KŌDAI MATH. SEM. REP.
19 (1967), 28—30

ON WIENER'S FORMULA FOR STOCHASTIC PROCESSES

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1. Let $\mathcal{E}(t)$ $(-\infty<t<\infty)$ be a weakly stationary stochastic process with the spectral representation:

$$\mathcal{E}(t)=\int_{-\infty}^{\infty} e^{it\lambda}dZ(\lambda), \tag{1}$$

and let

$$X(t)=f(t)+\mathcal{E}(t), \tag{2}$$

where $f(t)$ is a numerical valued function. Consider the stochastic integral

$$\int_{-\infty}^{\infty} X(t)aK(at)dt$$

with $K(t)\in L_1(-\infty, \infty)$. Kawata [1] has shown that under some conditions on $K(t)$ and $f(t)$ we have the following Wiener type formula:

$$\operatorname*{l.i.m.}_{a\to 0}\int_{-\infty}^{\infty} X(t)e^{-i\xi t}aK(at)dt=[M_\xi+Z(\xi+0)-Z(\xi-0)]\int_{-\infty}^{\infty} K(t)dt, \tag{3}$$

where ξ is a real constant and

$$M_\xi=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(t)e^{-i\xi t}dt.$$

The purpose of this paper is to prove the similar formula for the more general class of stochastic processes.

2. We state first the following

LEMMA. *Let* $\{f_\lambda(\cdot)\}_{\lambda\in\Lambda}$ *be a class of functions defined on* $(0, \infty)$. *If*

(i) $K(x)$ *is absolutely continuous in every finite interval,*

(ii) $|x^2K(x)|<H$, $K(x)\in L_1(0, \infty)$, *H being a constant,*

(iii) $\frac{1}{T}\int_0^T |f_\lambda(t)|dt\leqq G$, *$G$ being a constant independent of λ and T, and*

(iv) $\lim_{T\to\infty}\frac{1}{T}\int_0^T f_\lambda(t)dt=M_\lambda$, *uniformly in* $\lambda\in\Lambda$,

then

Received May 7, 1966.

$$\lim_{a\to 0}\int_0^\infty f_\lambda(t)aK(at)dt=M_\lambda\int_0^\infty K(t)dt$$

uniformly in $\lambda\in\Lambda$.

The proof of this Lemma will not be given here, since it is quite similar to the proof of well-known Wiener's formula (see [2], pp. 30–32).

Let $X(t)$ $(t\geqq 0)$ be a stochastic process which satisfies that

(v) $E\{|X(t)|^2\}<\infty$,

(vi) the stochastic integral

$$\int_b^a X(t)dt$$

exists for every finite interval $[a, b]$,

(vii) $\frac{1}{T}\int_0^T \sqrt{E\{|X(t)|^2\}}\,dt\leqq G$, G being a constant independent of T, and

(viii) there exists a random variable X_0 with $E\{|X_0|^2\}<\infty$ such that

$$\lim_{T\to\infty} E\left\{\left|\frac{1}{T}\int_0^T X(t)dt-X_0\right|^2\right\}=0.$$

We shall now prove the following

THEOREM. *Let* $X(t)$ $(t\geqq 0)$ *satisfy the conditions* (v), (vi), (vii) *and* (viii). *If* $K(t)$ *satisfies the conditions* (i) *and* (ii) *of Lemma, then*

$$\underset{a\to 0}{\text{l.i.m.}}\int_0^\infty X(t)aK(at)dt=X_0\int_0^\infty K(t)dt. \tag{4}$$

Proof. Denote by $\mathfrak{H}$ the Hilbert space consisting of all random variables Y with $E\{|Y|^2\}<\infty$. If $Z\in\mathfrak{H}$, then we have by (viii) that

$$\lim_{T\to\infty} E\left\{\frac{1}{T}\int_0^T X(t)dt\cdot\bar{Z}\right\}=\lim_{T\to\infty}\frac{1}{T}\int_0^T E\{X(t)\cdot\bar{Z}\}dt=E\{X_0\cdot\bar{Z}\}. \tag{5}$$

Therefore by Lemma we have from (5) and (vii) that for every $Z\in\mathfrak{H}$

$$\begin{aligned}&\lim_{a\to 0} E\left\{\int_0^\infty X(t)aK(at)dt\cdot\bar{Z}\right\}\\ &=\lim_{a\to 0}\int_0^\infty E\{X(t)\cdot\bar{Z}\}aK(at)dt \\ &=E\{X_0\cdot\bar{Z}\}\int_0^\infty K(t)dt,\end{aligned} \tag{6}$$

or

$$w\text{-}\lim_{a\to 0}\int_0^\infty X(t)aK(at)dt=X_0\cdot\int_0^\infty K(t)dt. \tag{7}$$

In order to prove (4), or equivalently to prove

$$(8)\qquad s\text{-}\lim_{a\to 0}\int_0^\infty X(t)aK(at)dt = X_0\cdot\int_0^\infty K(t)dt,$$

it is sufficient to show in addition to (7) that

$$(9)\qquad E\left\{\int_0^\infty X(t)aK(at)dt\cdot\int_0^\infty \overline{X(s)}b\overline{K(bs)}ds\right\}\to E\left\{X_0\cdot\int_0^\infty K(t)dt\cdot\int_0^\infty \overline{X(s)}b\cdot\overline{K(bs)}ds\right\}$$

as $a\to 0$ uniformly in $b\in U$ where U is a neighborhood of $b=0$. Since $\int_0^\infty X(s)bK(bs)ds$ converges weakly, $E\{|\int_0^\infty X(s)bK(bs)ds|^2\}$ is bounded for $b\in U$. Therefore it follows from (viii) that

$$(10)\qquad E\left\{\frac{1}{T}\int_0^T X(t)dt\cdot\int_0^\infty \overline{X(s)}b\overline{K(bs)}ds\right\}\to E\left\{X_0\int_0^\infty \overline{X(s)}b\overline{K(bs)}ds\right\}$$

as $T\to\infty$ uniformly in $b\in U$, that is,

$$(11)\qquad \lim_{T\to\infty}\frac{1}{T}\int_0^T\left(\int_0^\infty E\{X(t)\overline{X(s)}\}b\overline{K(bs)}ds\right)dt = \int_0^\infty E\{X_0\cdot\overline{X(s)}\}b\overline{K(bs)}ds$$

uniformly in $b\in U$. And we have for $b\in U$ that

$$(12)\qquad \frac{1}{T}\int_0^T\left|\int_0^\infty E\{X(t)\overline{X(s)}\}b\overline{K(bs)}ds\right|dt\leqq G\cdot\sup_{b\in U}\sqrt{E\left\{\left|\int_0^\infty X(s)bK(bs)ds\right|^2\right\}}.$$

Hence by Lemma we have that

$$(13)\qquad \lim_{a\to 0}\int_0^\infty\left(\int_0^\infty E\{X(t)\overline{X(s)}\}b\overline{K(bs)}ds\right)aK(at)dt = \int_0^\infty E\{X_0\cdot\overline{X(s)}\}b\overline{K(bs)}ds\cdot\int_0^\infty K(t)dt$$

uniformly in $b\in U$. But (13) is equivalent to (9), and thus theorem was proved.

REFERENCES

[1] KAWATA, T., Some convergence theorems for stationary stochastic processes. Ann. Math. Stat. **30** (1959), 1192–1214.

[2] BOCHNER, S., Vorlesungen über Fouriersche Integrale. Leibzig (1932).

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