$|\bar{N}, p_n|$ SUMMABILITY FACTORS OF INFINITE SERIES

BY S. M. MAZHAR

1. 1. Let $\sum a_n$ be a given infinite series with s_n as its n-th partial sum. Also let $\{p_n\}$ be a sequence of positive real constants such that P_n tends to infinity with n, where $P_n=\sum_{\nu=0}^{n} p_\nu$. We write

$$t_n=\frac{1}{P_n}\sum_{\nu=0}^{n} p_\nu s_\nu. \tag{1.1.1}$$

The series $\sum a_n$ is said to be absolutely summable $(\bar{N}, p_n)$ or, simply summable $|\bar{N}, p_n|$, if the sequence $\{t_n\}$ is of bounded variation.

If for some finite s

$$\sum_{\nu=1}^{n} |s_\nu - s| p_\nu = o(P_n),$$

as $n\to\infty$, then $\sum a_n$ is said to be strongly summable $(\bar{N}, p_n)$ or, simply summable $[\bar{N}, p_n]$. If

$$\sum_{\nu=1}^{n} |s_\nu| p_\nu = O(P_n),$$

as $n\to\infty$, then $\sum a_n$ is said to be bounded $[\bar{N}, p_n]$.

Writing $p_n=1/n$ in the above definitions we get summability $|R, \log n, 1|$,[1] summability $[R, \log n, 1]$ and bounded $[R, \log n, 1]$ respectively.

1. 2. Suppose $\sum a_n$ is summable $|\bar{N}, p_n|$. Then, since

$$s_{n+1}p_{n+1}=t_{n+1}P_{n+1}-t_nP_n,$$

we have

$$\sum_1^m |s_{n+1}| p_{n+1} = \sum_1^m |\varDelta t_n P_n|$$

$$\leqq \sum_1^m P_n|\varDelta t_n| + \sum_1^m |t_{n+1}||\varDelta P_n|$$

$$=O(P_m)+O\left(\sum_1^m |\varDelta P_n|\right)$$

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1) Summability $|R, \log n, 1|$ is equivalent to the summability $|\bar{N}, 1/n|$.

$$=O(P_{m+1}).$$

Thus if a series $\sum a_n$ is summable $|\bar{N}, p_n|$ it is necessarily bounded $[\bar{N}, p_n]$. However the converse is not true.

The object of this paper is to obtain a suitable summability factor $\{\lambda_n\}$ so that boundedness $[\bar{N}, p_n]$ of $\sum a_n$ may imply $|\bar{N}, p_n|$ summability of $\sum a_n\lambda_n$.

2. 1. In what follows we shall prove the following theorem.

THEOREM 1. *If $\sum a_n$ is bounded $[\bar{N}, p_n]$, where $\{p_{n+1}/p_n\}$ is bounded and if $\{\lambda_n\}$ is a bounded sequence satisfying the following conditions*:

(a) $$\sum_1^m |\Delta\lambda_n| = O(1),$$

(b) $$\sum_2^m \frac{p_n|\lambda_n|}{P_n} = O(1),$$

(c) $$\sum_2^m P_n|\Delta\lambda_{n+1}|\left|\Delta\frac{1}{p_n}\right| = O(1),$$

(d) $$\sum_2^m \frac{P_n}{p_n}|\Delta^2\lambda_n| = O(1),$$

as $m\to\infty$, then $\sum a_n\lambda_n$ is summable $|\bar{N}, p_n|$.

It may be remarked that the special case for $p_n = 1/n$ of this theorem has been recently considered by Kulshrestha [2].

2. 2. *Proof of Theorem* 1. Let $c_n = a_n\lambda_n$, $T_n = \sum_{\nu=0}^n c_\nu$ and

$$t_n^* = \frac{1}{P_n}\sum_{\nu=0}^n T_\nu p_\nu.$$

We have

$$\Delta t_n^* = \Delta\left(\frac{1}{P_n}\sum_{\nu=0}^n p_\nu T_\nu\right)$$

$$= \Delta\left(\frac{1}{P_n}\right)\sum_{\nu=0}^n p_\nu T_\nu - \frac{p_{n+1}T_{n+1}}{P_{n+1}}$$

$$= -\Delta\left(\frac{1}{P_n}\right)\sum_{\nu=0}^{n-1} P_\nu c_{\nu+1} + \Delta\left(\frac{1}{P_n}\right)P_n T_n - \frac{p_{n+1}T_{n+1}}{P_{n+1}}$$

