invariant. Hence it induces a transformation of Ξ^n. The group of transformations of Ξ^n induced from $\tilde{O}(n+2)$ is called the *Möbius group* of Ξ^n and denoted by $K(n)$. $K(n)$ is isomorphic with the factor group of $\tilde{O}(n+2)$ by the subgroup $\{e,-e\}$, where e denotes the identity of $\tilde{O}(n+2)$.

Let $y=(y^0,y^i,y^\infty)$ and $\bar{y}=(\bar{y}^0,\bar{y}^i,\bar{y}^\infty)$ with $\Sigma\tilde{\varepsilon}_{\alpha\beta}y^\alpha y^\beta=0$ $\Sigma\tilde{\varepsilon}_{\alpha\beta}\bar{y}^\alpha\bar{y}^\beta=0$ be two points in $\tilde{Q}$. Let f be a transformation of $\tilde{Q}$ given by $\bar{y}=f(y)$. Then there exists an element $s=(s^\alpha_\beta)$ in $\tilde{O}(n+2)$ such that $\bar{y}^\alpha=\Sigma s^\alpha_\beta y^\beta$. Corresponding with the transformation f of $\tilde{Q}$ we can induce a transformation of Ξ^n and denote it by the same letter f which is given by $\bar{x}=f(x)$ with $x^i=y^i/y^0$, $\bar{x}^i=\bar{y}^i/\bar{y}^0$. Then

$$\bar{x}^i=\frac{\Sigma s^i_\beta y^\beta}{\Sigma s^0_\beta y^\beta}=\frac{s^i_0y^0+\Sigma s^i_jy^j+s^i_\infty y^\infty}{s^0_0y^0+\Sigma s^0_jy^j+s^0_\infty y^\infty}=\frac{s^i_0+\Sigma s^i_j(y^j/y^0)+s^i_\infty(y^\infty/y^0)}{s^0_0+\Sigma s^0_j(y^j/y^0)+s^0_\infty(y^\infty/y^0)}.$$

On the other hand, the equation $\Sigma\tilde{\varepsilon}_{\alpha\beta}y^\alpha y^\beta=0$ implies $\Sigma\varepsilon_{jk}y^jy^k-2y^0y^\infty=0$, that is, $\Sigma\varepsilon_{jk}x^jx^k=2y^\infty/y^0$. Hence we have

$$(2)\qquad \bar{x}^i=\frac{s^i_0+\Sigma s^i_jx^j+\frac{1}{2}\Sigma s^i_\infty\varepsilon_{jk}x^jx^k}{s^0_0+\Sigma s^0_jx^j+\frac{1}{2}\Sigma s^0_\infty\varepsilon_{jk}x^jx^k}.$$

Under the conditions $\Sigma\tilde{\varepsilon}_{\lambda\mu}s^\lambda_\alpha s^\mu_\beta=\tilde{\varepsilon}_{\alpha\beta}$, components s^α_β of s are completely determined by s^0_0, s^i_0, s^0_j and s^i_j. Hence we set

$$(3)\qquad a^i=\frac{s^i_0}{s^0_0},\qquad a^i_j=\frac{s^i_j}{s^0_0},\qquad a_j=\frac{s^0_j}{s^0_0}$$

and we shall take (a^i,a^i_j,a_j) as a local coordinate system of $K(n)$ in the neighborhood of the identity defined by $s^0_0\neq 0$. We see, from the construction, that (a^i_j) is an element of $CO(n)$, the conformal group with respect to the metric ε. Hence the group $K(n)$ is a semidirect product of R^n, $CO(n)$ and $(\mathrm{R}^n)^*$.

PROPOSITION 7. *Let $\omega=(\omega^i,\omega^i_j,\omega_j)$ be the Maurer-Cartan forms on $K(n)$ which coincide with da^i, da^i_j, da_j at the identity. Then the equations of Maurer-Cartan of*

$K(n)$ are given by

$$
(4)\qquad
\begin{aligned}
&d\omega^i=-\Sigma\omega^i_k\wedge\omega^k,\\
&d\omega^i_j=-\Sigma\omega^i_k\wedge\omega^k_j-\omega^i\wedge\omega_j-\Sigma\varepsilon^{ik}\varepsilon_{jl}\omega_k\wedge\omega^l+\delta^i_j\Sigma\omega_k\wedge\omega^k,\\
&d\omega_j=-\Sigma\omega_k\wedge\omega^k_j,
\end{aligned}
$$

where $(\varepsilon^{ij})=(\varepsilon_{ij})^{-1}$.

Proof. If we set

$$(\bar\omega^\alpha_\beta)=s^{-1}ds\in\tilde{\mathfrak{o}}(n+2),\qquad \text{where}\quad s=(s^\alpha_\beta)\in\tilde{O}(n+2),$$

then we have $\Sigma\tilde\varepsilon_{\gamma\beta}\bar\omega^\gamma_\alpha+\Sigma\tilde\varepsilon_{\alpha\gamma}\omega^\gamma_\beta=0$, that is,

$$
(5)\qquad
\begin{aligned}
&\bar\omega^0_0+\bar\omega^\infty_\infty=0,\qquad \bar\omega^\infty_0=0,\qquad \bar\omega^\infty_j=\Sigma\varepsilon_{kj}\bar\omega^k_0,\\
&\Sigma\varepsilon_{kj}\bar\omega^k_i+\Sigma\varepsilon_{ik}\bar\omega^k_j=0,\qquad \bar\omega^i_\infty=\Sigma\varepsilon^{ki}\bar\omega^0_k,\qquad \bar\omega^0_\infty=0.
\end{aligned}
$$

Thus we have

$$\bar\omega=(\bar\omega^\alpha_\beta)=\begin{pmatrix}\bar\omega^0_0 & \bar\omega^0_j & 0\\ \bar\omega^i_0 & \bar\omega^i_j & \Sigma\varepsilon^{ik}\bar\omega^0_k\\ 0 & \Sigma\varepsilon_{kj}\bar\omega^k_0 & -\bar\omega^0_0\end{pmatrix}.$$

If we set $s=e$, then we get $\bar\omega^\alpha_\beta=ds^\alpha_\beta$. On the other hand, we get from (3)

$$
(6)\qquad
\begin{aligned}
&da^i=ds^i_0,\\
&da^i_j=ds^i_j-\delta^i_j ds^0_0,\\
&da_j=ds^0_j
\end{aligned}
$$

at the identity e. Moreover $\omega^i=da^i$, $\omega^i_j=da^i_j$, $\omega_j=da_j$ at the identity, hence we have

$$
(7)\qquad
\begin{aligned}
&\omega^i=\bar\omega^i_0,\\
&\omega^i_j=\bar\omega^i_j-\delta^i_j\bar\omega^0_0,\\
&\omega_j=\bar\omega^0_j.
\end{aligned}
$$

The equation $\bar\omega=s^{-1}ds$ implies $d\bar\omega^\alpha_\beta=-\Sigma\bar\omega^\alpha_\gamma\wedge\bar\omega^\gamma_\beta$ from which our proposition follows, since the Lie group $K(n)$ is isomorphic with $\tilde{O}(n+2)/\{e,-e\}$. (Q.E.D.)

The dual of Proposition 7 may be formulated as follows. Let $\mathfrak{m}=\mathrm{R}^n$, $\mathfrak{m}^*$ be its dual and let $\mathfrak{co}(n)$ be the Lie algebra of $CO(n)$.

PROPOSITION 8. *The Lie algebra $\mathfrak{k}(n)$ of $K(n)$ is the direct sum:*

$$\mathfrak{k}(n)=\mathfrak{m}+\mathfrak{co}(n)+\mathfrak{m}^*$$

with the following bracket operation; If $u, v\in\mathfrak{m}$, $u^*, v^*\in\mathfrak{m}^*$ *and* $U, V\in\mathfrak{co}(n)$, *then*

$$[u, v]=0, \qquad [u^*, v^*]=0,$$
$$[U, u]=Uu, \qquad [u^*, U]=u^*U,$$
$$[U, V]=UV-VU,$$
$$[u, u^*]=u\otimes u^*-\widetilde{u^*\otimes u}+u^*(u)\cdot I$$

where $\widetilde{u^*\otimes u}$ *denotes its dual under the isomorphism* $\mathfrak{m}^*\otimes\mathfrak{m}\to\mathfrak{m}\otimes\mathfrak{m}^*$ *and* I *denotes the identity matrix of degree* n.

The left invariant vector fields on $K(n)$ which coincide with $\partial/\partial a^i$, $\partial/\partial a^i_j$, $\partial/\partial a_j$ at the identity form a natural basis for $\mathfrak{m}$, $\mathfrak{co}(n)$ and $\mathfrak{m}^*$ respectively. Let 0 be the point of the Möbius space Ξ^n with coordinate $(0, \cdots, 0)$. Let H be the isotropy subgroup of $K(n)$ at 0 so that $\Xi^n=K(n)/H$. Then H is the semidirect product of $CO(n)$ and $(\mathrm{R}^n)^*$, and the Lie algera $\mathfrak{h}$ of H is given by $\mathfrak{co}(n)+\mathfrak{m}^*$. Proposition 8 implies that the homogeneous space $\Xi^n=K(n)/H$ is not weakly reductive.

