ON A CHARACTERIZATION OF REGULARLY BRANCHED THREE-SHEETED COVERING RIEMANN SURFACES

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§ **1.** Let R be an open Riemann surface. Let $\mathfrak{M}(R)$ be a family of non-constant meromorphic functions on R. Let f be a member of $\mathfrak{M}(R)$. Let $P(f)$ be the number of Picard's exceptional values of f, where we say α a Picard's exceptional value of f when α is not taken by f on R. Let $P(R)$ be a quantity defined by

$$\sup_{f\in\mathfrak{M}(R)} P(f).$$

When R is open, we have always $P(R)\geqq 2$, since there exists a non-constant regular function on R by the existence theorem due to Behnke-Stein and then it suffices to compose it to the exponential function.

Ozawa [2] gave the following criterion of non-existence of analytic mapping between two Riemann surfaces:

If $P(R)<P(S)$, then there is no analytic mapping from R into S.

In general it is very difficult to calculate $P(R)$ of a given open Riemann surface.

Let R be an ultrahyperelliptic surface, which is a proper existence domain of a two-valued function $\sqrt{g(z)}$ with an entire function $g(z)$ of z whose zeros are all simple and are infinite in number. Then by Selberg's generalization [5] of Nevanlinna's theory we have $P(R)\leqq 4$. Ozawa [2, 3] gave a characterization of R with $P(R)=4$, an example of R with $P(R)=3$ and several other interesting results.

We shall confine ourselves to the following Riemann surfaces:

Let R be a regularly branched three-sheeted covering Riemann surface, which is a proper existence domain of the three-valued algebroid function $\sqrt[3]{g(z)}$ with an entire function $g(z)$ of z whose zeros are all simple or double and are infinite in number. Then by Selberg's theory [5] we have $P(R)\leqq 6$. The existence of the surface with $P(R)=6$ is evident.

In the present paper we shall prove the following theorems:

THEOREM 1. *If $P(R)=6$, then there exist entire functions $f(z)$, $H(z)$ of z such that*

$$f(z)^3 g(z)=(e^{H(z)}-\gamma)(e^{H(z)}-\delta)^2, \qquad \gamma\neq\delta, \quad \gamma\delta\neq 0, \tag{1.1}$$

where $H(z)$ is a non-constant function with $H(0)=0$ and γ and δ are constants. The converse is also true.

THEOREM 2. *There is no regularly branched three-sheeted covering Riemann*

Received April 19, 1965.

surface with $P(R)=5$.

§ **2.** In order to prove the theorems it is necessary to establish a representation of regular functions on R. Let $g(z)$ be an entire function of z whose zeros are all simple or double and are infinite in number. Let R be a regularly branched three-sheeted covering Riemann surface formed by elements $p=(z, y)$ for each z, y which satisfy the equation

$$y^3=g(z). \tag{2.1}$$

Let f be a three-valued entire algebroid function of z, which is one-valued and regular on R, and let its defining equation be

$$F(z,f)\equiv f^3-S_1(z)f^2+S_2(z)f-S_3(z)=0, \tag{2.2}$$

where $S_1(z)$, $S_2(z)$ and $S_3(z)$ are entire functions of z. Then there exist two entire functions $f_1(z)$, $f_2(z)$ of z and an analytic function $f_3(z)$ being one-valued regular with the exception of all the double zeros of $g(z)$ at which $f_3(z)$ has simple poles and satisfying the following relations:

$$\begin{cases} S_1(z)=3f_1(z), \\ S_2(z)=3f_1(z)^2-3f_2(z)f_3(z)g(z), \\ S_3(z)=f_1(z)^3+f_2(z)^3g(z)+f_3(z)^3g(z)^2-3f_1(z)f_2(z)f_3(z)g(z). \end{cases} \tag{2.3}$$

Now we shall show the relation (2. 3). Let $\omega\neq 1$ be a cubic root of 1. We put $p_1=(z, y)$, $p_2=(z, \omega y)$ and $p_3=(z, \omega^2 y)$, and define the functions $f_1(z)$, $f_2(z)$ and $f_3(z)$ by

$$\begin{cases} f_1(z)\equiv\dfrac{1}{3}(f(p_1)+f(p_2)+f(p_3)), \\ f_2(z)\equiv\dfrac{1}{3y}(f(p_1)+\omega^2 f(p_2)+\omega f(p_3)), \\ f_3(z)\equiv\dfrac{1}{3y^2}(f(p_1)+\omega f(p_2)+\omega^2 f(p_3)). \end{cases} \tag{2.4}$$