$$= -\Delta\left(\frac{1}{P_n}\right)\sum_{\nu=0}^n P_\nu c_{\nu+1}$$

$$= -\Delta\left(\frac{1}{P_n}\right)\sum_{\nu=0}^{n-1}(s_{\nu+1} - a_0)\Delta(P_\nu\lambda_{\nu+1}) - \Delta\left(\frac{1}{P_n}\right)(s_{n+1} - a_0)P_n\lambda_{n+1}$$

$$=\Delta\left(\frac{1}{P_n}\right)\sum_{\nu=0}^{n-1}(s_{\nu+1}-a_0)\lambda_{\nu+1}p_{\nu+1}-\Delta\left(\frac{1}{P_n}\right)\sum_{\nu=0}^{n-1}(s_{\nu+1}-a_0)P_{\nu+1}\Delta\lambda_{\nu+1}$$

$$-\Delta\left(\frac{1}{P_n}\right)(s_{n+1}-a_0)P_n\lambda_{n+1}$$

$$=\Delta\left(\frac{1}{P_n}\right)\sum_{\nu=0}^{n-2}\Delta\lambda_{\nu+1}\sum_{\mu=0}^{\nu}(s_{\mu+1}-a_0)p_{\mu+1}+\Delta\left(\frac{1}{P_n}\right)\lambda_n\sum_{\mu=0}^{n-1}(s_{\mu+1}-a_0)p_{\mu+1}$$

$$-\Delta\left(\frac{1}{P_n}\right)\sum_{\nu=0}^{n-2}\Delta\left\{\frac{P_{\nu+1}}{p_{\nu+1}}\Delta\lambda_{\nu+1}\right\}\sum_{\mu=0}^{\nu}(s_{\mu+1}-a_0)p_{\mu+1}$$

$$-\Delta\left(\frac{1}{P_n}\right)\frac{P_n}{p_n}\Delta\lambda_n\sum_{\mu=0}^{n-1}(s_{\mu+1}-a_0)p_{\mu+1}-\Delta\left(\frac{1}{P_n}\right)(s_{n+1}-a_0)P_n\lambda_{n+1}$$

$$=L_1+L_2+L_3+L_4+L_5, \qquad \text{say.}$$

Now

$$\sum_2^m|L_1|=O\left(\sum_2^m\Delta\left(\frac{1}{P_n}\right)\sum_{\nu=0}^{n-2}|\Delta\lambda_{\nu+1}|P_{\nu+1}\right)$$

(2. 2. 1)
$$=O\left(\sum_0^{m-2}|\Delta\lambda_{\nu+1}|P_{\nu+1}\sum_{\nu+2}^{m}\Delta\left(\frac{1}{P_n}\right)\right)$$

$$=O\left(\sum_{\nu=0}^{m-2}|\Delta\lambda_{\nu+1}|\right)=O(1),$$

by virtue of the condition (a) of the hypotheses.

Next

$$\sum_2^m|L_2|=O\left(\sum_2^m\Delta\left(\frac{1}{P_n}\right)|\lambda_n|P_n\right)$$

$$=O\left(\sum_2^m|\lambda_n|\cdot\frac{p_{n+1}}{P_{n+1}}\right)$$

$$=O\left(\sum_2^m|\lambda_n|\cdot\frac{p_n}{P_n}\right)=O(1),$$

by the condition (b) and the hypothesis that $\{p_{n+1}/p_n\}$ is bounded.

Again

$$\sum_2^m|L_3|=O\left(\sum_2^m\Delta\left(\frac{1}{P_n}\right)\sum_{\nu=0}^{n-2}P_{\nu+1}\left|\Delta\left\{\frac{P_{\nu+1}}{p_{\nu+1}}\Delta\lambda_{\nu+1}\right\}\right|\right)$$

$$=O\left(\sum_2^m\Delta\left(\frac{1}{P_n}\right)\sum_{\nu=0}^{n-2}P_{\nu+1}\frac{p_{\nu+2}}{p_{\nu+1}}|\Delta\lambda_{\nu+1}|\right)+O\left(\sum_2^m\Delta\left(\frac{1}{P_n}\right)\sum_{\nu=0}^{n-2}\frac{P_{\nu+1}P_{\nu+2}}{p_{\nu+1}}|\Delta^2\lambda_{\nu+1}|\right)$$

