NORMAL STRUCTURE f SATISFYING $f^3+f=0$

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A structure on an n-dimensional differentiable manifold given by a non-null tensor field f of constant rank r satisfying $f^3+f=0$ is called an *f-structure* [2, 6, 7].[1] If $n=r$, then an f-structure gives an almost complex structure of the manifold and $n=r$ is necessarily even. If the manifold is orientable and $n-1=r$, then an f-structure gives an almost contact structure of the manifold and n is necessarily odd.

Sasaki and Hatakeyama [4] have introduced the notion of normality in the study of almost contact structure and characterized the normal almost contact structure by the vanishing of a tensor field constructed from the structure. On the other hand, it is well known [1, 5, 9] that an almost complex structure in the tangent bundle is determined by giving a linear connection in the tangent bundle. The almost complex structure in the tangent bundle is complex if and only if the linear connection determining the almost complex structure is locally flat [5].

When an n-dimensional manifold V admits a non-null f-structure f of rank r such that $n-r\geqq 1$, there exist two distributions L and M corresponding to the projection operators $l=-f^2$ and $m=f^2+1$ respectively. The operator f operating on the tangent bundle $T(V)$ of the manifold V acts as an almost complex structure on the distribution L and as a null-operator on the distribution M. It is now well known [1, 5, 9] that an almost complex structure is determined in the tangent bundle when a linear connection is given in the tangent bundle. By a similar device as that used in the study of almost complex structure in the tangent bundle, we shall show in § 3 of the present paper that an almost complex structure is determined in the vector bundle $M(V)$ by giving a connection ω in the vector bundle $M(V)$, $M(V)$ being the vector bundle consisting of all tangent vectors belonging to the distribution M.

When the almost complex structure in the vector bundle $M(V)$ is complex analytic, we say that the f-structure f is *normal* with respect to the given connection ω. We shall prove in § 5 that the f-structure f is normal with respect to a connection ω given in the vector bundle $M(V)$ if and only if the connection ω is of zero curvature and a tensor field constructed from f and ω vanishes identically (Theorem 2). The notion of normal f-structure seems to be very useful in study of certain submanifolds immersed in an almost complex space (cf. [8]).

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1) The numbers between brackets refer to the Bibliography at the end of the paper.

§ 1. f-structure.

Let there be given, in an n-dimensional differentiable manifold V of class C^∞, a non-null tensor field $\boldsymbol{f}$ of type (1. 1) and of class C^∞ satisfying

$$\boldsymbol{f}^3+\boldsymbol{f}=\boldsymbol{0}. \tag{1. 1}$$

We call such a structure $\boldsymbol{f}$ an *f-structure* of rank r, when the rank of $\boldsymbol{f}$ is constant everywhere and is equal to r, where r is necessarily even [2, 6, 7].

If we put

$$\boldsymbol{l}=-\boldsymbol{f}^2, \qquad \boldsymbol{m}=\boldsymbol{f}^2+\boldsymbol{1}, \tag{1. 2}$$

we have

$$\begin{aligned} &\boldsymbol{l}+\boldsymbol{m}=\boldsymbol{1}, \qquad \boldsymbol{l}^2=\boldsymbol{l}, \qquad \boldsymbol{m}^2=\boldsymbol{m}, \\ &\boldsymbol{f}\boldsymbol{l}=\boldsymbol{l}\boldsymbol{f}=\boldsymbol{f}, \qquad \boldsymbol{f}\boldsymbol{m}=\boldsymbol{m}\boldsymbol{f}=0, \end{aligned} \tag{1. 3}$$

where $\boldsymbol{1}$ denotes the unit tensor. These equations show that the operators $\boldsymbol{l}$ and $\boldsymbol{m}$ applied to the tangent space at each point of the manifold V are complementary projection operators. Then there exist in the manifold V two distributions L and M corresponding to the projection operators $\boldsymbol{l}$ and $\boldsymbol{m}$ respectively. When the rank of $\boldsymbol{f}$ is r, L is r-dimensional and M $(n-r)$-dimensional.

Let U be an arbitrary coordinate neighborhood of the manifold V admitting an f-structure $\boldsymbol{f}$ of rank r. If we take in U arbitrarily an ordered set $\{\boldsymbol{f}_x\}$ of $n-r$ (contravariant) vector fields $\boldsymbol{f}_x$ spanning the distribution M at each point, then there exists uniquely in U an ordered set $\{\boldsymbol{f}^y\}$ of $n-r$ covariant vector fields (1-forms) $\boldsymbol{f}^y$ such that[2)]

$$\sum_{x=n+1}^{2n-r} \boldsymbol{f}^x\otimes\boldsymbol{f}_x=\boldsymbol{m}, \qquad \boldsymbol{f}^x(\boldsymbol{f}_y)=\delta^x_y. \tag{1. 4}$$

Taking account of (1. 3), we have from (1. 4)

$$\boldsymbol{f}^y(\boldsymbol{f}\boldsymbol{X})=0, \qquad \boldsymbol{f}\boldsymbol{f}_x=0 \tag{1. 5}$$

for any vector $\boldsymbol{X}$ at each point of V. We call such an ordered set $\{\boldsymbol{f}_x\}$ an $(n-r)$-*frame* and the ordered set $\{\boldsymbol{f}^y\}$ an $(n-r)$-*coframe* being dual to $\{\boldsymbol{f}_x\}$.

If a covariant vector field $\boldsymbol{\phi}$, global or local, satisfies at each point

$$\boldsymbol{\phi}(\boldsymbol{X})=0$$

for any vector $\boldsymbol{X}$ belonging to the distribution L, $\boldsymbol{\phi}$ is said to be *transversal* to L. It is easily seen that any covariant vector field $\boldsymbol{\phi}$, being transversal to L, is expressible uniquely by

$$\boldsymbol{\phi}=\phi_y\boldsymbol{f}^y$$

as a linear combination of $\boldsymbol{f}^y$ in U. Similarly, any contravariant vector field $\boldsymbol{v}$ belonging to the distribution M is expressible uniquely by

2) The indices x, y, z, u run over the range $\{n+1, n+2, \cdots, 2n-r\}$.

$$\boldsymbol{v}=v^x\boldsymbol{f}_x$$

as a linear combination of $\boldsymbol{f}_x$ in U.

