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ON A CONTINUITY LEMMA OF EXTREMAL LENGTH AND ITS APPLICATIONS TO CONFORMAL MAPPING

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§1. Introduction.

1. The continuity of the extremal length of a curve family joining two disjoint compact sets in the plane with respect to their exhaustion was first discussed by Wolontis [10]. Later Strebel [7] showed the continuity for the Riemann surface and its two sets of compact boundary components (in the Stoilow compactification), and recently Marden and Rodin [4] generalized it for a wider class of curves. In the present paper we shall show the continuity of the extremal length with respect to increasing curve families. Such a property was already discussed for particular curve families in a problem of conformal mappings by Marden and Rodin [4], but we state the continuity in a general form.

As an application of the continuity lemma, we shall discuss a problem of conformal mapping from a domain onto a slit rectangle. The problem was first treated by Grötzsch [2] in the case of finite connectivity. In our former paper [8] we constructed a slit rectangle mapping function for a domain whose outer boundary was isolated. In the present paper we shall show that a plane domain with a preassigned boundary component given four distinct curves (vertices) can be mapped onto a horizontally slit rectangle with possible horizontal incisions, if the extremal length of the family of curves joining one pair of edges corresponding to vertical sides is finite.

§2. Preliminary.

2. We sum up some known results for extremal lengths. Let R be a Riemann surface and let Γ be a family of curves on R. We mean by a curve a collection of at most countable open connected arcs whose member is locally rectifiable. Let $\rho(z)|dz|$ be a nonnegative measurable metric. We call ρ *measurable* on Γ, if the integral of ρ along each $\gamma\in\Gamma$ exists. A metric ρ is called *admissible*, if it is measurable and its integral along each $\gamma\in\Gamma$ is not less than one. An admissible class, denoted by $P(\Gamma)$, is the collection of all admissible metrics. The closure of the intersection of $P(\Gamma)$ with the l_2-space of the metrics with finite norm is called a *generalized* admissible class and written by $P^*(\Gamma)$. The module of Γ is defined by

$$\inf_{\rho\in P(\Gamma)}\iint\rho^2 dxdy=\inf_{\rho\in P(\Gamma)}||\rho||^2$$

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and denoted by $\operatorname{mod}\Gamma$. There exists a unique metric ρ_0 within $P^*(\Gamma)$ called a (generalized) ***extremal metric***, which satisfies $\operatorname{mod}\Gamma=\|\rho_0\|^2$ so long as $P^*(\Gamma)\neq\phi$ [6]. The metric ρ_0 gives the minimum norm within the family $P^*(\Gamma)$ and the deviation of $\rho\in P^*(\Gamma)$ from the extremal metric ρ_0 is given by

$$(1) \qquad \|\rho-\rho_0\|^2\leqq\|\rho\|^2-\|\rho_0\|^2$$

[9]. The reciprocal of $\operatorname{mod}\Gamma$ is the extremal length of Γ, denoted by $\lambda(\Gamma)$.

Following Fugulede [1], we term a curve family with zero module an exceptional curve family. A statement is said to hold for almost all $\gamma\in\Gamma$, if it is false for an exceptional subfamily of Γ.

3. There is an alternative definition of the extremal length originally due to Fugulede [1] and used in the theory of functions by Marden and Rodin [4]. In their definition the admissible class of metrics for Γ, denoted by $P'(\Gamma)$, is the collection of all Borel measurable metrics which satisfy

$$\int_\gamma \rho|dz|\geqq 1 \qquad \text{for almost all } \gamma\in\Gamma.$$

Then the family $P'(\Gamma)$ is *equivalent* to the family $P^*(\Gamma)$ in the following sense: Every metric $\rho\in P^*(\Gamma)$ has an equivalent Borel measurable $\tilde\rho$ such that $\rho=\tilde\rho$ almost everywhere which is contained in $P'(\Gamma)$. On the other hand $P'(\Gamma)\subset P^*(\Gamma)$. In fact $\tilde\rho$ has the property that $\int_\gamma\rho|dz|\geqq 1$ for almost all $\gamma\in\Gamma$, which is shown as a property of an admissible metric measurable on Γ [9]. For the exceptional curve family with respect to $\rho\in P'(\Gamma)$, denoted by $\Lambda(\rho)$, we can select a sequence of metrics $\mu_n\in P(\Lambda(\rho))$ such that $\|\mu_n\|^2\to 0$. The sequence of metrics $\rho_n=\max(\rho,\mu_n)$ is admissible and tends to ρ, which implies $\rho\in P^*(\Gamma)$. Clearly the closure of the intersection of $P'(\Gamma)$ with the l_2-space coincides with $P^*(\Gamma)$. Fugulede proved the existence of an extremal metric in the $l_2(l_p)$-completion of $P'(\Gamma)$ [1]. It seems to us that the proof of the existence of the extremal metric is easy in $P^*(\Gamma)$ [5], but for the continuity lemma in § 3, the proof based on $P'(\Gamma)$ is much easier than on $P^*(\Gamma)$.

