ON A CONTINUITY LEMMA OF EXTREMAL LENGTH AND
ITS APPLICATIONS TO CONFORMAL MAPPING

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§ 1. Introduction.

1. The continuity of the extremal length of a curve family joining two
disjoint compact sets in the plane with respect to their exhaustion was first
discussed by Wolontis [10]. Later Strebel [7] showed the continuity for the Riemann
surface and its two sets of compact boundary components (in the Stoilow com-
 pactification), and recently Marden and Rodin [4] generalized it for a wider class
of curves. In the present paper we shall show the continuity of the extremal
length with respect to increasing curve families. Such a property was already
discussed for particular curve families in a problem of conformal mappings by
Marden and Rodin [4], but we state the continuity in a general form.

As an application of the continuity lemma, we shall discuss a problem of
conformal mapping from a domain onto a slit rectangle. The problem was first
we constructed a slit rectangle mapping function for a domain whose outer bounda-
ry was isolated. In the present paper we shall show that a plane domain with a
preassigned boundary component given four distinct curves (vertices) can be mapped
onto a horizontally slit rectangle with possible horizontal incisions, if the extremal
length of the family of curves joining one pair of edges corresponding to vertical
sides is finite.

§ 2. Preliminary.

2. We sum up some known results for extremal lengths. Let \( R \) be a Riemann
surface and let \( \Gamma \) be a family of curves on \( R \). We mean by a curve a collection
of at most countable open connected arcs whose member is locally rectifiable. Let
\( \rho(z)|dz| \) be a nonnegative measurable metric. We call \( \rho \) measurable on \( \Gamma \), if the
integral of \( \rho \) along each \( \gamma \in \Gamma \) exists. A metric \( \rho \) is called admissible, if it is
measurable and its integral along each \( \gamma \in \Gamma \) is not less than one. An admissible
class, denoted by \( P(\Gamma) \), is the collection of all admissible metrics. The closure of
the intersection of \( P(\Gamma) \) with the norm space of the metrics with finite norm is called
a generalized admissible class and written by \( P^*(\Gamma) \). The module of \( \Gamma \) is defined by

\[
\inf_{\rho \in P(\Gamma)} \int \rho^2 dx dy = \inf_{\rho \in P(\Gamma)} \|\rho\|^2
\]

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and denoted by \( \text{mod} \Gamma \). There exists a unique metric \( \rho_0 \) within \( P^*(\Gamma) \) called a (generalized) extremal metric, which satisfies \( \text{mod} \Gamma = \|\rho_0\|^p \) so long as \( P^*(\Gamma) \neq \emptyset \) [6]. The metric \( \rho_0 \) gives the minimum norm within the family \( P^*(\Gamma) \) and the deviation of \( \rho \in P^*(\Gamma) \) from the extremal metric \( \rho_0 \) is given by

\[
||\rho - \rho_0||^2 \leq ||\rho||^p - ||\rho_0||^p
\]

[9]. The reciprocal of \( \text{mod} \Gamma \) is the extremal length of \( \Gamma \), denoted by \( \lambda(\Gamma) \).

Following Fugulede [1], we term a curve family with zero module an exceptional curve family. A statement is said to hold for almost all \( \gamma \in \Gamma \), if it is false for an exceptional subfamily of \( \Gamma \).

3. There is an alternative definition of the extremal length originally due to Fugulede [1] and used in the theory of functions by Marden and Rodin [4]. In their definition the admissible class of metrics for \( \Gamma \), denoted by \( P'(\Gamma) \), is the collection of all Borel measurable metrics which satisfy

\[
\int \rho |dz| \geq 1 \quad \text{for almost all } \gamma \in \Gamma.
\]