Denoting by $f_b{}^a, f^a{}_y, f_b{}^x$ respectively the components of $\boldsymbol{f}, \boldsymbol{f}_y, \boldsymbol{f}^x$ with respect to local coordinates (η^a) defined in U,[3] we find from (1. 1), (1. 2), (1. 4) and (1. 5)

$$\begin{aligned} &f_b{}^c f_c{}^a=-\delta_b^a+f_b{}^y f^a{}_y, \qquad && f^c{}_y f_c{}^a=0, \\ &f^c f_c{}^x=0, && f^c{}_y f_c{}^x=\delta_y^x. \end{aligned} \tag{1. 6}$$

§ 2. Vector bundle $M(V)$.

Let there be given an f-structure $\boldsymbol{f}$ of rank r in an n-dimensional manifold V. Then the set of all tangent vectors belonging to the distribution M forms a vector bundle p: $M(V)\rightarrow V$ over V, which is a subbundle of the tangent bundle $T(V)$ of V.

Let $M^*(V)$ be the vector bundle which is dual to $M(V)$. If we take an element $\tilde{\boldsymbol{\phi}}$ belonging to the fibre M^*_{P} of the bundle $M^*(V)$ at a point P of V, then there exists at P uniquely a covector $\boldsymbol{\phi}$ of V, which is transversal to the distribution L, such that

$$\boldsymbol{\phi}(\boldsymbol{v})=\tilde{\boldsymbol{\phi}}(\boldsymbol{v}) \tag{2. 1}$$

for any element $\boldsymbol{v}$ belonging to the fibre M_{P} of $M(V)$ at P. Conversely, for any covector $\boldsymbol{\phi}$ transversal to L at P, there exists uniquely an element $\tilde{\boldsymbol{\phi}}$ of M^*_{P} satisfying (2. 1). In such a way, the vector bundle $M^*(V)$ can be identified naturally with the set of all covectors transversal to the distribution L. In this sense, the bundle $M^*(V)$ can be regarded as a subbundle of the cotangent bundle $T^*(V)$ of the manifold V.

If, in a coordinate neighborhood U of the manifold V, we take an $(n-r)$-frame $\{\boldsymbol{f}_x\}$ and an $(n-r)$-coframe $\{\boldsymbol{f}^y\}$ being dual to $\{\boldsymbol{f}_x\}$, then $\{\boldsymbol{f}_x\}$ is a basis of the fibre M_{P} of $M(V)$ at each point P of U and $\{\boldsymbol{f}^y\}$ is a basis of the fibre M^*_{P} of $M^*(V)$ at each point P of U and dual to $\{\boldsymbol{f}_x\}$. Taking in V a vector field $\boldsymbol{v}$ belonging to the distribution M and a covector field $\boldsymbol{\phi}$ transversal to the distribution L, then we have

$$\boldsymbol{v}=v^x\boldsymbol{f}_x, \qquad \boldsymbol{\phi}=\phi_y\boldsymbol{f}^y$$

in U with functions v^x and ϕ_y defined in U, where v^x and ϕ_y are determined uniquely respectively for $\boldsymbol{v}$ and $\boldsymbol{\phi}$. (v^x) and (ϕ_y) are called respectively the *components* of $\boldsymbol{v}$ and $\boldsymbol{\phi}$ with respect to the $(n-r)$-frame $\{\boldsymbol{f}_x\}$ in the vector bundle $M(V)$.

Let U and U' be two coordinate neighborhoods of the base space V such that $U\cap U'\neq\phi$. If $\{\boldsymbol{f}_x\}$ and $\{\boldsymbol{f}_{x'}\}$ be $(n-r)$-frames defined respectively in U and U', then we have

$$\boldsymbol{f}_{y'}=A_{y'}^y\boldsymbol{f}_y \tag{2. 2}$$

in $U\cap U'$, where the matrix $(A_{y'}^y)$ is a function in $U\cap U'$. Taking a vector field

3) The indices a, b, c, d, e, f run over the range $\{1, 2, \cdots, n\}$.

$\boldsymbol{v}$ belonging to M and a covector field $\boldsymbol{\phi}$ transversal to L, we have

$$\boldsymbol{v}=v^x\boldsymbol{f}_x, \qquad \boldsymbol{\phi}=\phi_y\boldsymbol{f}^y$$

and

$$\boldsymbol{v}=v^{x'}\boldsymbol{f}_{x'}, \qquad \boldsymbol{\phi}=\phi_{y'}\boldsymbol{f}^{y'}$$

respectively in U and in U', and

$$v^x=A^x_{x'}v^{x'}, \qquad \phi_y=A^{y'}_y\phi_{y'} \tag{2. 3}$$

in $U\cap U'$, where $(A^{x'}_x)=(A^x_{x'})^{-1}$.

Let there be given a connection ω^* in the vector bundle $M(V)$. Then ω^* has $n(n-r)^2$ components $\Gamma_c{}^x{}_y$ with respect to local coordinates (η^a) of the base space V and an $(n-r)$-frame $\{\boldsymbol{f}_x\}$ in any coordinate neighborhood U of V. We take now two coordinate neighborhoods U and U' of V such that $U\cap U'\neq\phi$ and $(n-r)$-frames $\{\boldsymbol{f}_x\}$ and $\{\boldsymbol{f}_{x'}\}$ respectively in U and in U'. Denoting by (η^a) and $(\eta^{a'})$ local coordinates defined respectively in U and U', we have (2. 2) in $U\cap U'$. Then, denoting by $\Gamma_a{}^x{}_y$ and $\Gamma_{a'}{}^{x'}{}_{y'}$ the components of the given connection ω^* respectively with respect to $\{\eta^a, \boldsymbol{f}_x\}$ and to $\{\eta^{a'}, \boldsymbol{f}_{x'}\}$, we find in $U\cap U'$

$$\Gamma_{c'}{}^{x'}{}_{y'}=\frac{\partial\eta^c}{\partial\eta^{c'}}A^{x'}_x(\Gamma_c{}^x{}_yA^y_{y'}+\partial_cA^x_{y'}), \tag{2. 4}$$

where $\partial_c=\partial/\partial\eta^c$. The equation (2. 4) is the so-called transformation law of components of a connection given in the vector bundle $M(V)$.