§ 3. A continuity lemma.

4. We now state

LEMMA 1. *Let $\{\Gamma_n\}$ be an increasing sequence of curve families. Put $\Gamma=\cup\Gamma_n$. Then we have*

$$\lim_{n\to\infty}\lambda(\Gamma_n)=\lambda(\Gamma).$$

Proof. We prove the lemma for the module. The module of a curve family is infinite if its generalized admissible class is void, and in the case that $P^*(\Gamma_n)=\phi$ for some n the statement is evident. If $P^*(\Gamma_n)\neq\phi$ for all n there exists a unique extremal metric ρ_n in every $P^*(\Gamma_n)$. The sequence $\{\operatorname{mod}\Gamma_n\}$ is increasing. Hence if $\lim \operatorname{mod} \Gamma_n=\infty$, $\operatorname{mod} \Gamma=\infty$. Suppose the sequence $\{\operatorname{mod} \Gamma_n\}$ is bounded. Then by the inequality (1) we have $\|\rho_m-\rho_n\|^2 \leqq \|\rho_m\|^2-\|\rho_n\|^2$ $(m>n)$. ρ_n tends to a metric ρ_0 from the boundedness of the sequence and we get $\|\rho\|^2 \geqq \|\rho_0\|^2$ for $\rho \in P^*(\Gamma)$.

We now show $\rho_0 \in P^*(\Gamma)$ which implies the extremality of ρ_0. From the definition of admissible metric we have $P(\Gamma)=\cap P(\Gamma_n)$. We prove $P^*(\Gamma)=\cap P^*(\Gamma_n)$. Clearly $P^*(\Gamma) \subset \cap P^*(\Gamma_n)$. Suppose $\rho \in P^*(\Gamma_n)$ for all n. If we take an equivalent metric measurable on Γ_n, denoted by the same notation ρ, we have

$$(2) \qquad \int_\gamma \rho|dz| \geqq 1 \qquad \text{for almost all } \gamma \in \Gamma_n.$$

Let $\Lambda_n(\rho)$ be the exceptional family of Γ_n for the inequality (2). Then the exceptional family $\Lambda(\rho)$ of Γ is equal to $\cup \Lambda_n(\rho)$. By Hersch's lemma [3] we have $\operatorname{mod} \Lambda(\rho) \leqq \Sigma_n \operatorname{mod} \Lambda_n(\rho)=0$, which implies that $\Lambda(\rho)$ is exceptional (see also [1]).

$P^*(\Gamma_n)$ contains the extremal metrics ρ_ν for $\nu \geqq n$ and hence $\rho_0 \in P^*(\Gamma_n)$. So we get $\rho_0 \in P^*(\Gamma)$.

§ 4. Application to conformal mappings.

5. Let Ω be a plane domain. A boundary component α of Ω is defined by a decreasing sequence of subdomains $\{\Delta_n\}$ of Ω which satisfies the conditions that each member has a single analytic relative boundary compact in Ω, $\Omega-\Delta_n$ is a domain, $\bar{\Delta}_{n+1} \subset \Delta_n$ and $\cap \Delta_n=\phi$. We call the sequence $\{\Delta_n\}$ a *boundary component* α of Ω. $\{\Delta_n\}$ is called a defining sequence of α. Two defining sequences $\{\Delta_n\}$ and $\{\Delta'_n\}$ are said equivalent when every Δ_n contains some Δ'_m and vice versa. We can assign to α a point set on the complex sphere defined by $\cap \mathrm{Cl}(\Delta_n)$, where $\mathrm{Cl}(\Delta_n)$ denotes the closure taken in the complex sphere, and the set is also written by α. A sequence of domains $\Omega_n=\Omega-\bar{\Delta}_n$ exhausts Ω and it is called an exaustion of Ω in the direction to α.