Then the family \( P'(\Gamma) \) is equivalent to the family \( P^*(\Gamma) \) in the following sense: Every metric \( \rho \in P^*(\Gamma) \) has an equivalent Borel measurable \( \tilde{\rho} \) such that \( \rho = \tilde{\rho} \) almost everywhere which is contained in \( P'(\Gamma) \). On the other hand \( P'(\Gamma) \subseteq P^*(\Gamma) \). In fact \( \tilde{\rho} \) has the property that \( \int \rho |dz| \geq 1 \) for almost all \( \gamma \in \Gamma \), which is shown as a property of an admissible metric measurable on \( \Gamma \) [9]. For the exceptional curve family with respect to \( \rho \in P'(\Gamma) \), denoted by \( \Lambda(\rho) \), we can select a sequence of metrics \( \mu_n \in P(\Lambda(\rho)) \) such that \( ||\mu_n||^p \rightarrow 0 \). The sequence of metrics \( \rho_n = \max(\rho, \mu_n) \) is admissible and tends to \( \rho \), which implies \( \rho \in P^*(\Gamma) \). Clearly the closure of the intersection of \( P'(\Gamma) \) with the \( L_\infty \)-space coincides with \( P^*(\Gamma) \). Fugulede proved the existence of an extremal metric in the \( L_\infty \)-completion of \( P'(\Gamma) \) [1]. It seems to us that the proof of the existence of the extremal metric is easy in \( P^*(\Gamma) \) [5], but for the continuity lemma in § 3, the proof based on \( P'(\Gamma) \) is much easier than on \( P^*(\Gamma) \).

§ 3. A continuity lemma.

4. We now state

**Lemma 1.** Let \( \{\Gamma_n\} \) be an increasing sequence of curve families. Put \( \Gamma = \cup \Gamma_n \). Then we have

\[
\lim_{n \to \infty} \lambda(\Gamma_n) = \lambda(\Gamma).
\]

**Proof.** We prove the lemma for the module. The module of a curve family is infinite if its generalized admissible class is void, and in the case that \( P^*(\Gamma_n) = \emptyset \) for some \( n \) the statement is evident. If \( P^*(\Gamma_n) = \emptyset \) for all \( n \) there exists a unique extremal metric \( \rho_n \) in every \( P^*(\Gamma_n) \). The sequence \( \{\text{mod} \Gamma_n\} \) is in-
creasing. Hence if \( \lim \text{mod} \Gamma = \infty \), \( \text{mod} \Gamma = \infty \). Suppose the sequence \( \{ m_n \} \) is bounded. Then by the inequality (1) we have \( \| \rho_m - \rho_n \| \leq \| \rho_m \| - \| \rho_n \| \) (\( m > n \)). \( \rho_n \) tends to a metric \( \rho_0 \) from the boundedness of the sequence and we get \( \| \rho \| \geq \| \rho_0 \| \) for \( \rho \in P^*(\Gamma) \).

We now show \( \rho_0 \in P^*(\Gamma) \) which implies the extremality of \( \rho_0 \). From the definition of admissible metric we have \( P(\Gamma) = \bigcap P(\Gamma_n) \). We prove \( P^*(\Gamma) = \bigcap P^*(\Gamma_n) \). Clearly \( P^*(\Gamma) \subseteq \bigcap P^*(\Gamma_n) \). Suppose \( \rho \in P^*(\Gamma) \) for all \( n \). If we take an equivalent metric measurable on \( \Gamma \), denoted by the same notation \( \rho \), we have

\[ \int_\gamma \rho |dz| \geq 1 \quad \text{for almost all } \gamma \in \Gamma_n. \]

Let \( \Lambda_n(\rho) \) be the exceptional family of \( \Gamma_n \) for the inequality (2). Then the exceptional family \( \Lambda(\rho) \) of \( \Gamma \) is equal to \( \bigcup \Lambda_n(\rho) \). By Hersch's lemma [3] we have \( \text{mod} \Lambda(\rho) \leq \sum \text{mod} \Lambda_n(\rho) = 0 \), which implies that \( \Lambda(\rho) \) is exceptional (see also [1]).

\( P^*(\Gamma_n) \) contains the extremal metrics \( \rho_v \) for \( v \not\equiv n \) and hence \( \rho_0 \in P^*(\Gamma_n) \). So we get \( \rho_0 \in P^*(\Gamma) \).

§ 4. Application to conformal mappings.