$$+O\Big(\sum_{2}^{m}\Delta\Big(\frac{1}{P_n}\Big)\sum_{\nu=0}^{n-2}P_{\nu+1}P_{\nu+2}|\Delta\lambda_{\nu+2}|\Big|\Delta\Big(\frac{1}{p_{\nu+1}}\Big)\Big|\Big)$$

$$=O\Big(\sum_{2}^{m}\Delta\Big(\frac{1}{P_n}\Big)\sum_{\nu=0}^{n-2}P_{\nu+1}|\Delta\lambda_{\nu+1}|\Big)+O\Big(\sum_{\nu=0}^{m-2}P_{\nu+1}|\Delta\lambda_{\nu+2}|\Big|\Delta\Big(\frac{1}{p_{\nu+1}}\Big)\Big|\Big)$$

$$+O\Big(\sum_{\nu=0}^{m-2}\frac{P_{\nu+1}}{p_{\nu+1}}|\Delta^2\lambda_{\nu+1}|\Big)=O(1),$$

by (2. 2. 1) and conditions (c) and (d) respectively.

Next

$$\sum_{2}^{m}|L_4|=O\Big(\sum_{2}^{m}\Delta\Big(\frac{1}{P_n}\Big)\frac{P_n}{p_n}|\Delta\lambda_n|P_n\Big)$$

$$=O\Big(\sum_{2}^{m}|\Delta\lambda_n|\Big)=O(1).$$

Finally, we have

$$\sum_{2}^{m}|L_5|\leqq\sum_{2}^{m}\Delta\Big(\frac{1}{P_n}\Big)P_n|\lambda_{n+1}||s_{n+1}-a_0|$$

$$=\sum_{2}^{m}\frac{p_{n+1}}{P_{n+1}}|\lambda_{n+1}||s_{n+1}-a_0|$$

$$=\sum_{2}^{m-1}\Delta\Big(\frac{|\lambda_{n+1}|}{P_{n+1}}\Big)\sum_{\mu=0}^{n}p_{\mu+1}|s_{\mu+1}-a_0|+\frac{|\lambda_{m+1}|}{P_{m+1}}\sum_{\mu=0}^{m}p_{\mu+1}|s_{\mu+1}-a_0|+O(1)$$

$$=O\Big(\sum_{2}^{m-1}\Big|\frac{(\Delta|\lambda_{n+1}|)}{P_{n+1}}\Big|P_{n+1}\Big)+O\Big(\sum_{2}^{m-1}|\lambda_{n+2}|\Delta\Big(\frac{1}{P_{n+1}}\Big)P_{n+1}\Big)+O(|\lambda_{m+1}|)+O(1)$$

$$=O\Big(\sum_{2}^{m-1}|\Delta\lambda_{n+1}|\Big)+O\Big(\sum_{2}^{m-1}|\lambda_{n+2}|\frac{p_{n+2}}{P_{n+2}}\Big)+O(1)=O(1),$$

by conditions (a) and (b) of the hypotheses.

This completes the proof of Theorem 1.

3. 1. The following theorem concerning summability $|C, 1|$ of $\sum a_n\lambda_n$ is a direct corollary of the above theorem (obtained by taking $p_n=1$).

THEOREM 2. *If*

$$\sum_{1}^{n}|s_\nu|=O(n) \qquad (3.\ 1.\ 1)$$

and $\{\lambda_n\}$ *is a bounded sequence such that*

$$\sum_{1}^{m}|\Delta\lambda_n|=O(1), \qquad \text{(a)}'$$

(b)′ $$\sum_{2}^{m}\left|\frac{\lambda_n}{n}\right|=O(1),$$

(c)′ $$\sum_{2}^{m} n|\Delta^2\lambda_n|=O(1),$$

as $m\to\infty$, *then* $\sum a_n\lambda_n$ *is summable* $|C, 1|$.

This is a generalisation of the following theorem of Pati [3].

THEOREM A. *If* $\sum a_n$ *is summable* $[C, 1]$ *then* $\sum a_n\lambda_n$ *is summable* $|C, 1|$, *where* $\{\lambda_n\}$ *is a convex sequence such that* $\sum \lambda_n/n<\infty$.

It may be observed that summability $[C, 1]$ implies (3. 1. 1). Also it is well known [1, 4] that if $\{\lambda_n\}$ is a convex sequence such that $\sum \lambda_n/n<\infty$, then λ_n necessarily satisfies all the above conditions of Theorem 2 but the converse is not true.

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DEPARTMENT OF MATHEMATICS AND STATISTICS,
ALIGARH MUSLIM UNIVERSITY, ALIGARH, INDIA.