We say a curve γ: $z=z(t)$ $(0<t<1)$ tends to α if each member of the defining sequence of α contains the image of a suitable interval $(s, 1)$ (resp. $(0, 1)$) under $z(t)$.

6. We denote by $\{\gamma_j\}_{j=1}^4$ four mutually disjoint piecewise analytic curves starting from the points of Ω, running within Ω and tending to α. We can choose such a defining sequence $\{\Delta_n\}$ the relative boundary of Δ_n intersects every γ_j precisely once. Let $\{\Omega_n\}$ be an exaustion of Ω in the direction to α. Let us denote by $p_j^{(n)}$ the intersection of γ_j with the relative boundary of Ω_n. We may assume that the sequence $\{p_j^{(n)}\}_{j=1}^4$ are arranged in the positive direction with respect to Ω_n. Then we say the curves γ_j mark vertices on α. The end parts of curves γ_j after $p_j^{(n)}$ divide the complementary domain $\Omega-\Omega_n$ into four subdomains, say $S_{12}^{(n)}$, $S_{23}^{(n)}$, $S_{34}^{(n)}$ and $S_{41}^{(n)}$, where the suffix numbers of $S^{(n)}$'s mean those of the corresponding subarcs of the curves γ_j as their boundary. We call the sequence $\{S_{12}^{(n)}\}$ a defining sequence of an *edge* α_{12} determined by γ_1 and γ_2. The equivalence of two defining sequences of an edge is defined as in no. **5**. The other three edges are defined similarly. A curve tending to an edge is also defined by its defining sequence.

Put $T_n = \Omega_n - \overline{S_{23}^{(n)} \cup S_{41}^{(n)}}$. $\{T_n\}$ makes an exaustion of Ω, called an exaustion of Ω *in the direction* to edges α_{23} and α_{41}. An exhaustion of Ω in the direction to α_{12} and α_{34} can be constructed similarly by means of $S_{12}^{(n)}$ and $S_{34}^{(n)}$. Taking a replica of T_n we can construct a double $\hat{T}_n$ of T_n with respect to the relative boundaries of T_n which are two piecewise analytic Jordan arcs. $\hat{T}_n$ has two boundary components which are the union of α_{12} and its counterpart $\tilde{\alpha}_{12}$, say $C_1^{(n)}$, and the union of α_{34} and $\tilde{\alpha}_{34}$ say $C_2^{(n)}$. If the family of curves joining $C_1^{(n)}$ and $C_2^{(n)}$ has finite extremal length, there exists a unique minimal radially slit annulus mapping function $f_n(z)$ except linear transformations with fixed points at zero and at infinity [7]. We state some properties of f_n:

i) *The images of* $C_1^{(n)}$ *and* $C_2^{(n)}$ *under* f_n *are two circles with center zero having possible radial incisions emanating from them, whose directions make a set of linear measure zero, and the images of the boundary components other than* $C_1^{(n)}$ *and* $C_2^{(n)}$ *are a quasi-minimal set of radial slits whose compact subset is minimal.*

ii) *The two relative boundaries of* T_n *are mapped into a straight line through the origin by* f_n *and the images of* T_n *and its counterpart under* f_n *are symmetric with respect to it.*

iii) *Let* $\hat{\Gamma}_n$ *be the family of curves joining* $C_1^{(n)}$ *and* $C_2^{(n)}$ *in* $\hat{T}_n$ *and let* Γ_n *be the subfamily of* $\hat{\Gamma}_n$ *whose member joins them in* $\overline{T}_n$. *Then we have* $\mathrm{mod}\,\hat{\Gamma}_n = 2\,\mathrm{mod}\,\Gamma_n$ *and the metric* $|f_n'/(f_n \log(r_1/r_2))|$ *is the extremal metric for the both module problems, where* r_1 *and* r_2 *are the radii of the images of* $C_0^{(n)}$ *and* $C_2^{(n)}$ $(r_1 > r_2)$.