5. Let \( \Omega \) be a plane domain. A boundary component \( \alpha \) of \( \Omega \) is defined by a decreasing sequence of subdomains \( \{ D_n \} \) of \( \Omega \) which satisfies the conditions that each member has a single analytic relative boundary compact in \( \Omega \), \( \Omega - D_n \) is a domain, \( \bigcup D_{n+1} \subseteq D_n \) and \( \bigcap D_n = \emptyset \). We call the sequence \( \{ D_n \} \) a boundary component \( \alpha \) of \( \Omega \). \( \{ D_n \} \) is called a defining sequence of \( \alpha \). Two defining sequences \( \{ D_n \} \) and \( \{ D'_n \} \) are said equivalent when every \( D_n \) contains some \( D'_m \) and vice versa. We can assign to \( \alpha \) a point set on the complex sphere defined by \( \bigcap \text{Cl} (D_n) \), where \( \text{Cl} (D_n) \) denotes the closure taken in the complex sphere, and the set is also written by \( \alpha \). A sequence of domains \( \Omega_n = \Omega - D_n \) exhausts \( \Omega \) and it is called an exaustion of \( \Omega \) in the direction to \( \alpha \).

We say, a curve \( \gamma \) : \( z = z(t) \) (\( 0 < t < 1 \)) tends to \( \alpha \) if each member of the defining sequence of \( \alpha \) contains the image of a suitable interval \( (s, 1) \) (resp. \( (0, 1) \)) under \( z(t) \).

6. We denote by \( \{ \gamma_j \}_{j=1}^4 \) four mutually disjoint piecewise analytic curves starting from the points of \( \Omega \), running within \( \Omega \) and tending to \( \alpha \). We can choose such a defining sequence \( \{ D_n \} \) the relative boundary of \( D_n \) intersects every \( \gamma_j \) precisely once. Let \( \{ \Omega_n \} \) be an exhaustion of \( \Omega \) in the direction to \( \alpha \). Let us denote by \( \gamma_j^{(n)} \) the intersection of \( \gamma_j \) with the relative boundary of \( \Omega_n \). We may assume that the sequence \( \{ \gamma_j^{(n)} \}_{j=1}^4 \) are arranged in the positive direction with respect to \( \Omega_n \). Then we say the curves \( \gamma_j \) mark vertices on \( \alpha \). The end parts of curves \( \gamma_j \) after \( \gamma_j^{(n)} \) divide the complementary domain \( \Omega - \Omega_n \) into four subdomains, say \( S_{\alpha_1}^{(n)}, S_{\alpha_2}^{(n)}, S_{\alpha_3}^{(n)} \) and \( S_{\alpha_4}^{(n)} \), where the suffix numbers of \( S^{(n)} \)'s mean those of the corresponding subarcs of the curves \( \gamma_j \) as their boundary. We call the sequence \( \{ S_{\alpha_i}^{(n)} \} \) a defining sequence of an edge \( \alpha_{12} \) determined by \( \gamma_1 \) and \( \gamma_2 \). The equivalence of two defining sequences of an edge is defined as in no. 5. The other three edges
are defined similarly. A curve tending to an edge is also defined by its defining sequence.

Put \( T_n = \Omega_n - \Sigma^{(i)}_{n} \cup \Sigma^{(s)}_{n} \). \( \{ T_n \} \) makes an exhaustion of \( \Omega \), called an exhaustion of \( \Omega \) in the direction to edges \( \alpha_{12} \) and \( \alpha_{41} \). An exhaustion of \( \Omega \) in the direction to \( \alpha_{12} \) and \( \alpha_{44} \) can be constructed similarly by means of \( \Sigma^{(i)}_{n} \) and \( \Sigma^{(s)}_{n} \). Taking a replica of \( T_n \) we can construct a double \( \hat{T}_n \) of \( T_n \) with respect to the relative boundaries of \( T_n \) which are two piecewise analytic Jordan arcs. \( \hat{T}_n \) has two boundary components which are the union of \( \alpha_{12} \) and its counterpart \( \alpha_{12} \), say \( Q_n \), and the union of \( \alpha_{34} \) and \( \alpha_{34} \), say \( C_n \). If the family of curves joining \( C^{(i)}_n \) and \( C^{(s)}_n \) has finite extremal length, there exists a unique minimal radially slit annulus mapping function \( f_n(z) \) except linear transformations with fixed points at zero and at infinity [7]. We state some properties of \( f_n \):