Taking a vector $\boldsymbol{X}$ and a covector $\boldsymbol{\rho}$ at a point P of the manifold V, we consider an element $\boldsymbol{T}_{\mathrm{P}}(\boldsymbol{X},\boldsymbol{\rho})$ of the fibre F_{P} of the vector bundle $M^*(V)\otimes M(V)$ at P and suppose for $\boldsymbol{T}_{\mathrm{P}}(\boldsymbol{X},\boldsymbol{\rho})$ to be bilinear with respect to its arguments $\boldsymbol{X}$ and $\boldsymbol{\rho}$. The correspondence $\boldsymbol{T}_{\mathrm{P}}$: $(\boldsymbol{X},\boldsymbol{\rho})\rightarrow\boldsymbol{T}_{\mathrm{P}}(\boldsymbol{X},\boldsymbol{\rho})$ is called an *F_{P}-valued tensor* of type (1, 1) at P. If there is given a correspondence $\boldsymbol{T}$: $\mathrm{P}\rightarrow\boldsymbol{T}_{\mathrm{P}}$, it is called an *$M^*(V)\otimes M(V)$-valued tensor field* of type (1, 1) and its differentiability is naturally defined. Let $\boldsymbol{v}$ and $\boldsymbol{\phi}$ be respectively a vector field and a covector field and $\boldsymbol{T}$ an $M^*(V)\otimes M(V)$-valued tensor field of type (1. 1). Denote by $\boldsymbol{T}(\boldsymbol{v},\boldsymbol{\phi})$ a cross-section of $M^*(V)\otimes M(V)$ such that its value at a point P is given by $\boldsymbol{T}_{\mathrm{P}}(\boldsymbol{v}_{\mathrm{P}},\boldsymbol{\phi}_{\mathrm{P}})$, where $\boldsymbol{v}_{\mathrm{P}}$ and $\boldsymbol{\phi}_{\mathrm{P}}$ are respectively the values of $\boldsymbol{v}$ and $\boldsymbol{\phi}$ at P. Then we have

$$\boldsymbol{T}(\sigma\boldsymbol{v},\tau\boldsymbol{\phi})=\sigma\tau\boldsymbol{T}(\boldsymbol{v},\boldsymbol{\phi})$$

for any two functions σ and τ. Let (η^a) and $\{\boldsymbol{f}_x\}$ be respectively local coordinates and an $(n-r)$-frame in a neighborhood U of V. Then in U the cross-section $\boldsymbol{T}(\boldsymbol{X},\boldsymbol{\rho})$ is expressed by

$$\boldsymbol{T}(\boldsymbol{X},\boldsymbol{\rho})=\sum_{b,c=1}^{n}X^c\rho_b\boldsymbol{T}_c{}^b,$$

where

$$\boldsymbol{T}_c{}^b=\sum_{x,y=n+1}^{2n-r}T_c{}^b{}_y{}^x\boldsymbol{f}^y\otimes\boldsymbol{f}_x$$

with functions $T_c{}^b{}_y{}^x$ defined in U and X^a, ρ_b are respectively components of $\boldsymbol{X}$ and $\boldsymbol{\rho}$ with respect to (η^a). $\boldsymbol{T}_c{}^b$ is an $M^*(V)\otimes M(V)$-valued tensor field of type (0, 0), i.e. an $M^*(V)\otimes M(V)$-valued scalar, in the neighborhood U for each pair (b, c) of indices. $\boldsymbol{T}_c{}^b$ or $T_c{}^b{}_y{}^x$ are called the *components* of $\boldsymbol{T}$. In a similar way, we consider tensor fields of any such mixed type.

Taking in the manifold V a vector field $\boldsymbol{v}$ belonging to the distribution M and a covector field $\boldsymbol{\phi}$ transversal to the distribution L, we put

$$\boldsymbol{v}=v^x\boldsymbol{f}_x \quad \text{and} \quad \boldsymbol{\phi}=\phi_y\boldsymbol{f}^y$$

in a coordinate neighborhood U of V. If we put

$$\begin{aligned}\nabla_c v^x&=\partial_c v^x+\Gamma_c{}^x{}_y v^y,\\ \nabla_c\phi_y&=\partial_c\phi_y-\Gamma_c{}^x{}_y\phi_x,\end{aligned} \tag{2. 5}$$

then it is easily verified by means of (2. 3) and (2. 4) that

$$(\nabla_c v^x)\boldsymbol{f}_x \quad \text{and} \quad (\nabla_c\phi_y)\boldsymbol{f}^y$$

are globally defined covariant vector fields in V which take their values respectively in $M(V)$ and in $M^*(V)$. In this sense, we call

$$\nabla_c\boldsymbol{v}=\nabla_c v^x\boldsymbol{f}_x \qquad \text{and} \qquad \nabla_c\boldsymbol{\phi}=\nabla_c\phi_y\boldsymbol{f}^y,$$

or simply $\nabla_c v^x$ and $\nabla_c\phi_y$, respectively the *covariant derivatives* of $\boldsymbol{v}$ and $\boldsymbol{\phi}$ with respect to the connection ω^*.