In terms of the local coordinate system (a^i, a^i_j, a_j) of $K(n)$ which is valid in a neighborhood containing H, the subgroup H is defined by $a^i=0$. For the elements of H we have from

$$\Sigma\tilde{\varepsilon}_{\lambda\mu}s^\lambda_\alpha s^\mu_\beta=\tilde{\varepsilon}_{\alpha\beta} \qquad \text{and} \qquad s^i_0=0$$

that

$$\begin{aligned} &s^\infty_0=0,\\ &s^\infty_j=0,\\ &s^0_0 s^\infty_\infty=1,\\ (8)\qquad &\Sigma\varepsilon_{kl}s^k_i s^l_j=\varepsilon_{ij},\\ &\Sigma\varepsilon_{kl}s^k_i s^l_\infty=s^0_i s^\infty_\infty,\\ &\Sigma\varepsilon_{kl}s^k_\infty s^l_\infty=2s^0_\infty s^\infty_\infty. \end{aligned}$$

We have also, from the equations (8),

$$s^i_\infty=\frac{1}{s^0_0}\Sigma\varepsilon^{jk}s^0_j s^i_k$$

and

$$s^0_\infty=\frac{1}{2s^0_0}\Sigma\varepsilon^{jk}s^0_j s^0_k.$$

Thus the transformation induced by an element of H is given by the equation of the form;

$$\bar{x}^i = \frac{\Sigma s^i_j x^j + (1/2s^0_0)\Sigma \varepsilon^{al} s^0_a s^i_l \varepsilon_{jk} x^j x^k}{s^0_0 + \Sigma s^0_j x^j + (1/4s^0_0)\Sigma \varepsilon^{al} s^0_a s^0_l \varepsilon_{jk} x^j x^k}$$
$$= \frac{\Sigma a^i_j x^j + (1/2)\Sigma \varepsilon^{al} \varepsilon_{jk} a_a a^i_l x^j x^k}{1 + \Sigma a_j x^j + (1/4)\Sigma \varepsilon^{al} \varepsilon_{jk} a_a a_l x^j x^k}$$

hence we have

$$(9) \qquad \bar{x}^i = \Sigma a^i_j x^j + \frac{1}{2}\Sigma(\varepsilon^{al}\varepsilon_{jk} a_a a^i_l - a^i_j a_k - a^i_k a_j) x^j x^k + \cdots.$$

§ 5. Cartan connections.

Let M be a manifold of dimension n, G a Lie group, H a closed subgroup of G with $\dim G/H = n$ and P a principal fibre bundle over M with structure group H.

Since H acts on P on the right, every element A of the Lie algebra $\mathfrak{h}$ of H, as is well known, induces in a natural manner a vector field on P, called the *fundamental vector field* corresponding to A. This vector field will be denoted by A^*. Since H acts along fibres, A^* is vertical, that is, tangent to the fibre at each point. For each element $a \in H$, the action of a on P will be denoted by R_a. We are now in position to define the notion of Cartan connection. It is a 1-form ω on P with value in the Lie algebra $\mathfrak{g}$ of G satisfying the following conditions:

(a) $\omega(A^*) = A$ for every $A \in \mathfrak{h}$

(b) $R^*_a \omega = ad(a^{-1}) \cdot \omega$, that is, $\omega(R_a X) = ad(a^{-1}) \cdot \omega(X)$ for every $a \in H$ and every vector X of P, where ad denotes the adjoint representation of H on $\mathfrak{g}$;

(c) $\omega(X) \neq 0$ for every non zero vector X of P.

The condition (c) means that ω defines an isomorphism of the tangent space at each point of P onto the Lie algebra $\mathfrak{g}$ and hence implies the absolute parallelizability of P.

Let G be the Möbius group $K(n)$ acting on an n-dimensional Möbius space and H be an isotropy subgroup of G so that G/H is the Möbius space. Let M be an arbitrary manifold of dimension n and P be a principal fibre bundle over M with structure group H. We fix the natural basis for the Lie algebra $\mathfrak{k}(n)$ as described in § 4.

A Cartan connection ω in P is then given, with respect to this basis, by a set of 1-forms $\omega^i, \omega^i_j, \omega_j$ on P.

The *structure equations* of the Cartan connection ω are given by

$$(\mathrm{I}) \qquad d\omega^i = -\Sigma \omega^i_k \wedge \omega^k + \Omega^i,$$

$$(\mathrm{II}) \qquad d\omega^i_j = -\Sigma \omega^i_k \wedge \omega^k_j - \omega^i \wedge \omega_j - \Sigma \varepsilon^{ik} \varepsilon_{jl} \omega_k \wedge \omega^l + \delta^i_j \Sigma \omega_k \wedge \omega^k + \Omega^i_j,$$

$$(\mathrm{III}) \qquad d\omega_j = -\Sigma \omega_k \wedge \omega^k_j + \Omega_j.$$

For the sake of simplicity, we shall take these equations as a definition of the 2-forms Ω^i, Ω^i_j, Ω_j. We call (Ω^i) the *torsion form* of the Cartan connection ω and (Ω^i_j, Ω_j) the *curvature form* of ω.

PROPOSITION 9. *The torsion and the curvature forms can be written as follows*:

$$\Omega^i=\frac{1}{2}\Sigma K^i{}_{kl}\omega^k\wedge\omega^l,$$

(10) $$\Omega^i_j=\frac{1}{2}\Sigma K^i{}_{jkl}\omega^k\wedge\omega^l,$$

$$\Omega_j=\frac{1}{2}\Sigma K_{jkl}\omega^k\wedge\omega^l$$

where $K^i{}_{kl}$, $K^i{}_{jkl}$ *and* K_{jkl} *are functions on* P.

Proof. Condition (c) implies that the algebra of differential forms on P is generated by ω^i, ω^i_j, ω_j and functions.

To show that the torsion and the curvature forms do not involves ω^i_j and ω_j, it is sufficient to prove the following three statements;

(i) The forms ω^i, restricted to each fibre of P, vanish identically;

(ii) The forms ω^i_j and ω_j, restricted to each fibre, remain linearly independent at every point of the fibre;

(iii) The torsion and curvature forms, restricted to each fibre, vanish identically.

Condition (a) implies (i) and (ii).

To prove (iii), consider the restriction of the structure equation (I) to a fibre, then by (i), the torsion form, restricted to the fibre, vanishes identically. By condition (a), the restriction of the structure equations (II) and (III) to a fibre must coincide with the Maurer-Cartan equation of H. It follows that the curvature form, restricted to the fibre, vanishes identically. (Q.E.D.)

In order that the form $\omega=(\omega^i, \omega^i_j, \omega_j)$ defines a Cartan connection in P, the following conditions must be imposed on ω^i and ω^i_j;

(a′) $\omega^i(A^*)=0$ and $\omega^i_j(A^*)=A^i_j$ for every $A=(A^i_j, A_j)\in\mathfrak{co}(n)+\mathfrak{m}^*=\mathfrak{h}$ where A^* is the fundamental vector field corresponding to A;

(b′) $R^*_a(\omega^i, \omega^i_j)=ad(a^{-1})(\omega^i, \omega^i_j)$ for every $a\in H$, where

$$ad(a^{-1}): \mathfrak{m}+\mathfrak{co}(n)\to\mathfrak{m}+\mathfrak{co}(n)$$

is the mapping

$$\mathfrak{k}(n)/\mathfrak{m}^*\to\mathfrak{k}(n)/\mathfrak{m}^*$$

induced by

$$ad(a^{-1}): \mathfrak{k}(n)\to\mathfrak{k}(n),$$

(c′) If X is a tangent vector to P such that $\omega^i(X)=0$, then X is vertical.

PROPOSITION 10. *Let P be a principal fibre bundle over M with structure group H. Given ω^i, and ω^i_j satisfying* (a′), (b′), (c′) *and*

(11) $$d\omega^i = -\Sigma \omega^i_k \wedge \omega^k$$

then there exists a unique Cartan connection $\omega = (\omega^i, \omega^i_j, \omega_j)$ *with the following properties*:

(12) $$\Sigma \Omega^i_i = 0, \qquad \text{i.e., } \Sigma K^i{}_{ijk} = 0,$$

(13) $$\Sigma K^i{}_{jil} = 0.$$

Proof. Uniqueness. We shall study first the relationship between two Cartan connections $\omega = (\omega^i, \omega^i{}_j, \omega_j)$ and $\bar\omega = (\omega^i, \omega^i_j, \bar\omega_j)$ with the given (ω^i, ω^i_j). By conditions (a) and (c), we can write

$$\bar\omega_j - \omega_j = \Sigma A_{jk}\omega^k,$$

where the coefficients A_{jk} are functions on P. Let

$$\Omega^i_j = \frac{1}{2}\Sigma K^i{}_{jkl}\omega^k \wedge \omega^l$$

and

$$\bar\Omega^i_j = \frac{1}{2}\Sigma \bar K^i{}_{jkl}\omega^k \wedge \omega^l$$

be defined by the structure equations (II) of the Cartan connections ω and $\bar\omega$ respectively. Then we have

$$\begin{aligned}\bar\Omega^i_j - \Omega^i_j &= \omega^i \wedge (\bar\omega_j - \omega_j) + \Sigma \varepsilon^{ik}\varepsilon_{jl}(\bar\omega_k - \omega_k)\wedge\omega^l - \delta^i_j \Sigma(\bar\omega_k - \omega_k)\wedge\omega^k \\ &= \Sigma A_{jk}\omega^i \wedge \omega^k + \Sigma \varepsilon^{ik}\varepsilon_{jl}A_{km}\omega^m \wedge \omega^l - \delta^i_j \Sigma A_{kl}\omega^l \wedge \omega^k \\ &= \Sigma(-\delta^i_l A_{jk} + \Sigma \varepsilon^{ia}\varepsilon_{jl}A_{ak} + \delta^i_j A_{kl})\omega^k \wedge \omega^l\end{aligned}$$

that is,

$$\bar K^i{}_{jkl} - K^i{}_{jkl} = -\delta^i_l A_{jk} + \delta^i_k A_{jl} + \Sigma \varepsilon^{ia}\varepsilon_{jl}A_{ak} - \Sigma \varepsilon^{ia}\varepsilon_{jk}A_{al} + \delta^i_j(A_{kl} - A_{lk}).$$

Hence

$$\Sigma \bar K^i{}_{ikl} - \Sigma K^i{}_{ikl} = n(A_{kl} - A_{lk}),$$

$$\Sigma \bar K^i{}_{jil} - \Sigma K^i{}_{jil} = (n-1)A_{jl} - A_{lj} + \varepsilon_{jl}\Sigma \varepsilon^{ka}A_{ak}.$$

The conditions (12) and (13) imply

(14) $$A_{kl} = A_{lk}$$

and

(15) $$(n-1)A_{jl} - A_{lj} + \varepsilon_{jl}\Sigma \varepsilon^{ka}A_{ak} = 0.$$

From (14) and (15), we have

$$(n-2)A_{jl}+\varepsilon_{jl}\Sigma\varepsilon^{ka}A_{ak}=0.$$

Multiplying by ε^{jl} and summing with respect to j and l, we obtain

$$(n-1)\Sigma\varepsilon^{ka}A_{ak}=0,$$

hence

$$\Sigma\varepsilon^{ka}A_{ak}=0 \qquad \text{if} \quad n>1.$$

Thus we get $A_{jl}=0$ if $n>2$, in other words, $\bar{\omega}=\omega$ if $n>2$.