iv) *Let* Λ_1 *be the family of curves joining* α_{12} *and a compact neighborhood* K *of a point of* T_n *and having the property* $\underline{\lim}_{t\to 1} |f_n(z(t))| < r_1$, *where* $z = z(t)$ *is a representation of a curve tending to* α_{12} *as* $t \to 1$. *Let* Λ_2 *be a similar family of curves tending to* α_{34} *and satisfying* $\overline{\lim}_{t\to 1} |f_n(z(t))| > r_2$. *Then the modules of* Λ_1 *and* Λ_2 *both vanish.*

The statement i) is found in [7]. The statement ii) is shown by the uniqueness of the mapping function. In fact $\hat{T}_n$ has a self anti-conformal mapping onto itself, denoted by $\tau(p)$ and $\bar{f}_n(\tau(p))$ is also a minimal radially slit annulus mapping. We have $\bar{f}_n(\tau(p)) = e^{i\theta} f_n(p)$. Since $p = \tau(p)$ on the relative boundaries of T_n, we can deduce the images of the relative boundaries lie on the line $\arg w = \theta/2$. The extremal metric in the minimal radially slit annulus was given in [7]. Since the mapping $\tau(p)$ fixes the curve family $\hat{\Gamma}_n$, the uniqueness of the extremal metric shows the invariance of the extremal metric under $\tau(p)$. On the other hand the extremal metric for the curve family Γ_n is extended to the replica of T_n symmetrically and the extended metric is admissible for $\hat{\Gamma}_n$, which implies the coincidence of the both extremal metrics in $\overline{T}_n$.

The property iv) for the domain $\hat{T}_n$ and $C_j^{(n)}$'s is easily verified by the same method as in [9] which is originally due to Ohtsuka [5] and the statement is evident because they are subfamilies of the above family.

From these statements we can construct a function $\varphi_n(z)$ which maps T_n onto a horizontally slit rectangle with the normalization that the relative boundary of

T_n containing subarcs of γ_2 and γ_3 is mapped into the bottom line $[0,1]$ in this order. Indeed $A\log f_n(z)+B$ with suitable constants is the desired function. The relative boundaries of T_n are mapped into two horizontal sides and the edges α_{12} and α_{34} are mapped onto two vertical sides with possible horizontal incisions with total area zero. The image of the other boundary components is a quasi-minimal set of horizontal slits [8].

6. We state

THEOREM 1. *Let $\varphi_n(z)$ be the mapping function constructed above which maps T_n onto a horizontally slit rectangle with bottom line $[0,1]$. If the heights of the image rectangles of T_n under φ_n is bounded then $\varphi_n(z)$ tends to a function $\varphi_0(z)$ in the sense that $\|\varphi_n'-\varphi_0'\|^2_{T_n}\to 0$. The image of α under φ_0 is a rectangle with possible horizontal incisions of 2-dimensional measure zero emanating from vertical sides. The images of other boundary components than α are a quasi-minimal set of horizontal slits. The module of the family of curves joining the edges α_{12} and α_{34} is equal to h, where h is the height of the image rectangle. The module of its subfamily of curves satisfying*

$$\varliminf_{t\to 1}\operatorname{Re}\varphi_0(z(t))-\varlimsup_{t\to 0}\operatorname{Re}\varphi_0(z(t))<1$$

vanishes, where the curve $z(t)$ tends to α_{12} and to α_{34} as $t\to 0$ as $t\to 1$ respectively.

Proof. From iii), the invariance of the module under a conformal mapping implies the module of the family Γ_n of curves joining α_{12} and α_{34} in $\overline{T}_n$ is equal to the height of its image rectangle, denoted by h_n and $\rho_n=|\varphi_n'|$ is an extremal metric. The boundedness of the sequence h_n and Lemma 1 show that ρ_n tends to an extremal metric ρ_0 for $\Gamma=\cup\Gamma_n$ strongly. Since $\{\varphi_n'\}$ is weakly compact in the complex Hilbert space, any subsequence of $\{\varphi_n'\}$ contains a convergent subsequence whose limit denotes φ_0'. The limit function is a strong limit from the convergence of the norms, is analytic and univalent and satisfies $\rho_0=|\varphi_0'|$. Put $h=\lim h_n$. Then the image domain of Ω under φ_0 is contained in the rectangle $0<\operatorname{Re} w<1$, $0<\operatorname{Im} w<h$ and its area is equal to h. Then the uniqueness of the strong limit ρ_0 implies the independence of φ_0' on the subsequences of $\{\varphi_n\}$ and hence $\|\varphi_n'-\varphi_0'\|_{T_n}\to 0$.