1) The images of \( C^{(i)}_n \) and \( C^{(s)}_n \) under \( f_n \) are two circles with center zero having possible radial incisions emanating from them, whose directions make a set of linear measure zero, and the images of the boundary components other than \( C^{(i)}_n \) and \( C^{(s)}_n \) are a quasi-minimal set of radial slits whose compact subset is minimal.

2) The two relative boundaries of \( T_n \) are mapped into a straight line through the origin by \( f_n \) and the images of \( T_n \) and its counterpart under \( f_n \) are symmetric with respect to it.

3) Let \( \hat{T}_n \) be the family of curves joining \( C^{(i)}_n \) and \( C^{(s)}_n \) in \( \hat{T}_n \) and let \( \Gamma_n \) be the subfamily of \( \hat{T}_n \) whose member joins them in \( \hat{T}_n \). Then we have \( \text{mod} \hat{T}_n = 2 \text{mod} \Gamma_n \) and the metric \( |f_n'(\log (r_1/r_2))| \) is the extremal metric for the both module problems, where \( r_1 \) and \( r_2 \) are the radii of the images of \( Q_n \) and \( Q_n \) (\( n > 2 \)).

4) Let \( \Lambda_1 \) be the family of curves joining \( \alpha_{12} \) and a compact neighborhood \( K \) of a point of \( T_n \) and having the property \( \lim_{t \to 1} |f_n(z(t))| < r_1 \), where \( z = z(t) \) is a representation of a curve tending to \( \alpha_{12} \) as \( t \to 1 \). Let \( \Lambda_2 \) be a similar family of curves tending to \( \alpha_{34} \) and satisfying \( \lim_{t \to 1} |f_n(z(t))| > r_2 \). Then the modules of \( \Lambda_1 \) and \( \Lambda_2 \) both vanish.

The statement 1) is found in [7]. The statement 2) is shown by the uniqueness of the mapping function. In fact \( \hat{T}_n \) has a self anti-conformal mapping onto itself, denoted by \( \tau(p) \) and \( f_n(\tau(p)) \) is also a minimal radially slit annulus mapping. We have \( f_n(\tau(p)) = e^{i\theta} f_n(p) \). Since \( p = \tau(p) \) on the relative boundaries of \( T_n \), we can deduce the images of the relative boundaries lie on the line \( \arg w = \theta/2 \). The extremal metric in the minimal radially slit annulus was given in [7]. Since the mapping \( \tau(p) \) fixes the curve family \( \hat{T}_n \), the uniqueness of the extremal metric shows the invariance of the extremal metric under \( \tau(p) \). On the other hand the extremal metric for the curve family \( \Gamma_n \) is extended to the replica of \( T_n \) symmetrically and the extended metric is admissible for \( \hat{T}_n \), which implies the coincidence of the both extremal metrics in \( \hat{T}_n \).

The property iv) for the domain \( \hat{T}_n \) and \( C^{(i)}_n \)'s is easily verified by the same method as in [9] which is originally due to Ohtsuka [5] and the statement is evident because they are subfamilies of the above family.

From these statements we can construct a function \( \varphi_n(z) \) which maps \( T_n \) onto a horizontally slit rectangle with the normalization that the relative boundary of
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$T_n$ containing subarcs of $\gamma_2$ and $\gamma_3$ is mapped into the bottom line $[0,1]$ in this order. Indeed $A \log f_n(z) + B$ with suitable constants is the desired function. The relative boundaries of $T_n$ are mapped into two horizontal sides and the edges $\alpha_{12}$ and $\alpha_{34}$ are mapped onto two vertical sides with possible horizontal incisions with total area zero. The image of the other boundary components is a quasi-minimal set of horizontal slits [8].