Existence. Assuming that there is at least one Cartan connection $\bar{\omega}=(\omega^i, \omega^i_j, \bar{\omega}_j)$ with the given (ω^i, ω^i_j) satisfying (11), we shall show the existence of a Cartan connection $\omega=(\omega^i, \omega^i_j, \omega_j)$ satisfying (12) and (13). If we define

$$A_{jk}=\frac{1}{n-2}\Sigma\bar{K}^i_{jik}-\frac{1}{n(n-2)}\Sigma\bar{K}^i{}_{ijk}-\frac{1}{2(n-1)(n-2)}\varepsilon_{jk}\Sigma\varepsilon^{al}\bar{K}^i{}_{ail} \tag{16}$$

and set

$$\omega_j=\bar{\omega}_j-\Sigma A_{jk}\omega^k$$

then $\omega=(\omega^i, \omega^i_j, \omega_j)$ is a Cartan connection with the required properties.

To complete the proof of the proposition, we have now only to prove that there exists at least one Cartan connection ω with the given (ω^i, ω^i_j). Let $\{U_\alpha\}$ be a locally finite open covering of M with a partition of unity $\{\varphi_\alpha\}$. If ω_α is a Cartan connection in $P|U_\alpha$ with the given (ω^i, ω^i_j), then $\Sigma(\varphi_\alpha\circ\pi)\omega_\alpha$ is a Cartan connection in P with the given (ω^i, ω^i_j) where π: $P\to M$ is the projection. Hence, our problem is reduced to the case where P is a trivial bundle. Fix a cross section σ: $M\to P$, and set $\omega_j(X)=0$ for every vector tangent to $\sigma(M)$. If Y is an arbitrary vector of P, then we can write uniquely

$$Y=R_aX+V$$

where X is a vector tangent to $\sigma(M)$ and $a\in H$ and V is a vector tangent to a fibre of P so that V can be extended to a unique fundamental vector field A^* of P with $A\in\mathfrak{h}$. By condition (a) and (b), a Cartan connection ω must satisfy the following condition:

$$\omega(Y)=ad(a^{-1})\cdot\omega(X)+A.$$

This determines $\omega_j(Y)$. (Q.E.D.)

PROPOSITION 11. *Let P be a principal fibre bundle over M with structure group H. If $\omega=(\omega^i, \omega^i_j, \omega_j)$ is a Cartan connection with the properties* (11), (12) *and* (13) *of Proposition* 10, *then its curvature forms possess the following properties*:

$$\Sigma\Omega^i_j\wedge\omega^j=0, \quad \textit{that is,} \quad K^i{}_{jkl}+K^i{}_{klj}+K^i{}_{ljk}=0. \tag{17}$$

$$\Sigma\Omega_j\wedge\omega^j=0, \quad \textit{that is,} \quad K_{jkl}+K_{klj}+K_{ljk}=0, \tag{18}$$

(19) *If* $\Omega^i_j=0$ *and* $\dim M>3$, *then* $\Omega_j=0$.

Proof. (17). From the structure equation (II) of a Cartan connection, we have

$$\begin{aligned}\Sigma\Omega^i_j\wedge\omega^j&=\Sigma d\omega^i_j\wedge\omega^j+\Sigma\omega^i_k\wedge\omega^k_j\wedge\omega^j+\Sigma\omega^i\wedge\omega_j\wedge\omega^j\\&\quad+\Sigma\varepsilon^{ik}\varepsilon_{jl}\omega_k\wedge\omega^l\wedge\omega^j-\Sigma\delta^i_j\omega_k\wedge\omega^k\wedge\omega^j\\&=\Sigma d\omega^i_j\wedge\omega^j+\Sigma\omega^i_k\wedge(-d\omega^k)\\&=d\Sigma(\omega^i_j\wedge\omega^j)\\&=d(-d\omega^i)\\&=0.\end{aligned}$$

(18). From the structure equation (III), we get

$$\begin{aligned}\Sigma\Omega_j\wedge\omega^j&=\Sigma d\omega_j\wedge\omega^j+\Sigma\omega_k\wedge\omega^k_j\wedge\omega^j\\&=\Sigma d\omega_j\wedge\omega^j+\Sigma\omega_k\wedge(-d\omega^k)\\&=d\Sigma(\omega_j\wedge\omega^j).\end{aligned}$$

On the other hand, taking the trace of the structure equation (II) and taking account of (12) we get

$$\Sigma d\omega^i_i=n\Sigma\omega_i\wedge\omega^i,$$

that is $\Sigma\omega_i\wedge\omega^i$ is a exact form, hence

$$\Sigma\Omega_j\wedge\omega^j=0.$$

(19). By applying exterior differentiation to the structure eqation (II) and setting $\Omega^i_j=0$, we obtain

$$\omega^i\wedge\Omega_j-\Sigma\varepsilon^{ik}\varepsilon_{jl}\Omega_k\wedge\omega^l+\delta^i_j\Sigma\Omega_k\wedge\omega^k=0.$$

This, together with (18), implies

$$\omega^i\wedge\Omega_j-\Sigma\varepsilon^{ik}\varepsilon_{jl}\Omega_k\wedge\omega^l=0,$$

that is,

$$\Sigma\varepsilon^{ik}\Omega_k\wedge\omega^j-\Sigma\varepsilon^{jk}\Omega_k\wedge\omega^i=0.$$

Then $\Sigma\varepsilon^{ik}\Omega_k\wedge\omega^j\wedge\omega^i=0$. Hence $\Sigma\varepsilon^{ik}\Omega_k\wedge\omega^i=0$ provided that $\dim M>3$. This, together with Proposition 9, implies that there exist 1-forms τ^i such that

$$\Sigma\varepsilon^{ik}\Omega_k=\tau^i\wedge\omega^i.$$

Thus we have

$$\begin{aligned}0&=\tau^i\wedge\omega^i\wedge\omega^j-\tau^j\wedge\omega^j\wedge\omega^i\\&=(\tau^i+\tau^j)\wedge\omega^i\wedge\omega^j.\end{aligned}$$

This implies that $\tau^i+\tau^j$ is a linear combination of ω^i and ω^j for any i and j $(i\neq j)$. Therefore we can easily see that τ^i is proportional to ω^i. Hence we have $\Omega_j=0$. (Q.E.D.)

§ 6. Conformal structures and conformal connections.

Let $H^2(n)$ be the subset of $G^2(n)$ consisting of elements (a^i_j, a^i_{jk}) with $\Sigma\varepsilon_{kl}a^k_ia^l_j=\rho\varepsilon_{ij}$ $(\rho>0)$, that is, $(a^i_j)\in CO(n)$, and $a^i_{jk}=\Sigma\varepsilon^{al}\varepsilon_{jk}a_aa^i_l-a^i_ja_k-a^i_ka_j$ for some (a_j)

PROPOSITION 12. *$H^2(n)$ forms a subgroup of $G^2(n)$ of dimension $n(n+1)/2+1$.*

Proof. Let (a^i_j, a^i_{jk}) and $(\bar{a}^i_j, \bar{a}^i_{jk})$ be in $H^2(n)$. By the consideration in § 3, we have

$$(\bar{a}^i_j, \bar{a}^i_{jk})(a^i_j, a^i_{jk})=(\Sigma\bar{a}^i_la^l_j, \Sigma\bar{a}^i_la^l_{jk}+\Sigma\bar{a}^i{}_{lm}a^l_ja^m_k).$$

Since $a^i_{jk}=\Sigma\varepsilon^{al}\varepsilon_{jk}a_aa^i_l-a^i_ja_k-a^i_ka_j$ and $\bar{a}^i_{jk}=\Sigma\varepsilon^{al}\varepsilon_{jk}\bar{a}_a\bar{a}^i_l-\bar{a}^i_j\bar{a}_k-\bar{a}^i_k\bar{a}_j$, we get

$$\Sigma\bar{a}^i_la^l_{jk}+\Sigma\bar{a}^i_{lm}a^l_ja^m_k=\Sigma\varepsilon^{al}\varepsilon_{jk}b_ab^i_l-b^i_jb_k-b^i_kb_j$$

where $b_j=a_j+\Sigma\bar{a}_ka^k_j$, $b^i_j=\Sigma\bar{a}^i_la^l_j\in CO(n)$. This implies $(\bar{a}^i_j, \bar{a}^i_{jk})(a^i_j, a^i_{jk})\in H^2(n)$. (Q.E.D.)

The Lie algebra $\mathfrak{h}^2(n)$ of $H^2(n)$ is the direct sum:

$$\mathfrak{h}^2(n)=\mathfrak{co}(n)+\mathfrak{co}(n)^{(1)}$$

with the following bracket operation; If (A^i_j), $(B^i_j)\in\mathfrak{co}(n)$ and (A^i_{jk}), $(B^i_{jk})\in\mathfrak{co}(n)^{(1)}$, then

$$[(A^i_j), (B^i_j)]=(\Sigma A^i_kB^k_j-\Sigma B^i_kA^k_j)\in\mathfrak{co}(n),$$
$$[(A^i_j), (B^i_{jk})]=(\Sigma A^i_lB^l_{jk}-\Sigma B^i_{lk}A^l_j-\Sigma B^i_{lj}A^l_k)\in\mathfrak{co}(n)^{(1)}$$

and

$$[(A^i_{jk}), (B^i_{jk})]=0.$$

As in § 4, let H be the isotropy subgroup at $0\in\Xi^n$ of $K(n)$ acting on the Möbius space Ξ^n.