Next we show the last statement which is a property of incisions. Let $\{T_n^\nu\}$ be an exhaustion of T_n in the direction to α_{12} and α_{34} and let $\{\varphi_{n\nu}\}$ be the sequence of normalized slit rectangle mappings. Then $\|\varphi_{n\nu}'-\varphi_n'\|_{T_n^\nu}\to 0$ as $\nu\to\infty$. Hence we can find a sequence of functions $\varphi_{n\nu(n)}$ satisfying $\|\varphi_{n\nu(n)}'-\varphi_0'\|_{T_n^{\nu(n)}}\to 0$. Denoting by Λ_n and Λ_n^k the subfamilies of Γ_n satisfying $\varliminf_{t\to 1}\operatorname{Re}\varphi_0(z(t))-\varlimsup_{t\to 0}\operatorname{Re}\varphi_0(z(t))<1$ and $<1-1/k$ respectively, we have $\operatorname{mod}\Lambda_n^k=0$ and $\operatorname{mod}\Lambda_n=\operatorname{mod}\cup_k\Lambda_n^k=0$. In fact similarly as in [9] we put $u=\operatorname{Re}\varphi_0$ and $u_m=\operatorname{Re}\varphi_{m\nu(m)}$ in $T_m^{\nu(m)}\cap T_n$ $(n\leqq m)$, $=0$ in $S_{12}^{(n)}-T_m^{\nu(m)}$ and $=1$ in $S_{24}^{(n)}-T_m^{\nu(m)}$. Then the metric $\rho_m=k|\operatorname{grad}(u-u_m)|$ is admissible for Λ_n^k and the convergence of $\varphi_{m\nu(m)}'$ implies $\operatorname{mod}\Lambda_n^k=0$. Since $\Lambda_n=\cup_k\Lambda_n^k$, we have also $\operatorname{mod}\Lambda_n=0$. The family Λ in the theorem is represented by $\cup_n\Lambda_n$ and hence we have the assertion.

We now discuss the shape of the image domain. The minimality of the image of the boundary components other than α follows from Lemma 5 in [8]. Put $\Gamma'=\Gamma-\Lambda$. The metric $\rho_0=|\varphi_0'|$ is an extremal metric for Γ' and continuous, then Lemma 1 in [8] implies

$$\inf_{\gamma\in\Gamma'_z}\int_\gamma \rho_0|dz|=1, \tag{3}$$

where Γ'_z is the subfamily of Γ' consisting of the curves through fixed point z in Ω. We can deduce from (3) that the image of α under φ_0 is a rectangle with possible horizontal incisions emanating from the vertical sides. Vanishing of the area of the incisions is obvious from the identity $\|\rho_0\|^2=h$. Thus the proof has been completed.

7. We discuss an extremal property of the slit rectangle mapping when the extremal distance of α_{12} and α_{34} is finite. We dealt with a similar problem for radial slit disc mappings [9] and Marden and Rodin [4] treated related problems for circular-radial slit mappings.

Let $f(z)$ be a univalent function in Ω. For a relatively compact open set K let $\Lambda(f,\alpha_{12})$ be the family of curves joining K and α_{12} in Ω and satisfying $\overline{\lim}_{t\to1}\operatorname{Re}f(z(t))>0$ ($z(t)\to\alpha_{12}$ as $t\to1$) and let $\Lambda(f,\alpha_{34})$ be family of curves joining K and α_{34} in Ω and satisfying $\underline{\lim}_{t\to1}\operatorname{Re}f(z(t))<1$. We denote by $\mathfrak{F}$ the family of univalent functions f which map α onto the outer boundary and satisfy $\operatorname{mod}\Lambda(f,\alpha_{12})=\operatorname{mod}\Lambda(f,\alpha_{34})=0$ and $\inf_{z\in\Omega}\operatorname{Im}f(z)=0$. The independence of the family $\mathfrak{F}$ on the choice of K is easily verified as in [9]. It is also shown as in the proof of Theorem 1 that $\varphi_0\in\mathfrak{F}$. Put $H(f)=\sup_{z\in\Omega}\operatorname{Im}f(z)$.