6. We state

**Theorem 1.** Let $\varphi_n(z)$ be the mapping function constructed above which maps $T_n$ onto a horizontally slit rectangle with bottom line $[0,1]$. If the heights of the image rectangles of $T_n$ under $\varphi_n$ is bounded then $\varphi_n(z)$ tends to a function $\varphi_0(z)$ in the sense that $\|\varphi_n(z) - \varphi_0(z)\|_{T_n} \to 0$. The image of $\alpha$ under $\varphi_n$ is a rectangle with possible horizontal incisions of 2-dimensional measure zero emanating from vertical sides. The images of other boundary components than $\alpha$ are a quasi-minimal set of horizontal slits. The module of the family of curves joining the edges $\alpha_{12}$ and $\alpha_{34}$ is equal to $h$, where $h$ is the height of the image rectangle. The module of its subfamily of curves satisfying

$$\lim_{t \to 1} \Re \varphi_0(z(t)) - \lim_{t \to 0} \Re \varphi_0(z(t)) < 1$$

vanishes, where the curve $z(t)$ tends to $\alpha_{12}$ and to $\alpha_{34}$ as $t \to 0$ as $t \to 1$ respectively.

**Proof.** From iii), the invariance of the module under a conformal mapping implies the module of the family $\Gamma_n$ of curves joining $\alpha_{12}$ and $\alpha_{34}$ in $T_n$ is equal to the height of its image rectangle, denoted by $h_n$ and $\rho_n = |\varphi_n'|$ is an extremal metric. The boundedness of the sequence $h_n$ and Lemma 1 show that $\rho_n$ tends to an extremal metric $\rho_0$ for $\Gamma = \bigcup \Gamma_n$ strongly. Since $\{\varphi'_n\}$ is weakly compact in the complex Hilbert space, any subsequence of $\{\varphi'_n\}$ contains a convergent subsequence whose limit denotes $\varphi'_0$. The limit function is a strong limit from the convergence of the norms, is analytic and univalent and satisfies $\rho_0 = |\varphi'_0|$. Put $h = \lim h_n$. Then the image domain of $\Omega$ under $\varphi_0$ is contained in the rectangle $0 < \Re w < 1, 0 < \Im w < h$ and its area is equal to $h$. Then the uniqueness of the strong limit $\rho_0$ implies the independence of $\varphi_0$ on the subsequences of $\{\varphi_n\}$ and hence $||\varphi'_n - \varphi'_0||_{T_n} \to 0$.

Next we show the last statement which is a property of incisions. Let $\{T_n\}$ be an exhaustion of $T_n$ in the direction to $\alpha_{12}$ and $\alpha_{34}$ and let $\{\varphi_n\}$ be the sequence of normalized slit rectangle mappings. Then $||\varphi'_n - \varphi'_0||_{T_n} \to 0$ as $n \to \infty$. Hence we can find a sequence of functions $\varphi_{n+\nu}(z)$ satisfying $||\varphi'_{n+\nu}(z) - \varphi'_0(z)||_{T_n} \to 0$. Denoting by $\Lambda_n$ and $\Lambda'_n$ the subfamilies of $\Gamma_n$ satisfying $\lim_{t \to 1} \Re \varphi_n(z(t)) - \lim_{t \to 0} \Re \varphi_0(z(t)) < 1$ and $<1-1/k$ respectively, we have $\mod \Lambda_n^k = 0$ and $\mod \Lambda_n = \mod \bigcup_k \Lambda_n^k = 0$. In fact similarly as in [9] we put $n = \Re \varphi_0$ and $u_m = \Re \varphi_{m+\nu}(z)$ in $T_{n+\nu} \cap T_n (n \leq m)$, $0$ in $S_{n+\nu}^k - T_{n+\nu}^k$ and $=1$ in $S_{n+\nu}^k - T_{n+\nu}^k$. Then the metric $\rho_m = k|\grad (u - u_m)|$ is admissible for $\Lambda_n^k$ and the convergence of $\varphi_{n+\nu}$ implies $\mod \Lambda_n^k = 0$. Since $\Lambda_n = \bigcup_k \Lambda_n^k$, we have also $\mod \Lambda_n = 0$. The family $\Lambda$ in the theorem is represented by $U_n \Lambda_n$ and hence we have the assertion.