PROPOSITION 13. *For each element $a\in H$, let f be the transformation of Ξ^n induced by a as in § 4. Then $a\rightarrow j^2_0(f)$ gives an isomorphism of H onto $H^2(n)$. Moreover if $a\in H$ has coordinate (a^i, a^i_j, a_j) where $a^i=0$, with respect to the local coordinate system in $K(n)$ induced in § 4, then the corresponding element of $H^2(n)$ has coordinate $(a^i_j, \Sigma\varepsilon^{al}\varepsilon_{jk}a_aa^i_l-a^i_ja_k-a^i_ka_j)$.*

Proof. This is evident from the explicit expression (9) of the transformation f. (cf. Proposition 2) (Q.E.D.)

The induced isomorphism of $\mathfrak{h}$ onto $\mathfrak{h}^2(n)$ is given by $(A^i_j, A_j) \to (A^i_j, \Sigma\varepsilon^{ia}\varepsilon_{jk}A_a - \delta^i_j A_k - \delta^i_k A_j)$.

From Proposition 13 and the proof of Proposition 12, we see that the multiplication in H is given by $(\bar{a}^i_j, \bar{a}_j)(a^i_j, a_j) = (\Sigma \bar{a}^i_k a^k_j, a_j + \Sigma \bar{a}_k a^k_j)$.

From Propositions 2, 3 and 13, a $CO(n)$-structure on a manifold M is equivalent to the reduction of the structure group $G^2(n)$ of $P^2(M)$ to the subgroup $H^2(n)$. (cf. [2]).[4)]

A *conformal structure* on a manifold M is, by definition, a sub-bundle P of $P^2(M)$ with structure group $H^2(n)$.

Let $\theta = (\theta^i, \theta^i_j)$ be the canonical form on $P^2(M)$. Given a conformal structure P on M, let us denote by the same letters the restriction of θ to P.

A *conformal connection* associated with a conformal structure P is, by definition, a Cartan connection $\omega = (\omega^i, \omega^i_j, \omega_j)$ in P such that $\omega^i = \theta^i$.

THEOREM 14. *For each conformal structure P of a manifold M, there is a unique conformal connection $\omega = (\omega^i, \omega^i_j, \omega_j)$ such that*

(i) $\qquad \omega^i = \theta^i$ *and* $\omega^i_j = \theta^i_j$ *so that* $d\omega^i = -\Sigma \omega^i_k \wedge \omega^k$,

(ii) $\qquad \Sigma \Omega^i_i = 0$,

(iii) $\qquad \Sigma K^i{}_{jil} = 0$.

Proof. This is an immediate consequence of Propositions 4, 6 and 10.

(Q.E.D.)

The unique conformal connection for P given in Theorem 14 is called the *normal conformal connection* associated with the conformal structure P.

The cohomology class determined by the torsion form (Ω^i) is called the *first order structure tensor* of the conformal structure P, and the cohomology classes determined by the curvature forms (Ω^i_j) and (Ω_j) are called the *second* and the *third order structure tensors* of P respectively.

A Möbius space $\Xi^n = K(n)/H$ of dimension n has a natural conformal structure. The normal conformal connection $(\omega^i, \omega^i_j, \omega_j)$ associated with it corresponds to the Maurer-Cartan form of the group $K(n)$ and its structure equations are nothing but the equations of Maurer-Cartan for the group $K(n)$ so that $\Omega^i = 0$, $\Omega^i_j = 0$ and $\Omega_j = 0$.

§ 7. Natural frames and coefficients of conformal connections.

Let P be a conformal structure on a manifold M and U a coordinate neighborhood in M with local coordinate system $(x^1, \cdots, x^n)$. Let σ: $U \to P$ be a local cross section given by $(x^i) \to (x^i, \sigma^i_j, \sigma^i_{jk})$ and $U \times H^2(n) \cong P|U$ the isomorphism induced by σ. Let (a^i_j, a^i_{jk}), with $\Sigma \varepsilon_{kl} a^k_i a^l_j = \rho \varepsilon_{ij}$ $(\rho > 0)$ and $a^i_{jk} = \Sigma \varepsilon^{al} \varepsilon_{jk} a_a a^i_l - a^i_j a_k - a^i_k a_j$, be the coordinate in $H^2(n)$. Then the natural coordinate system (u^i, u^i_j, u^i_{jk}) in $P|U$ can be written as

4) Every $CO(n)$-structure is 1-*flat* and hence has a unique prolonged subbundle of $P^2(M)$.

$$u^i = x^i,$$
$$u^i_j = \Sigma \sigma^i_k a^k_j,$$
$$u^i_{jk} = \Sigma \sigma^i_l a^l_{jk} + \Sigma \sigma^i_{lm} a^l_j a^m_k.$$

Let $\theta = (\theta^i, \theta^i_j)$ be the canonical form on $P^2(M)$ restricted to P and set

$$\phi^i = \sigma^* \theta^i,$$
$$\phi^i_j = \sigma^* \theta^i_j.$$

Then we obtain the following formulae (cf. § 3);

$$\theta^i = \Sigma b^i_k \phi^k,$$
$$(20) \qquad \theta^i_j = \Sigma b^i_k da^k_j - \Sigma \varepsilon^{il} \varepsilon_{jk} a_l \theta^k + a_j \theta^i + \delta^i_j \Sigma a_k \theta^k + \Sigma b^i_k \phi^k_l a^l_j,$$

where (b^i_j) denotes the inverse matrix of (a^i_j). Let $(\omega^i, \omega^i_j, \omega_j)$ be the normal conformal connection in P and set

$$\phi^i = \sigma^* \omega^i = \Sigma \Pi^i_k dx^k,$$
$$\phi^i_j = \sigma^* \omega^i_j = \Sigma \Pi^i_{kj} dx^k,$$
$$\phi_j = \sigma^* \omega_j = \Sigma \Pi_{kj} dx^k.$$

Then we obtain the following formulae:

$$\omega^i = \Sigma b^i_k \phi^k,$$
$$(21) \qquad \omega^i_j = \Sigma b^i_k da^k_j - \Sigma \varepsilon^{il} \varepsilon_{jk} a_l \omega^k + a_j \omega^i + \delta^i_j \Sigma a_k \omega^k + \Sigma b^i_k \phi^k_l a^l_j,$$
$$\omega_j = da_j - \Sigma a_k \omega^k_j + a_j \Sigma a_k \omega^k + \Sigma a^k_j \phi_k - \frac{1}{2} \Sigma \varepsilon^{ab} \varepsilon_{jk} a_a a_b \omega^k.$$

We call Π^i_k, Π^i_{jk} and Π_{jk} the coefficients of the normal conformal connection with respect to the local cross section σ.

PROPOSITION 15. *Let P be a conformal structure on M and $(\omega^i, \omega^i_j, \omega_j)$ the normal conformal connection in P. Let U be a coordinate neighborhood in M with local coordinate system $(x^1, \cdots, x^n)$. Then there is a unique local cross section σ: $U \to P^2(M)$ such that*

$$\sigma^* \omega^i = dx^i \quad \textit{and} \quad \sigma^* \Sigma \omega^i_i = 0.$$

If we set for such a σ

$$\sigma^* \omega^i_j = \Sigma \Pi^i_{kj} dx^k \quad \textit{and} \quad \sigma^* \omega_j = \Sigma \Pi_{kj} dx^k$$

then

$$\Pi^i_{jk} = \Pi^i_{kj} \quad \textit{and} \quad \Pi_{jk} = \Pi_{kj}.$$

Proof. For an arbitrary point u of P, we choose a local coordinate system $(x^1, \cdots, x^n)$ with origin $x=\pi(u)$ such that, in terms of the local coordinate system (u^i, u^i_j, u^i_{jk}) in $P^2(M)$ induced by $(x^1, \cdots, x^n)$, u is given by $(0, \delta^i_j, *)$. Let $\bar{\sigma}$: $U \to P^2(M)$ be the cross section given by

$$u^i=x^i, \qquad u^i_j=\delta^i_j, \qquad u^i_{jk}=-\Gamma^i_{jk},$$

where each Γ^i_{jk} is a certain function of $x^1, \cdots, x^n$. We take σ as the cross section given by

$$u^i=x^i, \qquad u^i_j=\delta^i_j, \qquad u^i_{jk}=-\Pi^i_{jk},$$

where

$$\Pi^i_{jk}=\Gamma^i_{jk}-\frac{1}{n}(\delta^i_j\Sigma\Gamma^h_{hk}+\delta^i_k\Sigma\Gamma^h_{hj}-\Sigma\varepsilon^{ia}\Gamma^h_{ha}\varepsilon_{jk}).$$

Then, from the expression for θ^i_j in terms of (u_i, u^i_j, u^i_{jk}) given in §3, we obtain

$$\sigma^*\omega^i_j=\Sigma\Pi^i_{kj}dx^k.$$

Clearly, σ is a cross section with the desired properties.

To prove the uniqueness, let $\tilde{\sigma}$: $U \to P^2(M)$ be another cross section with the desired properties and set

$$\tilde{\sigma}^*\omega^i_j=\Sigma\tilde{\Pi}^i_{kj}dx^k.$$

From $(21)_2$ and $\sigma^*\omega^i=\tilde{\sigma}^*\omega^i=dx^i$, we obtain

$$\sigma^*\omega^i_j=\Sigma\Pi^i_{kj}dx^k=(\sigma^*a_j)dx^i+\delta^i_j\Sigma(\sigma^*a_k)dx^k-\Sigma\varepsilon^{il}\varepsilon_{jk}(\sigma^*a_l)dx^k+\psi^i_j,$$

$$\tilde{\sigma}^*\omega^i_j=\Sigma\tilde{\Pi}^i_{kj}dx^k=(\tilde{\sigma}^*a_j)dx^i+\delta^i_j\Sigma(\tilde{\sigma}^*a_k)dx^k-\Sigma\varepsilon^{il}\varepsilon_{jk}(\tilde{\sigma}^*a_l)dx^k+\psi^i_j.$$

Hence we have

$$\tilde{\Pi}^i_{kj}-\Pi^i_{kj}=\delta^i_k\varphi_j+\delta^i_j\varphi_k-\Sigma\varepsilon^{il}\varepsilon_{jk}\varphi_l.$$

where we set $\varphi_j=(\tilde{\sigma}^*a_j)-(\sigma^*a_j)$. From

$$\sigma^*\Sigma\omega^i_i=\tilde{\sigma}^*\Sigma\omega^i_i=0,$$

we obtain

$$\varphi_1=\cdots=\varphi_n=0.$$

The remaining assertions are immediate consequences of the facts that $\Omega^i=0$ and $\Sigma\Omega^i_i=0$. (Q.E.D.)