Let there be given a linear connection ω in the base space V and denote by $\Gamma_c{}^a{}_b$ its components with respect to local coordinates (η^a) in a coordinate neighborhood U of V. If we consider now an $M(V)\otimes M^*(V)$-valued vector field $\boldsymbol{T}^a$, then we can put

$$\boldsymbol{T}^a=T^a{}_y{}^x\boldsymbol{f}_x\otimes\boldsymbol{f}^y$$

in U and $T^a{}_y{}^x$ are components of $\boldsymbol{T}^a$ with respect to $\{\eta^a, \boldsymbol{f}_x\}$ in U. On putting in U

$$\nabla_c T^a{}_y{}^x=\partial_c T^a{}_y{}^x+\Gamma_c{}^a{}_b T^b{}_y{}^x+\Gamma_c{}^x{}_z T^a{}_y{}^z-\Gamma_c{}^z{}_y T^a{}_z{}^x, \tag{2. 6}$$

then the tensor field

$$\nabla_c\boldsymbol{T}^a=(\nabla_c T^a{}_y{}^x)\boldsymbol{f}_x\otimes\boldsymbol{f}^y$$

defined in each neighborhood U determines globally in the manifold V a tensor field of type (1. 1) which takes its values in $M(V)\otimes M^*(V)$. In this sense, we call $\nabla_c\boldsymbol{T}^a$, or simply $\nabla_c T^a{}_y{}^x$, the *covariant derivative* of $\boldsymbol{T}^a$ with respect to connections ω and ω^*. In the same way, we can define the covariant derivatives of tensor fields of any mixed type. Summing up, if there are given connections ω and ω^* respectively in the manifold V and in the vector bundle $M(V)$, we can introduce the covariant differentiation ∇_c operating on tensor fields $T_{b\cdots}{}^{a\cdots}{}_{y\cdots}{}^{x\cdots}$ of any mixed type. In general case, two connections ω and ω^* may be given independently. However, if there is given a linear connection ω in the manifold V, then there exists in the vector bundle $M(V)$ a connection ω^* defined by components

$$\Gamma_c{}^x{}_y=(\partial_c f^e{}_y+\Gamma_c{}^e{}_d f^d{}_y)f_e{}^x,$$

where $\Gamma_c{}^a{}_b$ are the components of the given linear connection ω.

§ 3. Almost complex structure in $M(V)$.

By identifying each tangent space of the fibre of the vector bundle p: $M(V)\to V$ with the fibre itself, the tangent space $T_\sigma(M(V))$ of the manifold $M(V)$ at a point σ of $M(V)$ is expressible as a direct sum by

$$T_\sigma(M(V))=T_{\mathrm{P}}(V)\oplus F_{\mathrm{P}}=L_{\mathrm{P}}\oplus M_{\mathrm{P}}\oplus F_{\mathrm{P}},$$

P being the point $p(\sigma)$ of V, where $T_{\mathrm{P}}(V)$, F_{P}, L_{P} and M_{P} are at the point P respectively the tangent space of V, the fibre of $M(V)$, the tangent plane belonging to L and the tangent plane belonging to M. There exists naturally an identification j: $M_{\mathrm{P}}\to F_{\mathrm{P}}$.

Let there be given a connection ω^* in the vector bundle $M(V)$. Taking a tangent vector $\boldsymbol{X}$ of the base space V at P, we denote by $\boldsymbol{X}^L$ the horizontal lift of $\boldsymbol{X}$ at each point σ of the fibre $p^{-1}(\mathrm{P})$ with respect to the connection ω^*. We define a linear operator $\boldsymbol{J}_\sigma$ applied to the tangent space $T_\sigma(M(V))$ of the manifold $M(V)$ at a point σ by $\boldsymbol{J}_\sigma(\boldsymbol{X}^L)=(\boldsymbol{f}\boldsymbol{X})^L$, $\boldsymbol{J}_\sigma(\boldsymbol{Y}^L)=j(\boldsymbol{Y})$, $\boldsymbol{J}_\sigma(\boldsymbol{Z})=-(j^{-1}(\boldsymbol{Z}))^L$, where $\boldsymbol{X}$, $\boldsymbol{Y}$ and $\boldsymbol{Z}$ belong respectively to L_{P}, M_{P} and F_{P}, P being the point $p(\sigma)$. It is easily verified that the operators $\boldsymbol{J}_\sigma$ defined in each tangent space $T_\sigma(M(V))$ determine an almost complex structure $\boldsymbol{F}$ in the manifold $M(V)$, i.e. that $\boldsymbol{F}^2=-\boldsymbol{I}$, $\boldsymbol{I}$ being the unit operator.

We shall now obtain the tensor representation $F_i{}^h$ of the almost complex structure $\boldsymbol{F}$. Let $\{U\}$ be an open covering of the base space V. The fibre space of $M(V)$ being the $(n-r)$-dimensional vector space R^{n-r}, the collection $p^{-1}(U)=U\times R^{n-r}$ of local product representation of the bundle $M(V)$ over U's forms an open covering of $M(V)$. In a local product representation $p^{-1}(U)=U\times R^{n-r}$, any element $\boldsymbol{v}$ of $M(V)$ such that $p(\boldsymbol{v})\in U$ is expressed by (η^a, v^x), where (η^a) are coordinates of the point $p(\boldsymbol{v})$ and $\boldsymbol{v}=v^x\boldsymbol{f}_x$, $\{\boldsymbol{f}_x\}$ being an $(n-r)$-frame in U. Any tangent vector of the bundle space $M(V)$ is expressed by $\begin{pmatrix}V^a\\V^x\end{pmatrix}$, if the tangent spaces of R^{n-r} are identified with R^{n-r} itself. That is to say, (η^a, v^x) are local coordinates defined in each neighborhood $p^{-1}(U)=U\times R^{n-r}$ of the bundle space $M(V)$.