We now discuss the shape of the image domain. The minimality of the image
of the boundary components other than \( \alpha \) follows from Lemma 5 in [8]. Put \( \Gamma' = \Gamma - \Lambda \). The metric \( \rho_0 = |\gamma'_0| \) is an extremal metric for \( \Gamma' \) and continuous, then Lemma 1 in [8] implies

\[
(3) \quad \inf_{\gamma \in \Gamma'} \int_{\gamma} |\rho_0|dz = 1,
\]

where \( \Gamma' \) is the subfamily of \( \Gamma' \) consisting of the curves through fixed point \( z \) in \( \Omega \). We can deduce from (3) that the image of \( \alpha \) under \( \varphi_0 \) is a rectangle with possible horizontal incisions emanating from the vertical sides. Vanishing of the area of the incisions is obvious from the identity \( \|\rho_0\|^q = h \). Thus the proof has been completed.

7. We discuss an extremal property of the slit rectangle mapping when the extremal distance of \( \alpha_{12} \) and \( \alpha_{34} \) is finite. We dealt with a similar problem for radial slit disc mappings [9] and Marden and Rodin [4] treated related problems for circular-radial slit mappings.

Let \( f(z) \) be a univalent function in \( \Omega \). For a relatively compact open set \( K \) let \( \Lambda(f, \alpha_{12}) \) be the family of curves joining \( K \) and \( \alpha_{12} \) in \( \Omega \) and satisfying \( \lim_{t\to0} \text{Re} f(z(t)) > 0 \) \( (z(t) \to \alpha_{12} \text{ as } t \to 1) \) and let \( \Lambda(f, \alpha_{34}) \) be family of curves joining \( K \) and \( \alpha_{34} \) in \( \Omega \) and satisfying \( \lim_{t\to1} \text{Re} f(z(t)) < 1 \). We denote by \( \mathfrak{F} \) the family of univalent functions \( f \) which map \( \alpha \) onto the outer boundary and satisfy \( \text{mod} \Lambda(f, \alpha_{12}) = \text{mod} \Lambda(f, \alpha_{34}) = 0 \) and \( \inf \varphi_0 \text{Im} f(z) = 0 \). The independence of the family \( \mathfrak{F} \) on the choice of \( K \) is easily verified as in [9]. It is also shown as in the proof of Theorem 1 that \( \varphi_0 \in \mathfrak{F} \). Put \( H(f) = \sup \varphi_0 \text{Im} f(z) \).

**Theorem 2.** If the extremal distance of \( \alpha_{12} \) and \( \alpha_{34} \) is finite, then \( \varphi_0 \) is the unique function which minimize the quantity \( H(f) \) among \( \mathfrak{F} \).

**Proof.** Let \( \Gamma \) denote the family of curves joining \( \alpha_{12} \) and \( \alpha_{34} \) and let \( \Lambda(f) \) denote subfamily of \( \Gamma \) satisfying \( \lim_{t\to1} \text{Re} f(z(t)) > 0 \) or \( \lim_{t\to0} \text{Re} f(z(t)) < 1 \) \( (z(t) \to \alpha_{12} \text{ as } t \to 0) \). Then the normalization of \( \mathfrak{F} \) implies \( \text{mod} \Lambda(f) = 0 \). In fact we take a curve joining \( \alpha_{34} \) and \( \alpha_{12} \) and consider its open covering \( K \) in \( \Omega \). \( K \) can be represented by relatively compact open sets \( K_n \) in \( \Omega \) in the form \( K = \bigcup_n K_n \). Denoting by \( \Lambda_n(f) \) the subfamily of \( \Lambda(f) \) intersecting \( K_n \), we have \( \text{mod} \Lambda_n(f) = 0 \) from the remark before this theorem, since any member of \( \Lambda_n(f) \) contains a curve of \( \Lambda(f, \alpha_{12}) \) or \( \Lambda(f, \alpha_{34}) \) for \( K_n \) as a subarc. Thus \( \text{mod} \Lambda(f) = \text{mod} \bigcup_n \Lambda_n(f) = 0 \).