We call σ in Proposition 15 the *natural cross section* or the *natural frame* of P associated with $(x^1, \cdots, x^n)$.

§ 8. Riemannian connections and conformal connections.

The group $G^1(n)=GL(n, R)$ can be considered as the subgroup of $G^2(n)$ consisting of the elements (a^i_j, a^i_{jk}) with $a^i_{jk}=0$. Thus $O(n)\subset CO(n)\subset H^2(n)\subset G^2(n)$. Since $G^2(n)$ acts on $P^2(M)$, the subgroups $O(n)$ and $H^2(n)$ act on $P^2(M)$. We consider the associated bundle $P^2(M)/O(n)$ and $P^2(M)/H^2(n)$ with fibres $G^2(n)/O(n)$ and $G^2(n)/H^2(n)$ respectively.

PROPOSITION 16 *The cross sections $M\rightarrow P^2(M)/O(n)$ are in one-to-one correspondence with the Riemannian connection of M.*

Proof. Let (u^i, u^i_j, u^i_{jk}) be the local coordinate system in $P^2(M)$ induced from a local coordinate system (x^i) in M as in § 3. We introduce a local coordinate system (z^i, z^i_j, z^i_{jk}) in $P^2(M)/O(n)$ in such a way that the natural mapping $P^2(M)\rightarrow P^2(M)/O(n)$ is given by the equations.

$$z^i=u^i,$$

$$z^i_j=*,$$

$$z^i_{jk}=\Sigma u^i_{pq}v^p_jv^q_k \qquad \text{where} \quad (v^i_j)=(u^i_j)^{-1}.$$

Then a cross section Γ: $M\rightarrow P^2(M)/O(n)$ is given, locally, by a set of functions $\Gamma^i_{jk}=\Gamma^i_{jk}\ (x^1, \cdots, x^n)$ with $\Gamma^i_{jk}=\Gamma^i_{kj}$ as follows:

$$(z^i, z^i_j, z^i_{jk})=(x^i, *, -\Gamma^i_{jk}).$$

Then we can see without difficulty that the behavior of the functions Γ^i_{jk} under the change of coordinate systems of M is the same as that of Christoffel's symbols. (Q.E.D.)

Since the reduction of structure group to $H^2(n)$ and the cross sections $M\rightarrow P^2(M)/H^2(n)$ are in one-to-one correspondence, the conformal structures of M are in one-to-one correspondence with the cross sections $M\rightarrow P^2(M)/H^2(n)$.

Every Riemannian connection Γ: $M\rightarrow P^2(M)/O(n)$, composed with the natural mapping ϖ: $P^2(M)/O(n)\rightarrow P^2(M)/H^2(n)$, gives a conformal structure $M\rightarrow P^2(M)/H^2(n)$.

$$\begin{array}{ccc} P^2(M)/O(n) & \xrightarrow{\ \varpi\ } & P^2(M)/H^2(n) \\ {\scriptstyle\Gamma}\nwarrow & \circlearrowright & \nearrow \\ & M & \end{array}$$

A Riemannian connection is said to *belong to a conformal structure* P if Γ induces P in the manner described above. We say that two Riemannian connections are *conformally related* if they belong to the same conformal structure.

PROPOSITION 17. *Two Riemannian connections whose Christoffel's symbols are given by $\left\{ {i \atop jk} \right\}$ and $\left\{ {\bar{i} \atop jk} \right\}$ are conformally related if and only if there exists a* 1-

form with components φ_i *such that*

$$\overline{\left\{ {i \atop jk} \right\}} = \left\{ {i \atop jk} \right\} + \delta^i_j \varphi_k + \delta^i_k \varphi_j - g_{jk} \Sigma g^{il} \varphi_l.$$

Proof. Let P be a conformal structure on M. An element $(a^i_j, \Sigma \varepsilon^{al} \varepsilon_{jk} a_a a^i_l - a^i_j a_k - a^i_k a_j)$ of $H^2(n)$ induces the transformation of $P^2(M)$ given by

$$(u^i, u^i_j, u^i_{jk}) \to (u^i, \Sigma u^i_p a^p_j, \Sigma u^i_p(\Sigma \varepsilon^{al} \varepsilon_{jk} a_a a^p_l - a^p_j a_k - a^p_k a_j) + \Sigma u^i_{pq} a^p_j a^q_k).$$

It induces the transformation of $P^2(M)/O(n)$ given by

$$(z^i, *, z^i_{jk}) \to (z^i, *, z^i_{jk} + \Sigma \varepsilon^{il} \varepsilon_{jk} a_p b^p_q v^q_l - \delta^i_j \Sigma a_p b^p_q v^q_k - \delta^i_k \Sigma a_p b^p_q v^q_j)$$

where $(b^i_j) = (a^i_j)^{-1}$ and $(v^i_j) = (u^i_j)^{-1}$. If we put $\varphi_j = \Sigma a_p b^p_q v^q_j$, then

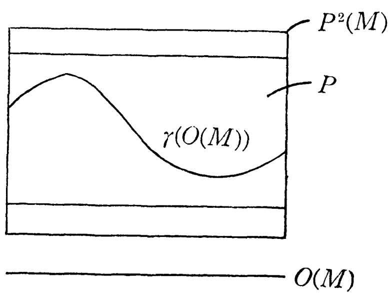
$$\bar{z}^i_{jk} = z^i_{jk} + \Sigma \varepsilon^{il} \varepsilon_{jk} \varphi_l - \delta^i_j \varphi_k - \delta^i_k \varphi_j.$$

Let $CO(M)$ be the principal fibre boundle over M with structure group $CO(n)$ and we call it the conformal bundle of M. Let M^* be the kernel of the natural homomorphism $H^2(n) \to CO(n)$ so that $CO(M) = P/M^*$. Let $u' \in CO(M)$ be the image of $u \in P$ under the natural projection $P \to CO(M)$. Then u' induces a conformal isomorphism $E^n \to T_x(M)$ where $x = \pi(u)$. Thus our assertion is clear. (Q.E.D.)

Two Riemannian metrics $g = (g_{ij})$ and $\bar{g} = (\bar{g}_{ij})$ on M is said to be conformally related if there exists a function $\rho > 0$ on M such that $\bar{g} = \rho^2 g$. If $\bar{g} = (\bar{g}_{ij})$ is conformally related to $g = (g_{ij})$ then there exists a 1-form $\varphi = (\varphi_j)$ such that

$$\overline{\left\{ {i \atop jk} \right\}} = \left\{ {i \atop jk} \right\} + \delta^i_j \varphi_k + \delta^i_k \varphi_j - g_{jk} \Sigma g^{il} \varphi_l$$

where $\left\{ {i \atop jk} \right\}$ and $\overline{\left\{ {i \atop jk} \right\}}$ denote the Christoffel's symbols of g and $\bar{g}$ respectively. Thus conformally related Riemannian metrics define conformally related Riemannian connections. This implies that a conformal structure is given by a class of conformally related Riemannian metrics.



Let Γ: $M \to P^2(M)/O(n)$ be a Riemannian connection. It corresponds naturally to a reduction of the structure group to $O(n)$. In other words, it induces an isomorphism γ of the orthonormal frame bundle $O(M)$ into $P^2(M)$. Thus a Riemannian connection Γ belongs to a conformal structure P if and only if the corresponding subbundle $\gamma(O(M))$ of $P^2(M)$ with structure group $O(n)$ is contained in P.

PROPOSITIONS 18. *Let* Γ *be a Riemannian connection of* M *belonging to the conformal structure* P *and* γ: $O(M) \to P \subset P^2(M)$ *the corresponding isomorphism. Let*

(θ^i, θ^i_j) be the canonical form of $P^2(M)$ restricted to P. Then $(\gamma^*\theta^i)$ is the canonical form of $P^1(M)$ restricted to $O(M)$ and $(\gamma^*\theta^i_j)$ is the connection form of Γ.

Proof. Let U be a coordinate neighborhood in M with local coordinate system $(x^1, \cdots, x^n)$. Let (u'^i, u'^i_j) and (u^i, u^i_j, u^i_{jk}) be local coordinate systems in $O(M)\subset P^1(M)$ and in $P\subset P^2(M)$ respectively, induced from $(x^1, \cdots, x^n)$. Let $\{{}^{\,i}_{jk}\}$ be the Christoffel's symbols of the Riemannian connection Γ with respect to the local coordinate system $(x^1, \cdots, x^n)$. Then γ: $O(M)\to P$ is given, locally, by

$$u^i=u'^i,$$

$$u^i_j=u'^i_j,$$

$$u^i_{jk}=-\Sigma\left\{ {i \atop pq} \right\} u'^p_j u'^q_k.$$

Let σ: $U\to P^2(M)$ be the natural cross section of P. Let σ': $U\to P^1(M)$ be the natural cross section, that is, the local cross section given by $(x^i)\to(x^i, \delta^i_j)$. Then, from the expression for θ^i_j in terms of (u^i, u^i_j, u^i_{jk}) given in § 3, we obtain

$$\gamma^*\theta^i_j=\Sigma v'^i_k du'^k_j+\Sigma v'^i_k\left\{ {k \atop pq} \right\} u'^p_h u'^q_j v'^h_l du'^l.$$

Hence we have

$$\sigma'^*(\gamma^*\theta^i_j)=\Sigma\left\{ {i \atop kj} \right\} dx^k. \qquad \text{(Q.E.D.)}$$

Let P be a conformal structure on M. We shall explain *Weyl's conformal curvature tensor* of P. Let $CO(M)$ denotes the principal fibre bundle over M with structure group $CO(n)$ and we call it the conformal bundle of M associated with P. Let $(\omega^i, \omega^i_j, \omega_j)$ be the normal conformal connection associated with P. Let M^* be the kernel of the natural homomorphism $H^2(n)\to CO(n)$ so that $CO(M)=P/M^*$. Let $\mathfrak{m}^*$ be the Lie algebra of M^*, then $\mathfrak{m}^*$ is nothing but $\mathfrak{co}(n)^{(1)}$ and hence isomorphic with $(R^n)^*$.