THEOREM 2. *If the extremal distance of α_{12} and α_{34} is finite, then φ_0 is the unique function which minimize the quantity $H(f)$ among $\mathfrak{F}$.*

Proof. Let Γ denote the family of curves joining α_{12} and α_{34} and let $\Lambda(f)$ denote subfamily of Γ satisfying $\overline{\lim}_{t\to0}\operatorname{Re}f(z(t))>0$ or $\underline{\lim}_{t\to1}\operatorname{Re}f(z(t))<1$ ($z\to\alpha_{12}$ as $t\to0$). Then the normalization of $\mathfrak{F}$ implies $\operatorname{mod}\Lambda(f)=0$. In fact we take a curve joining α_{23} and α_{41} and consider its open covering K in Ω. K can be represented by relatively compact open sets K_n in Ω in the form $K=\cup_n K_n$. Denoting by $\Lambda_n(f)$ the subfamily of $\Lambda(f)$ intersecting K_n, we have $\operatorname{mod}\Lambda_n(f)=0$ from the remark before this theorem, since any member of $\Lambda_n(f)$ contains a curve of $\Lambda(f,\alpha_{12})$ or $\Lambda(f,\alpha_{34})$ for K_n as a subarc. Thus $\operatorname{mod}\Lambda(f)=\operatorname{mod}\cup\Lambda_n(f)=0$.

We define a metric ρ from f by

$$\rho=\begin{cases}|f'(z)| & (0<\operatorname{Re}f(z)<1),\\ 0 & (\text{elsewhere}).\end{cases}$$

We show $\rho\in P^*(\Gamma)$. Let γ be a curve of $\Gamma-\Lambda(f)$. Then considering the oscillation of $\operatorname{Re}f(z)$, we have

$$\int_\gamma \rho|dz|\geqq1$$

from the facts $\overline{\lim}_{t\to 0}\operatorname{Re} f(z(t))\leqq 0$ and $\underline{\lim}_{t\to 1}\operatorname{Re} f(z(t))\geqq 1$. Since $\operatorname{mod} \Lambda(f)=0$, from the same reason in no. **3** we have $\rho\in P^*(\Gamma)$. The extremality and uniqueness of φ_0 follow immediately from the inequality (1)

$$\|\rho-\rho_0\|^2\leqq\|\rho\|^2-\|\rho_0\|^2\leqq H(f)-h,$$

where ρ_0 is the extremal metric $|\varphi_0'|$ for Γ defined in Theorem 1, because the carrier of ρ is contained in the closed rectangle $0\leqq\operatorname{Re} f(z)\leqq 1$, $0\leqq\operatorname{Im} f(z)\leqq H(f)$ in the image plane.

We can take many other functionals for which φ_0 is extremal. For instance the area of the image domain $\|f'\|^2$ is intimately connected with the extremal length and was dealt with by Marden and Rodin [4] and by others.

8. Following the notations in no. **6**, we construct an exhaustion $\{\tilde{T}_n\}$ of Ω in the direction to α_{12} and α_{34}, where $\tilde{T}_n=\Omega_n-\overline{S_{12}^{(n)}\cup S_{34}^{(n)}}$. Under the assumption that both the extremal length and the module of the family of the curves joining the relative boundaries of $\tilde{T}_n$ are finite, there exists a unique normalized horizontal slit rectangle mapping $\psi_n(z)$ of $\tilde{T}_n$ such that the edge α_{23} corresponds to a subcontinuum (or a point) of the bottom line $[0,1]$. We can deduce that ψ_n tends to a function ψ_0 strongly as $n\to\infty$, if the extremal lengths are bounded, but we have no informations about the shape of the image domain except the minimality of the images of boundary components other than α. However we can conclude that ψ_0 coincides with φ_0, if we assume the accessibility of the four curves γ_j and the separability of edges. Here the accessibility of γ_j is in the sense that there exists a point z_j whose arbitrary neighborhood in the complex sphere contains a suitable end part of γ_j.