Let there be given a linear connection ω^* in the vector bundle $M(V)$ and $\Gamma_c{}^x{}_y$ its components with respect to local coordinates (η^a) and an $(n-r)$-frame $\{\boldsymbol{f}_x\}$ in a neighborhood U of the manifold V. Then, in the tangent space of the bundle space $M(V)$ at any point (η^a, v^x) of $p^{-1}(U)=U\times R^{n-r}$, the horizontal plane is defined by a linear equation

$$V^x+\Gamma_a{}^x V^a=0, \tag{3.1}$$

$\Gamma_a{}^x$ being defined by

$$\Gamma_a{}^x=\Gamma_a{}^x{}_y v^y, \tag{3.2}$$

and the vertical plane is defined by a linear equation

$$V^a=0.$$

If, in each tangent space of the bundle space $M(V)$, we consider a frame consisting of $2n-r$ vectors $\boldsymbol{V}_{(i)}$ with components $V^h_{(i)}$ such that[4)]

$$(V^h_{(b)})=\begin{pmatrix} V^a_{(b)} \\ V^x_{(b)} \end{pmatrix}=\begin{pmatrix} \delta^a_b \\ -\Gamma_b{}^x \end{pmatrix}, \quad (V^h_{(y)})=\begin{pmatrix} V^a_{(y)} \\ V^x_{(y)} \end{pmatrix}=\begin{pmatrix} 0 \\ \delta^x_y \end{pmatrix},$$

then $\boldsymbol{V}_{(b)}$ are horizontal by virtue of (3. 1) and $\boldsymbol{V}_{(y)}$ are vertical. We now define in each tangent space of the bundle space $M(V)$ a linear operator $\boldsymbol{F}$ by

$$\text{(3. 3)} \qquad \begin{aligned} \boldsymbol{F}(\boldsymbol{V}_{(b)})&=\sum_{a=1}^{n} f_b{}^a \boldsymbol{V}_{(a)}+\sum_{x=n+1}^{2n-r} f_b{}^x \boldsymbol{V}_{(x)}, \\ \boldsymbol{F}(\boldsymbol{V}_{(y)})&=-\sum_{a=1}^{n} f^a{}_y \boldsymbol{V}_{(a)}. \end{aligned}$$

Then, the linear operators $\boldsymbol{F}$ thus defined determine a tensor field of type (1, 1) in the bundle space $M(V)$. If we denote by $F_i{}^h$ the components of the tensor field $\boldsymbol{F}$ with respect to local coordinates (η^a, v^x) defined in neighborhood $p^{-1}(U)=U\times R^{n-r}$, then we find from (3. 3)

$$\text{(3. 4)} \qquad (F_i{}^h)=\begin{pmatrix} \delta^a_b & 0 \\ -\Gamma_b{}^x & \delta^x_y \end{pmatrix}\begin{pmatrix} f_b{}^a & -f_y{}^a \\ f_b{}^x & 0 \end{pmatrix}\begin{pmatrix} \delta^a_b & 0 \\ -\Gamma_b{}^x & \delta^x_y \end{pmatrix}^{-1},$$

i.e.

$$\text{(3. 5)} \qquad (F_i{}^h)=\begin{pmatrix} f_b{}^a-\Gamma_b{}^y f^a{}_y & -f^a{}_y \\ f_b{}^x-f_b{}^e\Gamma_e{}^x+\Gamma_b{}^z f^e{}_z\Gamma_e{}^x & f^e{}_y\Gamma_e{}^x \end{pmatrix}.$$

Taking account of (1. 6), we can easily verify

$$\begin{pmatrix} f_b{}^a & -f_y{}^a \\ f_b{}^x & 0 \end{pmatrix}^2=-\begin{pmatrix} \delta^a_b & 0 \\ 0 & \delta^x_y \end{pmatrix}.$$

Therefore, by virtue of (3. 4) we find

$$\boldsymbol{F}^2=-\boldsymbol{I},$$

$\boldsymbol{I}$ being the unit tensor. Consequently, the tensor field $\boldsymbol{F}$ thus defined is an almost complex structure in the bundle space $M(V)$. Summing up, we have

THEOREM 1. *If a differentiable manifold V admits an f-structure* $\boldsymbol{f}$ *of rank r, then there exist almost complex structures in the bundle space of the $(n-r)$-dimensional vector bundle $M(V)$ over V. Given a connection with components $\Gamma_c{}^x{}_y$ in $M(V)$, then an almost complex structure* $\boldsymbol{F}=(F_i{}^h)$ *is determined by* (3. 5).

This seems to be a generalization of the following theorem: *There exists almost*

4) The indices h, i, j, k, l run over the range $\{1, 2, \cdots, n, n+1, \cdots, 2n-r\}$.

complex structures in the tangent bundle $T(V)$ of any differentiable manifold V and an almost complex structures is determined by giving a connection in $T(M)$. (Cf. Tachibana and Okumura [5], for example.)

§ 4. Nijenhuis tensor of F.

Let there be given an f-structure $\boldsymbol{f}$ of rank r in a differentiable manifold V and a connection ω^* in the vector bundle $M(V)$. Let $\Gamma_c{}^x{}_y$ be the components of the connection ω^* with respect to $\{\eta^a, \boldsymbol{f}_x\}$, where (η^a) are local coordinates and $\{\boldsymbol{f}_x\}$ is an $(n-r)$-frame in a neighborhood U of the manifold V. Denoting by $f_b{}^a$ the components of the f-structure $\boldsymbol{f}$ with respect to (η^a), we define in U a tensor field $S_{cb}{}^a$ of type (1. 2) by

$$S_{cb}{}^a=N_{cb}{}^a+(\partial_c f_b{}^z-\partial_b f_c{}^z)f_z{}^a-(f_c{}^u\Gamma_b{}^z{}_u-f_b{}^u\Gamma_c{}^z{}_u)f_z{}^a, \tag{4. 1}$$

$(f^a{}_x)$ and $(f_b{}^y)$ being respectively the components of $\boldsymbol{f}_x$ and $\boldsymbol{f}^y$, where the Nijenhuis tensor $N_{cb}{}^a$ of the f-structure $f_b{}^a$ is by definition

$$N_{cb}{}^a=f_c{}^e\partial_e f_b{}^a-f_b{}^e\partial_e f_c{}^a-(\partial_c f_b{}^e-\partial_b f_c{}^e)f_e{}^a.$$