PROPOSITION 19.

(i) $\qquad \iota_{A^*}\Omega^i_j=0 \quad$ *for every* $\quad A\in\mathfrak{m}^*$,

(ii) $\qquad L_{A^*}\Omega^i_j=0 \quad$ *for every* $\quad A\in\mathfrak{m}^*$

where ι_{A^*} and L_{A^*} denote the interior product and the Lie differentiation with respect to the fundamental vector field A^* corresponding to $A\in\mathfrak{m}^*$.

Proof. The equation (i) follows from Proposition 9.
We have

$$L_{A^*}\Omega^i_j=d\iota_{A^*}\Omega^i_j+\iota_{A^*}d\Omega^i_j=\iota_{A^*}d\Omega^i_j$$

by (i). By taking exterior derivative of the structure equation (II) and using the facts that $\Omega^i=0$, we have

$$d\Omega^i_j=\Sigma\Omega^i_k\wedge\omega^k_j-\Sigma\omega^i_k\wedge\Omega^k_j-\omega^i\wedge\Omega_j+\Sigma\varepsilon^{ik}\varepsilon_{jl}\Omega_k\wedge\omega^l-\delta^i_j\Sigma\Omega_k\wedge\omega^k.$$

The right hand side of this equation vanishes for fundamental vector fields A^* corresponding to $A\in\mathfrak{m}^*$, hence $\iota_{A^*}d\Omega^i_j=0$. This proves (ii). (Q.E.D)

By the Proposition above, we see that 2-form (Ω^i_j) can be projected down to the bundle $CO(M)=P/M^*$. It follows that (Ω^i_j) defines a tensor field of type (1. 3) on M. This tensor field is called the *conformal curvature tensor of Weyl*; it depends only on the conformal structure P.

§ 9. Geodesics and completeness.

Let P be a conformal structure on a manifold M and $(\omega^i, \omega^i_j, \omega_j)$ the normal conformal connection associated with P. With each element $\xi=(\xi^1, \cdots, \xi^n)$ of E^n, we can associate a unique vector field ξ^* of P with the following properties:

$$\omega^i(\xi^*)=\xi^i, \qquad \omega^i_j(\xi^*)=0, \qquad \omega_j(\xi^*)=0.$$

We call ξ^* the *standard horizontal vector field* corresponding to ξ.

A curve x_t in M is called a "*geodesic*" of the given conformal structure if

$$x_t=\pi((\exp t\xi^*)u_0)$$

for some standard horizontal vector field ξ^* and for some point $u_0\in P$, where π: $P\to M$ is the projection. We call t a *canonical parameter* of the geodesic x_t. On the other hand, a curve $x_s=(x^1(s), \cdots, x^n(s))$ in M is called a *conformal circle* of the given conformal structure if

$$\frac{d^3x^i}{ds^3}+3\Sigma\Pi^i_{jk}\frac{d^2x^j}{ds^2}\frac{dx^k}{ds}+\Sigma\frac{d\Pi^i_{jk}}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds}+\Sigma\Pi^i_{al}\Pi^a_{jk}\frac{dx^l}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds}$$
$$-\Sigma\Pi_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}\frac{dx^i}{ds}+\Sigma\varepsilon_{jk}\left(\frac{d^2x^j}{ds^2}+\Sigma\Pi^j_{ab}\frac{dx^a}{ds}\frac{dx^b}{ds}\right)\left(\frac{d^2x^k}{ds^2}+\Sigma\Pi^k_{lm}\frac{dx^l}{ds}\frac{dx^m}{ds}\right)\frac{dx^i}{ds}$$
$$+\Sigma\varepsilon^{ia}\Pi_{ka}\frac{dx^k}{ds}=0$$

for some parameter s, where Π^i_{jk} and Π_{jk} are the coefficients of the normal conformal connection.

THEOREM 20. *Let P be a conformal structure on M. If we disregard parametrizations, then the "geodesics" of P are the same as the conformal circles of P.*

Proof. Let U be a coordinate neighborhood in M with local coordinate system $(x^1, \cdots, x^n)$. Let σ: $U\to P$ be a cross section such that $\sigma^*\omega^i=dx^i$ and let $U\times H=P|U$ the isomorphism induced by σ. Let (a^i_j, a_j) be the coordinate system in H introduced in § 4. We may take (x^i, a^i_j, a_j) as a coordinate system in $P|U$.

Let (B^i, B^i_j, B_j) be the components of the standard horizontal vector field ξ^*, $\xi=(\xi^1, \cdots, \xi^n)\in E^n$, with respect to the natural basis $\partial/\partial x^i$, $\partial/\partial a^i_j$, $\partial/\partial a_j$. From (21) and the definition of the standard horizontal vector field we have

$$B^i=\Sigma a^i_k\xi^k,$$

$$B^i_j=\Sigma\varepsilon^{al}\varepsilon_{jk}a^i_a a_l\xi^k-a_j\frac{dx^i}{dt}-a^i_j\Sigma a_k\xi^k-\Sigma\Pi^i_{kl}a^l_j\frac{dx^k}{dt},$$

$$B_j=-a_j\Sigma a_k\xi^k-\Sigma a^l_j\Pi_{kl}\frac{dx^k}{dt}+\frac{1}{2}\Sigma\varepsilon^{ab}\varepsilon_{jk}a_a a_b\xi^k.$$

Set $u_t=(\exp t\xi^*)u_0=(x^i(t), a^i_j(t), a_j(t))$, then we get

$$\frac{dx^i}{dt}=B^i,$$

$$\frac{da^i_j}{dt}=B^i_j,$$

$$\frac{da_j}{dt}=B_j.$$

Hence we have

$$\begin{aligned}&\frac{d^3x^i}{dt^3}+3\Sigma\Pi^i_{jk}\frac{d^2x^j}{dt^2}\frac{dx^k}{dt}+\Sigma\frac{d\Pi^i_{jk}}{dt}\frac{dx^j}{dt}\frac{dx^k}{dt}+\Sigma\Pi^i_{al}\Pi^a_{jk}\frac{dx^l}{dt}\frac{dx^j}{dt}\frac{dx^k}{dt}\\&-2\Sigma\Pi_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}\frac{dx^i}{dt}+3\Sigma a_l\xi^l\left(\frac{d^2x^i}{dt^2}+\Sigma\Pi^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}\right)+\Sigma\varepsilon^{ia}\Pi_{ka}\frac{dx^k}{dt}\\&+\frac{3}{2}\Sigma\varepsilon^{ab}a_a a_b\frac{dx^i}{dt}=0.\end{aligned}$$

If we make a change of parameter $t=t(s)$ satisfying the differential equation

$$\{t, s\}=\frac{1}{2}\Sigma\varepsilon_{jk}\left(\frac{d^2x^j}{ds^2}+\Sigma\Pi^j_{ab}\frac{dx^a}{ds}\frac{dx^b}{ds}\right)\left(\frac{d^2x^k}{ds^2}+\Sigma\Pi^k_{lm}\frac{dx^l}{ds}\frac{dx^m}{ds}\right)-\Sigma\Pi_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds},$$

where

$$\{t, s\}=\frac{d^3t}{ds^3}\Big/\frac{dt}{ds}-\frac{3}{2}\left(\frac{d^2t}{ds^2}\Big/\frac{dt}{ds}\right)^2,$$

then the given geodesic of P is a conformal circle of P and *vice versa*. (Q.E.D.)

The conformal structure P is called *complete* if every standard horizontal vector field is complete, that is, generates a 1-parameter group of global transformations.

§ 10. Conformal transformations and flat conformal structures.

Let P and P' be conformal structures on manifolds M and M' of the same dimension n respectively. A diffeomorphism $f: M \to M'$ is called conformal (with respect to P and P') if f, prolonged to a mapping of $P^2(M)$ onto $P^2(M')$, maps P onto P'. In particular, a transformation f of M is called conformal (with respect to P) if it maps P onto itself.

A conformal structure P on a manifold M is called *flat* if, for each point of M, there exists a neighborhood U and a conformal diffeomorphism of U onto an open subset of a Möbius space. Every vector field X on M generates a 1-parameter local group of local transformations. This local group, prolonged to $P^2(M)$, induces a vector field on $P^2(M)$, which will be denoted by $\tilde{X}$. We call X an *infinitesimal conformal transformation* (with respect to P) if the local 1-parameter group of local transformations generated by X in a neighborhood of each point of M consists of local conformal transformations.

PROPOSITION 21. *Let $\omega=(\omega^i, \omega^i_j, \omega_j)$ be the normal conformal connection associated with P. For a vector field X on M, the following conditions are mutually equivalent*:

(i) *X is an infinitesimal conformal transformation of M;*
(ii) *$\tilde{X}$ is tangent to P at every point of P;*
(iii) $L_{\tilde{X}}\omega=0$;
(iv) $L_{\tilde{X}}\xi^*=0$ *for every $\xi \in E^n$, where ξ^* is the standard horizontal vector field corresponding to ξ.*

Proof. (i)⇒(ii). Let φ_t and $\tilde{\varphi}_t$ be the local 1-parameter groups of local transformations generated by X and $\tilde{X}$ respectively. If X is an infinitesimal conformal transformation, then φ_t is a local conformal transformation and hence $\tilde{\varphi}_t$ maps P into itself. Thus $\tilde{X}$ is tangent to P at every point of P.

(ii)⇒(i). If $\tilde{X}$ is tangent to P at every point of P, the integral curve of $\tilde{X}$ through each point of P is contained in P and hence each $\tilde{\varphi}_t$ maps P into itself. This means that each φ_t is a local conformal transformation and hence X is an infinitesimal conforml transformation.

(i)⇒(iii). Since the normal conformal connection $\omega=(\omega^i, \omega^i_j, \omega_j)$ is canonically associated with P, every conformal transformation, prolonged to P, leaves ω invariant. Hence we have (iii).