THEOREM 3. *Let $\tilde{T}_n$ be an exhaustion of Ω in the direction to α_{12} and α_{34} and let $\tilde{\Gamma}_n$ be the family of curves joining two relative boundaries of $\tilde{T}_n$. Suppose the sequence of the extremal lengths $\lambda(\tilde{\Gamma}_n)$ is positive and bounded. Then the normalized slit rectangle mapping ψ_n of $\tilde{T}_n$ constructed as in Theorem* 1[1] *tends to a function ψ_0 in the sense that $\|\psi_n'-\psi_0'\|^2_{T_n}\to 0$.*

If four curves γ_j defining the vertices are accessible and if the two edges α_{12} and α_{34} have disjoint neighborhoods, then ψ_0 coincides with φ_0 defined in Theorem 1.

Proof. The metric $\rho_n=|\psi_n'|$ is extremal for $\tilde{\Gamma}_n$ and is contained in $P(\tilde{\Gamma}_m)$ for all $m>n$, where ρ_n is extended as zero outside of $\tilde{T}_n$. Put $h_n=\operatorname{mod}\tilde{\Gamma}_n$ which is decreasing and is bounded away from zero. Then we have from (1) $\|\rho_n-\rho_m\|^2\leqq\|\rho_n\|^2-\|\rho_m\|^2=h_n-h_m$ for $m>n$ and ρ_n tends to a metric ρ_0 strongly. The same reason as in the proof of Theorem 1 shows that ψ_n tends to a univalent function ψ_0 in the sense that $\|\psi_n'-\psi_0'\|^2\to 0$.

Next we show the continuity of the module of $\tilde{\Gamma}_n$. Our proof is originally due to Strebel [7], using a classical method of the proof of Phragmén—Lindelöf's theorem. Let ρ be a metric of $P(\Gamma)$, where Γ is the family of curves joining α_{12} and α_{34} in Ω. We set

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1) Here we regard γ_j's lying partly on the boundary as the curves defining vertices.

$$L(\rho, \Gamma)=\inf_{\gamma\in\Gamma}\int_{\gamma}\rho|dz|$$

Let $\tilde{\Omega}$ denote the complementary domain of α with respect to the complex sphere containing Ω. By our assumptions, there exist four neighborhoods of the vertices of α, denoted by V_j ($j=1, 2, 3, 4$), such that the module of each family χ_j of curves joining V_j and its opposite edge is less than $\varepsilon/8$, since a point has infinite extremal distance from any compact disjoint set. Then we can construct metrics $\mu_i\in P(\chi_j)$ ($j=1, 2, 3, 4$) such that $\|\mu_i\|^2<\varepsilon/4$. We set $\rho_\varepsilon=\max_{1\leqq j\leqq 4}(\rho, \mu_j)$ which is a member of $P(\Gamma)$. Then we have $\underline{\lim}\, L(\rho_\varepsilon, \tilde{\Gamma}_n)\geqq 1$. In fact, contrary to the assertion, there exists a curve $\gamma_n\in\tilde{\Gamma}_n$ joining two relative boundaries of $\tilde{T}_n$ which satisfies

$$\int_{\gamma_n}\rho_\varepsilon|dz|\leqq 1-\delta.$$

γ_n intersects both the relative boundaries of $\tilde{T}_\nu$ for $\nu\leqq n$. Let $\zeta_{n\nu}^{(1)}$ and $\zeta_{n\nu}^{(2)}$ be two points of the intersections of γ_n with the relative boundaries of $\tilde{T}_\nu$ separating α_{12} and α_{34} from $\tilde{T}_n$ respectively. Then the points $\zeta_{n\nu}^{(1)}$ and $\zeta_{n\nu}^{(2)}$ have at least one cluster point on each relative boundary of $\tilde{T}_\nu$ and outside of V_j's. Therefore we can construct a curve β joining α_{12} and α_{34} such that

$$\int_{\beta}\rho_\varepsilon|dz|\leqq 1-\frac{\delta}{2},$$

by Strebel's method [7]. We get a contradiction to the admissibility of ρ_ε.

For any $k>1$, $k\rho_\varepsilon$ is admissible for Γ_n with sufficiently large n and we have $\|\rho_n\|^2\leqq\|k\rho_\varepsilon\|^2$. Thus we get

$$\|\rho_0\|^2\leqq\|\rho_\varepsilon\|^2\leqq\|\rho\|^2+\varepsilon.$$

Clearly $\rho_0\in P^*(\Gamma)$ and ρ_0 is extremal for Γ. We obtain the continuity of modules with respect to this exhaustion.

The uniqueness of the extremal metric ρ_0 implies $\psi_0=\varphi_0$.

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