We define next in U an $M(V)$-valued tensor field

$$\boldsymbol{S}_{cb}=S_{cb}{}^x\boldsymbol{f}_x$$

of type (0, 2) by

$$S_{cb}{}^x=f_c{}^e(\partial_e f_b{}^x-\partial_b f_e{}^x)-f_b{}^e(\partial_e f_c{}^x-\partial_c f_e{}^x)-(f_c{}^z f_b{}^e-f_b{}^z f_c{}^e)\Gamma_e{}^x{}_z, \tag{4. 2}$$

an $M^*(V)$-valued tensor field

$$\boldsymbol{S}_c{}^a=S_c{}^a{}_y\boldsymbol{f}^y$$

of type (1, 1) by

$$S_c{}^a{}_y=f^e{}_y\partial_e f_c{}^a-f_c{}^e\partial_e f^a{}_y+f_e{}^a\partial_c f^e{}_y+f_c{}^e f^a{}_z\Gamma_e{}^z{}_y, \tag{4. 3}$$

an $M(V)\otimes M^*(V)$-valued tensor field

$$\boldsymbol{S}_c=S_{cy}{}^x\boldsymbol{f}_x\otimes\boldsymbol{f}^y$$

of type (0, 1) by

$$S_{cy}{}^x=f^e{}_y(\partial_e f_c{}^x-\partial_c f_e{}^x)+f_c{}^z f^e{}_y\Gamma_e{}^x{}_z-\Gamma_c{}^x{}_y \tag{4. 4}$$

and an $M^*(V)\otimes M^*(V)$-valued tensor field

$$\boldsymbol{S}^a=S_{xy}{}^a\boldsymbol{f}^x\otimes\boldsymbol{f}^y$$

of type (1, 0) by

$$S^a{}_{xy}=f_x{}^e\partial_e f^a{}_y-f_y{}^e\partial_e f_x{}^a-(f_x{}^e\Gamma_e{}^z{}_y-f_y{}^e\Gamma_e{}^z{}_x)f^a{}_z. \tag{4. 5}$$

They have the following properties:

$$S_{cb}{}^a+S_{bc}{}^a=0,\qquad S_{cb}{}^x+S_{bc}{}^x=0,\qquad S^a{}_{xy}+S^a{}_{yx}=0.$$

The tensor fields $S_{cb}{}^a$, $\boldsymbol{S}_{cb}$, $\boldsymbol{S}_c{}^a$, $\boldsymbol{S}_c$ and $\boldsymbol{S}^a$, defined above in each neighborhood

U, determine globally tensor field in the manifoid V, respectively. In fact, taking arbitrarily a symmetric linear connection $\Gamma_c{}^a{}_b$ in the manifold V, we can easily verify that these tensor fields are expressed respectively as following:

$$
\begin{aligned}
&S_{cb}{}^a = N_{cb}{}^a + (\nabla_c f_b{}^z - \nabla_b f_c{}^z) f_z{}^a,\\
&S_{cb}{}^x = f_c{}^e(\nabla_e f_b{}^x - \nabla_b f_e{}^x) - f_b{}^e(\nabla_e f_c{}^x - \nabla_c f_e{}^x),\\
&S_c{}^a{}_y = f^e{}_y \nabla_e f_c{}^a - f_c{}^e \nabla_e f^a{}_y + f_e{}^a \nabla_c f^e{}_y,\\
&S_{cy}{}^x = f^e{}_y(\nabla_e f_c{}^x - \nabla_c f_e{}^x),\\
&S^a{}_{xy} = f_x{}^e \nabla_e f^a{}_y - f_y{}^e \nabla_e f_x{}^a
\end{aligned}
\tag{4.6}
$$

by making use of covariant differentiation ∇_c introduced by (2. 6). Thus these local tensor fields determine global tensor fields in the manifold V, respectively. These tensors S's are seemed to be generalizations of those introduced in [4].

As we have known in Theorem 1, there exists for a given linear connection an almost complex structure $\boldsymbol{F}$ in the bundle space $M(V)$ and its components $F_i{}^h$ is given by (3. 5). The Nijenhuis tensor $H_{ji}{}^h$ of the almost complex structure $\boldsymbol{F}$ is by definition

$$
H_{ji}{}^h = F_j{}^l \partial_l F_i{}^h - F_i{}^l \partial_l F_j{}^h - (\partial_j F_i{}^l - \partial_i F_j{}^l) F_l{}^h
\tag{4.7}
$$

with respect to local coordinates (η^a, v^x) in each neighborhood $p^{-1}(U) = U \times R^{n-r}$ of the bundle space $M(V)$, where we have put $\eta^x = v^x$ and $\partial_i = \partial/\partial\eta^i$. If we substitute (3. 5) in the right-hand side of (4. 7), taking account of (3. 2), we find