(iii)⇒(iv). If $L_{\tilde{X}}\omega=0$, then

$$0=\tilde{X}\cdot(\omega^i(\xi^*))=(L_{\tilde{X}}\omega^i)(\xi^*)+\omega^i(L_{\tilde{X}}\xi^*)=\omega^i(L_{\tilde{X}}\xi^*),$$

$$0=\tilde{X}\cdot(\omega^i_j(\xi^*))=(L_{\tilde{X}}\omega^i_j)(\xi^*)+\omega^i_j(L_{\tilde{X}}\xi^*)=\omega^i_j(L_X\xi^*)$$

and

$$0=\tilde{X}\cdot(\omega_j(\xi^*))=(L_{\tilde{X}}\omega_j)(\xi^*)+\omega_j(L_{\tilde{X}}\xi^*)=\omega_j(L_{\tilde{X}}\xi^*).$$

On the other hand, the $(n+1)(n+2)/2$ 1-forms $(\omega^i, \omega^i_j, \omega_j)$ are linearly independent everywhere on P and define an absolute parallelism on P. Hence we have $L_{\tilde{X}}\xi^*=0$.

(iv)⇒(i). Let $P(u_0)$ be the set of points in P which can be joined to u_0 by an integral curve of a standard horizontal vector field. Then $\bigcup_{u_0\in P}P(u_0)=P$. From $L_{\tilde{X}}\xi^*=0$, $\tilde{\varphi}_t$ leaves each $P(u_0)$ invariant and hence leaves P invariant, that is, φ_t is a local conformal transformation. Hence X is an infinitesimal conformal transformation. (Q.E.D.)

THEOREM 22. *Let P be a conformal structure on a manifold M of dimension n. Then*

(i) *The set of all infinitesimal conformal transformations of M, denoted by $\bar{\mathfrak{c}}(M)$, is a Lie algebra of dimension at most $(n+1)(n+2)/2=\dim P$;*

(ii) *The subset of $\bar{\mathfrak{c}}(M)$ consisting of complete vector fields, denoted by $\mathfrak{c}(M)$, is a subalgebra of $\bar{\mathfrak{c}}(M)$;*

(iii) *The group of conformal transformations of M, denoted by $\mathfrak{C}(M)$, is a Lie transformation group with Lie algebra $\mathfrak{c}(M)$;*

(iv) *If the conformal structure P is complete, every infinitesimal conformal transformation is complete, i.e., $\mathfrak{c}(M)=\bar{\mathfrak{c}}(M)$.*

Proof. (i). Since the normal conformal connection $(\omega^i, \omega^i_j, \omega_j)$ is canonically associated with a conformal structure P, every conformal transformation, prolonged to P, leaves $(\omega^i, \omega^j_j, \omega_j)$ invariant. Let $\bar{\mathfrak{c}}(P)$ be the set of vector fields X on P prolonged from $X\in\bar{\mathfrak{c}}(M)$. Then $\bar{\mathfrak{c}}(M)$ is isomorphic with $\bar{\mathfrak{c}}(P)$ under the correspondence $X\rightarrow\tilde{X}$. Let u be an arbitrary point of P. The following lemma implies that the linear mapping φ: $\bar{\mathfrak{c}}(P)\rightarrow T_u(P)$ defined by $\varphi(\tilde{X})=\tilde{X}_u$ is injective so that $\dim\bar{\mathfrak{c}}(P)\leqq\dim T_u(P)=(n+1)(n+2)/2$.

LEMMA. *If an element $\tilde{X}$ of $\bar{\mathfrak{c}}(P)$ vanishes at some point of P, then it vanishes identically on P.*

Proof of Lemma. If $\tilde{X}_u=0$, then $\tilde{X}_{ua}=0$ for every $a\in H^2(n)$. Let U be the set of points $x=\pi(u)\in M$ such that $\tilde{X}_u=0$. Then U is closed in M. Since M is connected, it suffices to show that U is open. Assume $\tilde{X}_u=0$. Let b_t be a local 1-parameter group of local transformations generated by a standard horizontal vector field ξ^* in a neighborhood of u. Since $[\tilde{X}, \xi^*]=0$ by Proposition 21, $\tilde{X}$ is invariant by b_t and hence $\tilde{X}_{b_tu}=0$. On the other hand, the points of the form $\pi(b_tu)$ cover a neighborhood of $x=\pi(u)$ when ξ and t vary. This proves that U is open.

(ii) is clear.

(iii) Every 1-parameter subgroup of $\mathfrak{C}(M)$ induces an infinitesimal conformal transformation which is complete on M and, conversely, every complete infinitesimal conformal transformation generates a 1-parameter subgroup of $\mathfrak{C}(M)$.

(iv) It suffices to show that every element $\tilde{X}$ of $\bar{\mathfrak{c}}(P)$ is complete. Let u_0 be an arbitrary point of P and let $\tilde{\varphi}_t$ $(|t|<\delta)$ be a local 1-parameter group of local transformations generated by $\tilde{X}$. We shall prove that $\tilde{\varphi}_t(u)$ is defined for every $u\in P$ and $|t|<\delta$. Then it follows that $\tilde{X}$ is complete. For any point u of P, there are a finite number of standard horizontal vector fields $\xi_1^*, \cdots, \xi_k^*$ and an element

$a \in H^2(n)$ such that

$$u=(b^1_{t_1}\circ b^2_{t_2}\circ\cdots\circ b^k_{t_k}u_0)a,$$

where each b^i_t is the 1-parameter group of transformations of P generated by ξ^*_i. Then we define $\tilde{\varphi}_t(u)$ by

$$\tilde{\varphi}_t(u)=(b^1_{t_1}\circ b^2_{t_2}\circ\cdots\circ b^k_{t_k}(\tilde{\varphi}_t(u_0)))a \qquad \text{for} \quad |t|<\delta.$$

From (iv) of Proposition 21, it follows that the above definition is independent of the choice of $\xi^*_1, \cdots, \xi^*_k$. (Q.E.D.)

THEOREM 23. *If the Lie algebra $\mathfrak{c}(M)$ of infinitesimal conformal transformations of M is of dimension $(n+1)(n+2)/2$, then the normal conformal connection of P has vanishing curvature.*

Proof. Let E be the identity matrix in $\mathfrak{co}(n)$ and E^* the fundamental vector field on P corresponding to E. Let ξ^* and ξ'^* be the standard horizontal vector fields on P. Then we have

$$[E^*, \xi^*]=\xi^* \qquad \text{and} \qquad [E^*, \xi'^*]=\xi'^*.$$

The exterior differentiation applied to the structure equations (II) and (III) yields

$$0=-\Sigma\Omega^i_k\wedge\omega^k_j+\Sigma\omega^i_k\wedge\Omega^k_j+\omega^i\wedge\Omega_j-\Sigma\varepsilon^{ik}\varepsilon_{jl}\Omega_k\wedge\omega^l+d\Omega^i_j,$$

$$0=-\Sigma\Omega_k\wedge\omega^k_j+\Sigma\omega_k\wedge\Omega^k_j+d\Omega_j.$$

Hence we have

$$L_{E^*}\Omega^i_j=(d\circ\iota_{E^*}+\iota_{E^*}\circ d)\Omega^i_j=\iota_{E^*}d\Omega^i_j=0$$

and

$$L_{E^*}\Omega_j=(d\circ\iota_{E^*}+\iota_{E^*}\circ d)\Omega_j=\iota_{E^*}d\Omega_j=\Omega_j,$$

where L_{E^*} and ι_{E^*} denote the Lie differentiation and the interior product with respect to E^* respectively. Therefore,

$$\begin{aligned} E^*\cdot\Omega^i_j(\xi^*, \xi'^*)&=(L_{E^*}\Omega^i_j)(\xi^*, \xi'^*)+\Omega^i_j([E^*, \xi^*], \xi'^*)+\Omega^i_j(\xi^*, [E^*, \xi'^*]) \\ &=2\Omega^i_j(\xi^*, \xi'^*) \end{aligned}$$

and

$$\begin{aligned} E^*\cdot\Omega_j(\xi^*, \xi'^*)&=(L_{E^*}\Omega_j)(\xi^*, \xi'^*)+\Omega_j([E^*, \xi^*], \xi'^*)+\Omega_j(\xi^*, [E^*, \xi'^*]) \\ &=3\Omega_j(\xi^*, \xi'^*). \end{aligned}$$

On the other hand, if $\tilde{X}$ is the infinitesimal transformation of P induced by an infinitesimal conformal transformation $X\in\mathfrak{c}(M)$, then from

$$L_{\tilde{X}}\Omega^i_j = L_{\tilde{X}}(d\omega^i_j + \Sigma\omega^i_k\wedge\omega^k_j + \omega^i\wedge\omega_j + \Sigma\varepsilon^{ik}\varepsilon_{jl}\omega_k\wedge\omega^l - \delta^i_j\Sigma\omega_k\wedge\omega^k) = 0,$$

$$L_{\tilde{X}}\Omega_j = L_{\tilde{X}}(d\omega_j + \Sigma\omega_k\wedge\omega^k_j) = 0$$

and from (iv) of Proposition 21, we obtain

$$\tilde{X}\cdot\Omega^i_j(\xi^*, \xi'^*) = (L_{\tilde{X}}\Omega^i_j)(\xi^*, \xi'^*) + \Omega^i_j([\tilde{X}, \xi^*], \xi'^*) + \Omega^i_j(\xi^*, [\tilde{X}, \xi'^*]) = 0$$

and

$$\tilde{X}\cdot\Omega_j(\xi^*, \xi'^*) = (L_{\tilde{X}}\Omega_j)(\xi^*, \xi'^*) + \Omega_j([\tilde{X}, \xi^*], \xi'^*) + \Omega_j(\xi^*, [\tilde{X}, \xi'^*]) = 0.$$

Since $\dim \mathfrak{c}(M) = \dim P$, for every point u of P, there exists an element X of $\mathfrak{c}(M)$ such that $\tilde{X}_u = E^*_u$. We have therefore

$$2(\Omega^i_j(\xi^*, \xi'^*))_u = (E^*\cdot\Omega^i_j(\xi^*, \xi'^*))_u = (\tilde{X}\cdot\Omega^i_j(\xi^*, \xi'^*))_u = 0$$

and

$$3(\Omega_j(\xi^*, \xi'^*))_u = (E^*\cdot\Omega_j(\xi^*, \xi'^*))_u = (\tilde{X}\cdot\Omega_j(\xi^*, \xi'^*))_u = 0.$$

Since u is an arbitrary point of P, we have $\Omega^i_j = 0$ and $\Omega_j = 0$. (Q.E.D.)