In general f_1, f_2 and f_3 above defined are eventually multi-valued functions of z and their multi-valuedness might occur when z moves around a certain zero point of $g(z)$. We introduce a suitable local parameter around a branch point of R which lies over a zero point of $g(z)$ and expand f in its neighborhood with respect to the above local parameter. Then we can see that f_1, f_2 and f_3 are one-valued functions having the desired properties. Therefore putting $p=(z, y)$ we have the following representation of all the regular functions on R:

$$f(p)=f_1(z)+f_2(z)y+f_3(z)y^2. \tag{2.5}$$

Conversely $f(p)$ defined by (2. 5) with f_1, f_2 and f_3 having the described properties is clearly a regular function on R. From (2.5) we have

$$f_1-f+f_2y+f_3y^2=0,$$

$$f_3g+(f_1-f)y+f_2y^2=0,$$

$$f_2g+f_3gy+(f_1-f)y^2=0.$$

Eliminating y and y^2 we have

$$f^3-3f_1f^2+(3f_1^2-3f_2f_3g)f-(f_1^3+f_2^3g+f_3^3g^2-3f_1f_2f_3g)=0.$$

Comparing this with the equation (2.2) we obtain the desired relations (2.3).

Let $D(z)$ be the discriminant of the cubic equation (2.2). Then from (2.4) we have

$$D(z)=-27g(z)^2(f_2(z)^3-f_3(z)^3g(z))^2, \tag{2.6}$$

and from (2.2)

$$D(z)=-4S_1(z)^3S_3(z)+S_1(z)^2S_2(z)^2+18S_1(z)S_2(z)S_3(z)-27S_3(z)^2-4S_2(z)^3. \tag{2.7}$$

Eliminating f_1 and f_2 or f_1 and f_3 from (2.3) we see that $f_3^3g^2$ and f_2^3g are two roots of a quadratic equation

$$X^2-\left(S_3+\frac{2}{27}S_1^3-\frac{1}{3}S_1S_2\right)X+\frac{1}{27}\left(\frac{1}{3}S_1^2-S_2\right)^3=0. \tag{2.8}$$

Let $D_1(z)$ be the discriminant of the quadratic equation (2.8). Then from (2.7) we have

$$D_1(z)=-\frac{1}{27}D(z). \tag{2.9}$$

§ **3.** Now we shall prove theorem 1 in § 3 and § 4. Let R be a three-sheeted covering Riemann surface defined by the equation (2.1) and suppose that $P(R)=6$. Then there exists a moromorphic function $f\in\mathfrak{M}(R)$ with $P(f)=6$. Further we may assume that six Picard's exceptional values of f are 0, a_1, a_2, a_3, a_4 and ∞. Then f becomes a three-valued entire algebroid function of z which is regular on R and satisfies (2.2) and (2.3). By Rémoundos' method of proof of his celebrated generalization of Picard's theorem [4] pp. 25–27, it is sufficient to consider the following two cases:

$$\text{(i)}\quad \begin{pmatrix} F(z,0) \\ F(z,a_1) \\ F(z,a_2) \\ F(z,a_3) \\ F(z,a_4) \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \beta_1e^{H_1} \\ \beta_2e^{H_2} \\ \beta_3e^{H_3} \end{pmatrix}, \qquad \text{(ii)}\quad \begin{pmatrix} \beta_1e^{H_1} \\ c_1 \\ c_2 \\ \beta_2e^{H_2} \\ \beta_3e^{H_3} \end{pmatrix},$$

where c_1, c_2, β_1, β_2 and β_3 are non-zero constants and H_1, H_2 and H_3 are non-constant entire functions of z satisfying $H_1(0)=H_2(0)=H_3(0)=0$.

Case (i). We have

$$\begin{cases} -S_3=c_1, & (1)\\ a_1^3-a_1^2S_1+a_1S_2-S_3=c_2, & (2)\\ a_2^3-a_2^2S_1+a_2S_2-S_3=\beta_1e^{H_1}, & (3)\\ a_3^3-a_3^2S_1+a_3S_2-S_3=\beta_2e^{H_2}, & (4)\\ a_4^3-a_4^2S_1+a_4S_2-S_3=\beta_3e^{H_3}. & (5)\end{cases}$$