$$
\begin{aligned}
H_{cb}{}^a =\ & S_{cb}{}^a - (\Gamma_c{}^z S_b{}^a{}_z - \Gamma_b{}^z S_c{}^a{}_z) + \Gamma_c{}^z \Gamma_b{}^u S_{zu}{}^a\\
& - (f_c{}^e R_{eb}{}^z - f_b{}^e R_{ec}{}^z) f^a{}_z + (\Gamma_c{}^z f^e{}_z R_{eb}{}^u - \Gamma_b{}^z f^e{}_z R_{ec}{}^u) f^a{}_u,\\
H_{cb}{}^x =\ & S_{cb}{}^x - (\Gamma_c{}^z S_{bz}{}^x - \Gamma_b{}^z S_{cz}{}^x) - S_{cb}{}^e \Gamma_e{}^x\\
& + (\Gamma_c{}^z S_b{}^e{}_z - \Gamma_b{}^z S_c{}^e{}_z) \Gamma_e{}^x - \Gamma_c{}^z \Gamma_b{}^y S_{zy}{}^e \Gamma_e{}^x\\
& + (R_{cb}{}^x - f_c{}^e f_b{}^d R_{ed}{}^x) + (\Gamma_c{}^z f^e{}_z f_b{}^d - \Gamma_b{}^z f^e{}_z f_c{}^d) R_{ed}{}^x - \Gamma_c{}^z f^e{}_z \Gamma_b{}^u f^d{}_u R_{ed}{}^x\\
& + (f_c{}^e R_{eb}{}^z - f_b{}^e R_{ec}{}^z) f^d{}_z \Gamma_d{}^x - (\Gamma_c{}^z f^e{}_z R_{eb}{}^u - \Gamma_b{}^z f^e{}_z R_{ec}{}^u) f^e{}_u \Gamma_d{}^x,\\
H_{cy}{}^a =\ & S_c{}^a{}_y + \Gamma_c{}^z S_{zy}{}^a + R_{ce}{}^z f^e{}_y f^a{}_z,\\
H_{cy}{}^x =\ & S_{cy}{}^x - S_c{}^e{}_y \Gamma_e{}^x + \Gamma_c{}^z S_{yz}{}^e \Gamma_e{}^x\\
& + f_c{}^e f_y{}^d R_{ed}{}^x + \Gamma_c{}^z f_z{}^d f^e{}_y R_{ed}{}^x + f^e{}_y R_{ec}{}^z f_z{}^d \Gamma_d{}^x,\\
H_{zy}{}^a =\ & S^a{}_{zy},\\
H_{zy}{}^x =\ & -S^e{}_{zy} \Gamma_e{}^x - f^e{}_z f^d{}_y R_{ed}{}^x,
\end{aligned}
\tag{4.8}
$$

where

$$
R_{cb}{}^x = R_{cby}{}^x v^y
\tag{4.9}
$$

and $R_{cby}{}^x$ is the curvature tensor of the linear connection $\Gamma_c{}^x{}_y$, $R_{cby}{}^x$ being defined by

$$R_{cby}{}^{x}=\partial_c\Gamma_b{}^x{}_y-\partial_b\Gamma_c{}^x{}_y+\Gamma_c{}^x{}_z\Gamma_b{}^z{}_y-\Gamma_b{}^x{}_z\Gamma_c{}^z{}_y.$$

Generally, the Nijenhuis tensor $H_{ji}{}^h$ of an almost complex structure $F_i{}^h$ is hybrid with respect to the indices (h, i) and pure with respect to the indices (j, i), that is,

$$H_{jl}{}^hF_i{}^l+H_{ji}{}^kF_k{}^h=0,\qquad H_{jl}{}^hF_i{}^l-H_{li}{}^hF_j{}^l=0. \tag{4.10}$$

Taking account of (3. 5), (4. 8) and (4. 9), we see that the left-hand sides of the equations (4. 10) are regarded as polynomials with respect to the variables v^x. Then, putting the terms independent of v^x equal to zero in each equation of (4. 10), we get equations containing $S_{cb}{}^a$, $S_{cb}{}^x$, $S_c{}^a{}_y$, $S_{cy}{}^x$, $S^a{}_{zy}$, from which we find

$$\begin{aligned}
&S_{cb}{}^x=S_{ce}{}^af^e{}_bf_a{}^x-S_{ey}{}^xf_c{}^ef_b{}^y,\\
&S_b{}^a{}_y=S_{de}{}^af_b{}^df^e{}_y-S^e{}_{zy}f_b{}^zf_e{}^a,\\
&S_{cy}{}^x=S_{de}{}^xf_c{}^df^e{}_y+S^e{}_{zy}f_c{}^zf_e{}^x,\\
&S^a{}_{zy}=-S_{cb}{}^af_z{}^cf_y{}^b.
\end{aligned} \tag{4.11}$$

Identities (4. 11) show that

$$S_{cb}{}^a=0 \quad\text{implies}\quad S_{cb}{}^x=0,\quad S_c{}^a{}_y=0,\quad S_{cy}{}^x=0,\quad S^a{}_{zy}=0.$$

(Cf. Nakagawa [3], Yano and Ishihara [8].)

§ 5. Normal *f*-structure.

When the almost complex structure $\boldsymbol{F}$ defined by (3. 5) is complex in the bundle space $M(V)$, we say that the given f-structure $\boldsymbol{f}$ is *normal* with respect to a connection ω^* given in the vector bundle $M(V)$. Thus a necessary and sufficient condition for the f-structure $\boldsymbol{f}$ to be normal with respect to ω^*, as is well known, the Nijenhuis tensor $H_{ji}{}^h$ of $\boldsymbol{F}$ given by (4. 8) vanishes identically in the bundle space $M(V)$.

The equations obtained by putting each components $H_{ji}{}^h$ to be equal to zero are equations with respect to the variables v^x by virtue of (4. 8). Therefore, the condition $H_{ji}{}^h=0$ implies

$$\begin{aligned}
&S_{cb}{}^a=0,\qquad S_{cb}{}^x=0,\\
&S_{cy}{}^a=0,\qquad S_{cy}{}^x=0,\qquad S^a{}_{zy}=0,
\end{aligned} \tag{5.1}$$

because these S's contain none of v^x. Substituting (5. 1) in the equations $H_{ji}{}^h=0$, we have linear equations with respect to $R_{cb}{}^x$, which imply

$$R_{cb}{}^x=0,$$

i.e.

$$R_{cby}{}^x=0 \tag{5.2}$$

as a consequence of $R_{cb}{}^x=R_{cby}{}^xv^y$. Conversely, if we assume the conditions (5. 1) and (5. 2), we have evidently $H_{ji}{}^{h}=0$. Consequently, a necessary and sufficient condition for $H_{ji}{}^{h}$ to vanish identically is that (5. 1) and (5. 2) are valid.