THEOREM 24. *A conformal structure P on a manifold M is flat if and only if the normal conformal connection has vanishing curvature.*

Proof. Since the normal conformal connection of the conformal structure on a Möbius space has vanishing curvature, the normal conformal connection of a flat conformal structure has also vanishing curvature.

To prove the converse, let P be a conformal structure on M whose normal conformal connection $(\omega^i, \omega^i_j, \omega_j)$ has vanishing curvature. The structure equations on P reduce to the equations of Maurer-Cartan for the group $K(n)$. It follows that, given a point u of P, there exists a diffeomorphism h of a neighborhood N' of the identity of $K(n)$ onto a neighborhood N of u which sends $(\omega^i, \omega^i_j, \omega_j)$ into the Maurer-Cartan forms of $K(n)$. In an obvious manner, we extend h to a diffeomorphism h: $N'\cdot H \to N\cdot H^2(n)$. Let $U' = \pi'(N')$ and $U = \pi(N)$, where π': $K(n)\to K(n)/H$ and π: $P\to M$. Then $\pi'^{-1}(U') = N'\cdot H$ and $\pi^{-1}(U) = N\cdot H^2(n)$. By construction, h: $\pi'^{-1}(U')\to\pi^{-1}(U)$ is a bundle isomorphism. If we consider $K(n)$ as the natural conformal structure on the Möbius space $K(n)/H$ (cf. § 6), then we see that h sends the normal conformal connection of P into that of $K(n)$. In a unique manner, we can extend h to a bundle isomorphism h: $P^2(U')\to P^2(U)$. We see that h^* sends the canonical form of $P^2(U)$ into that of $P^2(U')$. By Proposition 5, h is induced by a diffeomorphism of U' onto U. (Q.E.D.)

COROLLARY. *A conformal structure P on a manifold of dimension >3 is flat if and only if the conformal curvature tensor of Weyl vanishes.*

Proof. This follows from Proposition 11 and the definition of the conformal curvature tensor of Weyl (cf. § 8). (Q.E.D.)

THEOREM 25. *Let P be a complete flat conformal structure on a simply connected manifold M of dimension n. Then there is a conformal diffeomorphism of M onto a Möbius space of dimension n.*

Proof. This follows from the definition of flatness and the standard continuation argument. (Q.E.D.)

§ 11. Conformal connections on Riemannian manifolds.

In this section M will denote always a Riemannian manifold with metric g. Let $O(M)$ be the orthonormal frame bundle over M determined by the metric g and Γ the Riemannian connection on $O(M)$. Let P be the conformal structure on M naturally associated with $O(M)$ as in § 8. Let U be a coordinate neighborhood in M with local coordinate system $(x^1, \cdots, x^n)$. Let (θ^i, θ^i_j) be the canonical from on $P^2(M)$ restricted to P and σ: $U \rightarrow P^2(M)$ a local cross section and set

$$\phi^i = \sigma^*\theta^i = \Sigma \Pi^i_k dx^k,$$

$$\phi^i_j = \sigma^*\theta^i_j = \Sigma \Pi^i_{kj} dx^k.$$

PROPOSITION 26. *There exists a cross section σ: $U \rightarrow P^2(M)$ such that*

$$\Pi^i_j = \delta^i_j,$$

$$\Pi^i_{jk} = \left\{ \begin{matrix} i \\ j\,k \end{matrix} \right\},$$

where $\{^{\,i}_{jk}\}$ denote the Christoffel's symbols of the Riemannian connection Γ.

Proof. This is an immediate consequence of Proposition 18. (Q.E.D.)

PROPOSITION 27. *Let $(\omega^i, \omega^i_j, \omega_j)$ be the normal conformal connection associated with P and σ: $U \rightarrow P^2(M)$ the cross section given in Proposition 26. If we set for such a σ*

$$\phi_j = \sigma^*\omega_j = \Sigma \Pi_{kj} dx^k,$$

then

$$\Pi_{jk} = -\frac{1}{n-2} R_{jk} + \frac{R}{2(n-1)(n-2)} g_{jk}, \tag{22}$$

where R_{jk} and R denote the components of the Ricci tensor and the scalar curvature of g respectively.

Proof. From Proposition 26 and the equation (21) we have

$$\omega^i = \Sigma b^i_k dx^k$$

$$\omega^i_j = \Sigma b^i_k da^k_j - \Sigma g^{il} g_{jk} a_l \omega^k + a_j \omega^i + \delta^i_j \Sigma a_k \omega^k + \Sigma b^i_k \left\{ \begin{matrix} k \\ a\,l \end{matrix} \right\} a^l_j dx^a.$$

Set

$$\bar{\omega}_j = da_j - \Sigma a_k \omega_j^k + a_j \Sigma a_k \omega^k + \Sigma a_j^k \left(-\frac{1}{n-2} R_{kl} + \frac{R}{2(n-1)(n-2)} g_{kl} \right) dx^l$$
$$-\frac{1}{2} \Sigma g^{ab} g_{jk} a_a a_b \omega^k.$$

Then

$$\phi^i = \sigma^* \omega^i = dx^i,$$

$$\phi_j^i = \sigma^* \omega_j^i = \Sigma \left\{ {i \atop kj} \right\} dx^k,$$

$$\phi_j = \sigma^* \bar{\omega}_j = \Sigma \left(-\frac{1}{n-2} R_{kj} + \frac{R}{2(n-1)(n-2)} g_{kj} \right) dx^k.$$

Since the normal conformal connection is uniquely associated with P, it suffices to prove that $(\omega^i, \omega_j^i, \bar{\omega}_j)$ is the normal conformal connection. Let Ω_j^i be the curvature form of the connection $(\omega^i, \omega_j^i, \bar{\omega}_j)$. From the structure equation (II) we have

$$\sigma^* \Omega_j^i = \frac{1}{2} \Sigma \left(R^i{}_{jkl} - \frac{1}{n-2} (\delta_k^i R_{jl} - \delta_l^i R_{jk} + \Sigma g^{ia} g_{jl} R_{ak} - \Sigma g^{ia} g_{jk} R_{al}) \right.$$
$$\left. + \frac{R}{(n-1)(n-2)} (\delta_k^i g_{jl} - \delta_l^i g_{jk}) \right) dx^k \wedge dx^l,$$

where R_{jkl}^i denote the components of the curvature tensor of the Riemannian connection Γ. If we set

$$\Omega_j^i = \frac{1}{2} \Sigma K_{jkl}^i \omega^k \wedge \omega^l$$

and

$$C_{jkl}^i = R_{jkl}^i - \frac{1}{n-2} (\delta_k^i R_{jl} - \delta_l^i R_{jk} + \Sigma g^{ia} g_{jl} R_{ak} - \Sigma g^{ia} g_{jk} R_{al})$$
(23)
$$- \frac{R}{(n-1)(n-2)} (\delta_k^i g_{jl} - \delta_l^i g_{jk}),$$

then

$$\sigma^* K_{jkl}^i = C_{jkl}^i.$$

We can easily see that $\Sigma C_{ikl}^i = 0$ and $\Sigma C_{jil}^i = 0$. Hence $\Sigma K_{ikl}^i = 0$ and $\Sigma K_{jil}^i = 0$. This proves that $(\omega^i, \omega_j^i, \bar{\omega}_j)$ is the normal conformal connection. (Q.E.D.)

The C_{jkl}^i are the components of the conformal curvature tensor of Weyl of the Riemannian manifold M.

PROPOSITION 28. *If* $\dim M=3$, *then* $\Omega^i_j=0$, *that is, the conformal curvature tensor of Weyl vanishes identically.*

Proof. Let C^i_{jkl} be the components of the conformal curvature tensor of Weyl and set $C_{ijkl}=\Sigma g_{ia}C^a{}_{jkl}$. Then

$$C_{ijkl}=-C_{jikl}=-C_{ijlk} \quad \text{and} \quad C_{ijkl}=C_{klij}.$$

Let 0 be an arbitrary point of M. By choosing a coordinate system such that $g_{ij}=\delta_{ij}$ at 0, together with (13), we have $\Sigma C_{ijil}=0$ at 0. Hence

$$C_{2121}+C_{3131}=0, \quad C_{1212}+C_{3232}=0, \quad C_{1313}+C_{2323}=0,$$

$$C_{3132}=0, \quad C_{2123}=0 \quad \text{and} \quad C_{1213}=0 \quad \text{at} \quad 0.$$

This implies $C_{ijkl}=0$ at 0. Since C_{ijkl} are components of a tensor field and 0 is an arbitrary point of M, $C_{ijkl}=0$ at every point of M. (Q.E.D.)

THEOREM 29. *The conformal structure P on a Riemannian manifold of dimension* 3 *is flat if and only if* $\Omega_j=0$.

Proof. This is an immediate consequence of Theorem 24 and Proposition 28. (Q.E.D.)

Let $(\omega^i, \omega^i_j, \omega_j)$ be the normal conformal connection associated wifh P. Let σ be the local cross section given in Proposition 26 and set $\sigma^*\Omega_j=(1/2)\Sigma C_{jkl}dx^k\wedge dx^l$. From the structure equation (III) and Proposition 27 we have

$$C_{jkl}=\frac{1}{n-2}(R_{jk;l}-R_{jl;k})-\frac{1}{2(n-1)(n-2)}\left(g_{jk}\frac{\partial R}{\partial x^l}-g_{jl}\frac{\partial R}{\partial x^k}\right), \tag{24}$$

where $R_{jk;l}$ denote the components of the covariant derivative of the Ricci tensor with respect to the Riemannian connection Γ.

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