We define a metric \( \rho \) from \( f \) by

\[
\rho = \begin{cases} 
|f''(z)| & (0 < \text{Re} f(z) < 1), \\
0 & \text{(elsewhere)}. 
\end{cases}
\]

We show \( \rho \in \mathcal{P}^*(\Gamma) \). Let \( \gamma \) be a curve of \( \Gamma' - \Lambda(f) \). Then considering the oscillation of \( \text{Re} f(z) \), we have

\[
\int_{\gamma} |\rho|dz \geq 1.
\]
from the facts \( \lim_{t \to 0} \Re f(z(t)) \leq 0 \) and \( \lim_{t \to 0} \Re f(z(t)) \geq 1 \). Since \( \mod A(f) = 0 \), from the same reason in no. 3 we have \( \rho \in P^*(\Gamma) \). The extremality and uniqueness of \( \phi_0 \) follow immediately from the inequality (1)

\[
\|\rho - \rho_0\|^2 \leq \|\rho\|^2 - \|\rho_0\|^2 \leq H(f) - \rho,
\]

where \( \rho_0 \) is the extremal metric \( |\phi_0| \) for \( \Gamma \) defined in Theorem 1, because the carrier of \( \rho \) is contained in the closed rectangle \( 0 \leq \Re f(z) \leq 1, 0 \leq \Im f(z) \leq H(f) \) in the image plane.

We can take many other functionals for which \( \phi_0 \) is extremal. For instance the area of the image domain \( ||f'||^2 \) is intimately connected with the extremal length and was dealt with by Marden and Rodin [4] and by others.

8. Following the notations in no. 6, we construct an exhaustion \( \{T_n\} \) of \( \Omega \) in the direction to \( \alpha_{12} \) and \( \alpha_{34} \), where \( T_n = \Omega_n - S_n^{34} \cup S_n^{12} \). Under the assumption that both the extremal length and the module of the family of curves joining the relative boundaries of \( T_n \) are finite, there exists a unique normalized horizontal slit rectangle mapping \( \phi_n(z) \) of \( T_n \) such that the edge \( \alpha_{34} \) corresponds to a subcontinuum (or a point) of the bottom line \([0,1]\). We can deduce that \( \phi_n \) tends to a function \( \phi_0 \) strongly as \( n \to \infty \), if the extremal lengths are bounded, but we have no informations about the shape of the image domain except the minimality of the images of boundary components other than \( \alpha \). However we can conclude that \( \phi_0 \) coincides with \( \phi_0 \), if we assume the accessibility of the four curves \( \gamma_j \) and the separability of edges. Here the accessibility of \( \gamma_j \) is in the sense that there exists a point \( z_j \) whose arbitrary neighborhood in the complex sphere contains a suitable end part of \( \gamma_j \).

**Theorem 3.** Let \( \tilde{T}_n \) be an exhaustion of \( \Omega \) in the direction to \( \alpha_{12} \) and \( \alpha_{34} \) and let \( \tilde{\Gamma}_n \) be the family of curves joining two relative boundaries of \( \tilde{T}_n \). Suppose the sequence of the extremal lengths \( \lambda(\Gamma_n) \) is positive and bounded. Then the normalized slit rectangle mapping \( \phi_n(z) \) of \( \tilde{T}_n \) constructed as in Theorem 1 tends to a function \( \phi_0 \) in the sense that \( ||\phi_n - \phi_0||^2 \to 0 \).

If four curves \( \gamma_j \) defining the vertices are accessible and if the two edges \( \alpha_{12} \) and \( \alpha_{34} \) have disjoint neighborhoods, then \( \phi_0 \) coincides with \( \phi_0 \) defined in Theorem 1.