Summing up, we have

PROPOSITION 1. *A necessary and sufficient condition for an f-structuae* $\boldsymbol{f}$ *to be normal in a differentiable manifold* V *with respect to a connection* ω^* *given in the vector bundle* $M(V)$ *is that the tensor fields*

$$S_{cb}{}^{a},\ S_{cb}{}^{x},\ S_{c}{}^{a}{}_{y},\ S_{cy}{}^{x},\ S^{a}{}_{zy}$$

vanish identically and the connection ω^* *is of zero curvature.*

Taking account of the identities (4. 11), we see that, if the tensor field $S_{cb}{}^{a}$ vanishes identically, the other tensor fields S's, i.e. $S_{cb}{}^{x}$, $S_{c}{}^{a}{}_{y}$, $S_{cy}{}^{x}$, $S^{a}{}_{zy}$ are equal identically to zero. Thus, by virtue of Proposition 1 we have

THEOREM 2. *A necessary and sufficient condition for an f-structure* $\boldsymbol{f}$ *to be normal in a differentiable manifold* V *with respect to a connection* ω^* *given in the vector bundle* $M(V)$ *is that the tensor field* $S_{cb}{}^{a}$ *vanishes identically and the connection* ω^* *is of zero curvature.*

If an f-structure in a manifold V is normal with respect to a connection given in the vector bundle $M(V)$, then the connection has zero curvature by means of Theorem 2. Thus, V assumed to be simply connected, the vector bundle $M(V)$ is trivial, i.e. it is a product bundle. Therefore, we have

PROPOSITION 2. *If an f-structure in a manifold* V *is normal with respect to a connection given in the vector bundle* $M(V)$, *then the vector bundle* $M(\tilde{V})$ *induced from* $M(V)$ *by the covering projection* π: $\tilde{V}\rightarrow V$ *is trivial, where* $\tilde{V}$ *is the universal covering space of* V.

Let there be given an f-structure $\boldsymbol{f}$ of rank r in a manifold V. If the vector bundle $M(V)$ is trivial, then it is naturally identified with the product space $V\times R^{n-r}$. Then there exists naturally a connection ω_0^* of zero curvature in the vector bundle $M(V)=V\times R^{n-r}$, that is, all of the components of ω_0^* vanish with respect to each local coordinates (η^a, v^x) of $V\times R^{n-r}$, (η^a) being local coordinates in V and (v^x) being Cartesian coordinates in R^{n-r}. We assume that the f-structure $\boldsymbol{f}$ is normal with respect to ω_0^*. If this is the case, the tensor fields S's are given by (4. 1), (4. 2), (4. 3), (4. 4) and (4. 5) with vanishing $\Gamma_{c}{}^{z}{}_{y}$, respectively and each of S's is a tensor field of corresponding type in the manifold V for fixed indices x, y and z. By means of Theorem 2, in this case, *a necessary and sufficient condition for* $\boldsymbol{f}$ *to be normal with respect to* ω_0^* *is*

$$S_{cb}{}^{a}=N_{cb}{}^{a}+(\partial_c f_b{}^z-\partial_b f_c{}^z)f_z{}^a=0.$$

(Cf. Nakagawa [3].)

Let W be the subset $\{z=x+\sqrt{-1}y\mid |x|\leqq 1\}$ in the plane of all complex numbers $z=x+\sqrt{-1}y$ and R the set of all real numbers t. On putting

$$\Sigma=W\times R=\{(z,t)\mid |x|\leqq 1,\ -\infty<t<\infty\},$$

if we identify each two points $(1+\sqrt{-1}y, t)$ and $(-1+\sqrt{-1}y, -t)$ of Σ then we obtain a new manifold $\tilde{\Sigma}$ of three dimensions. The manifold $\tilde{\Sigma}$ admits an f-structure $\tilde{\boldsymbol{f}}$ of rank 2. In fact, there exists in Σ an f-structure $\boldsymbol{f}$ defined by components

$$(f_b{}^a)=\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with respect to coordinates (x, y, t). It is easily seen that the f-structure $\boldsymbol{f}$ induces naturally an f-structure $\tilde{\boldsymbol{f}}$ in $\tilde{\Sigma}$. On the other hand, there exists in the vector bundle $M(\tilde{\Sigma})$ a connection $\tilde{\omega}^*$ of zero curvature, whose components are all zero with respect to local coordinates (x, y, t). f-structure $\tilde{\boldsymbol{f}}$ is trivially normal with respect to the connection $\tilde{\omega}^*$. However, the f-structure $\tilde{\boldsymbol{f}}$ in $\tilde{\Sigma}$ is not almost contact, because $\tilde{\Sigma}$ is not orientable.

BIBLIOGRAPHY

[1] DOMBROWSKI, P., On the geometry of the tangent bundle. J. Reine u. Angew. Math. **210** (1962), 73–88.

[2] ISHIHARA, S., AND K. YANO, On integrability of a structure f satisfying $f^3+f=0$. Quart. J. Math. Oxford (2), **15** (1964), 217–222.

[3] NAKAGAWA, H., f-structures induced on submanifolds in spaces, almost Hermitian or Kaehlerian. To appear in Kōdai Math. Sem. Rep. **18** (1966).

[4] SASAKI, S., AND Y. HATAKEYAMA, On differentiable manifolds with certain structure which are closely related to almost contact structure, II. Tôhoku Math. J. **13** (1961), 281–294.

[5] TACHIBANA, S., AND M. OKUMURA, On the almost-complex structure of tangent bundles of Riemannian spaces. Tôhoku Math. J. **14** (1962), 156–161.

[6] YANO, K., On structure f satisfying $f^3+f=0$. Tech. Rep., No. 2, June 20 (1961), Univ. of Washington.

[7] ———, On a structure defined by a tensor field f of type (1, 1) satisfying $f^3+f=0$. Tensor, N. S., **14** (1963), 99–109.

[8] YANO, K., AND S. ISHIHARA, The f-structure induced on submanifolds of complex and almost complex spaces. To appear in Kōdai Math. Sem. Rep. **18** (1966).

[9] YANO, K., AND A. J. LEDGER, Linear connections of tangent bundles. J. London Math. Soc. **39** (1964), 495–500.

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