**Proof.** The metric \( \rho_n = |\phi_n'| \) is extremal for \( \tilde{T}_n \) and is contained in \( P(\tilde{T}_n) \) for all \( m > n \), where \( \rho_n \) is extended as zero outside of \( \tilde{T}_n \). Put \( h_n = \mod \tilde{T}_n \), which is decreasing and is bounded away from zero. Then we have from (1) \( ||\rho_n - \rho_m||^2 \leq ||\rho_n||^2 - ||\rho_m||^2 = h_n - h_m \) for \( m > n \) and \( \rho_n \) tends to a metric \( \rho_0 \) strongly. The same reason as in the proof of Theorem 1 shows that \( \phi_n \) tends to a univalent function \( \phi_0 \) in the sense that \( ||\phi_n - \phi_0||^2 \to 0 \).

Next we show the continuity of the module of \( \tilde{T}_n \). Our proof is originally due to Strebel [7], using a classical method of the proof of Phragmén—Lindelöf’s theorem. Let \( \rho \) be a metric of \( P(\Gamma) \), where \( \Gamma \) is the family of curves joining \( \alpha_{12} \) and \( \alpha_{34} \) in \( \Omega \). We set

1) Here we regard \( r_j \)’s lying partly on the boundary as the curves defining vertices.
\[ L(\rho, \Gamma) = \inf_{r \in F} \int_{\mathcal{F}} \rho|dz| \]

Let \( \mathcal{O} \) denote the complementary domain of \( \alpha \) with respect to the complex sphere containing \( \Omega \). By our assumptions, there exist four neighborhoods of the vertices of \( \alpha \), denoted by \( V_j \) \((j = 1, 2, 3, 4)\), such that the module of each family \( \chi_j \) of curves joining \( V_j \) and its opposite edge is less than \( \varepsilon/8 \), since a point has infinite extremal distance from any compact disjoint set. Then we can construct metrics \( \rho_j \in P(\chi_j) \) \((j = 1, 2, 3, 4)\) such that \( ||\mu_i||^2 < \varepsilon/4 \). We set \( \rho_i = \max_{1 \leq j \leq 4}(\rho_j, \mu_j) \) which is a member of \( P(\Gamma) \). Then we have \( \lim L(\rho_n, \tilde{T}) = 1 \). In fact, contrary to the assertion, there exists a curve \( \gamma_n \) joining two relative boundaries of \( \tilde{T} \) which satisfies

\[ \int_{\gamma_n} \rho_n|dz| \leq 1 - \delta. \]

\( \gamma_n \) intersects both the relative boundaries of \( \tilde{T} \), for \( n \leq n \). Let \( \zeta^{(n)} \) and \( \zeta_{\alpha}^{(n)} \) be two points of the intersections of \( \gamma_n \) with the relative boundaries of \( \tilde{T} \), separating \( \alpha_{12} \) and \( \alpha_{44} \) from \( \tilde{T} \) respectively. Then the points \( \zeta^{(n)} \) and \( \zeta_{\alpha}^{(n)} \) have at least one cluster point on each relative boundary of \( \tilde{T} \), and outside of \( V_j \)'s. Therefore we can construct a curve \( \beta \) joining \( \alpha_{12} \) and \( \alpha_{44} \) such that

\[ \int_{\beta} \rho_n|dz| \leq 1 - \frac{\delta}{2}, \]

by Strebel's method [7]. We get a contradiction to the admissibility of \( \rho_n \).

For any \( k > 1 \), \( k\rho_i \) is admissible for \( \Gamma_n \) with sufficiently large \( n \) and we have \( ||\rho_n||^2 \leq ||k\rho_i||^2 \). Thus we get

\[ ||\rho_0||^2 \leq ||\rho_i||^2 \leq ||\rho||^2 + \varepsilon. \]

Clearly \( \rho_0 \in P^*(\Gamma) \) and \( \rho_0 \) is extremal for \( \Gamma \). We obtain the continuity of modules with respect to this exhaustion.

The uniqueness of the extremal metric \( \rho_0 \) implies \( \psi_0 = \varphi_0 \).

**References**


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