Eliminating S_1, S_2 and S_3 from (1), (3), (4) and (5) we have

$$\begin{vmatrix} a_2^2 & a_2 & a_2^3+c_1-\beta_1e^{H_1}\\ a_3^2 & a_3 & a_3^3+c_1-\beta_2e^{H_2}\\ a_4^2 & a_4 & a_4^3+c_1-\beta_3e^{H_3}\end{vmatrix}=0,$$

i.e.

$$a_3a_4(a_3-a_4)\beta_1e^{H_1}-a_2a_4(a_2-a_4)\beta_2e^{H_2}+a_2a_3(a_2-a_3)\beta_3e^{H_3}$$
$$=(c_1+a_2a_3a_4)(a_2-a_3)(a_2-a_4)(a_3-a_4).$$

By the impossibility of Borel's identity [1] we obtain

$$H_1\equiv H_2\equiv H_3\equiv H,\qquad a_3a_4(a_3-a_4)\beta_1-a_2a_4(a_2-a_4)\beta_2+a_2a_3(a_2-a_3)\beta_3=0,$$
$$c_1+a_2a_3a_4=0.$$

From (1), (2), (3) and (4) we have

$$\begin{vmatrix} a_1^2 & a_1 & a_1^3+c_1-c_2\\ a_2^2 & a_2 & a_2^3+c_1-\beta_1e^{H}\\ a_3^2 & a_3 & a_3^3+c_1-\beta_2e^{H}\end{vmatrix}=0,$$

i.e.

$$(-a_1a_3(a_1-a_3)\beta_1+a_1a_2(a_1-a_2)\beta_2)e^H$$
$$=a_2a_3(a_2-a_3)c_2-(c_1+a_1a_2a_3)(a_1-a_2)(a_1-a_3)(a_2-a_3).$$

From this we have

$$-a_1a_3(a_1-a_3)\beta_1+a_1a_2(a_1-a_2)\beta_2=0,$$
$$a_2a_3(a_2-a_3)c_2-(c_1+a_1a_2a_3)(a_1-a_2)(a_1-a_3)(a_2-a_3)=0.$$

Therefore we obtain

$$c_1=-a_2a_3a_4, \qquad c_2=(a_1-a_2)(a_1-a_3)(a_1-a_4),$$

$$\beta_2=\frac{a_3(a_1-a_3)}{a_2(a_1-a_2)}\beta_1, \qquad \beta_3=\frac{a_4(a_1-a_4)}{a_2(a_1-a_2)}\beta_1,$$

and from (1), (2) and (3)

$$(3.1)\qquad \begin{cases} S_1=\dfrac{1}{a_2(a_1-a_2)}\beta_1e^H+a_2+a_3+a_4, \\ S_2=\dfrac{a_1}{a_2(a_1-a_2)}\beta_1e^H+a_2a_3+a_3a_4+a_4a_2, \\ S_3=a_2a_3a_4. \end{cases}$$

Case (ii). Similarly by the impossibility of Borel's identity we obtain

$$H_1\equiv H_2\equiv H_3\equiv H, \qquad c_1=a_1(a_1-a_3)(a_1-a_4), \qquad c_2=a_2(a_2-a_3)(a_2-a_4),$$

$$\beta_2=\frac{(a_1-a_3)(a_2-a_3)}{a_1a_2}\beta_1, \qquad \beta_3=\frac{(a_1-a_4)(a_2-a_4)}{a_1a_2}\beta_1;$$

$$(3.2)\qquad \begin{cases} S_1=-\dfrac{1}{a_1a_2}\beta_1e^H+a_3+a_4, \\ S_2=-\dfrac{a_1+a_2}{a_1a_2}\beta_1e^H+a_3a_4, \\ S_3=-\beta_1e^H. \end{cases}$$

§ **4.** Ozawa [3] proved the following lemma:

LEMMA. *Let $H(z)$ be a non-constant entire function of z. Then the function $e^H-\gamma$, $\gamma\neq 0$ has an infinite number of simple zeros in such a manner that*

$$\varlimsup_{r\to\infty}\frac{N_2(r,0,e^H-\gamma)}{T(r,e^H)}=1,$$

where $T(r,f)$ is the Nevanlinna characteristic function of f and $N_2(r,0,f)$ is the counting function of simple zeros of f.

From now on we shall proceed under the infiniteness of simple zeros of the function $e^H-\gamma$ $(\gamma\neq 0)$ ensured by Ozawa's lemma. We substitute (3.1) and (3.2) into (2.7) respectively, then the coefficient of e^{4H} in $D(z)$ is

$$\frac{a_1^4}{a_2^4(a_1-a_2)^4}\beta_1^4\neq 0$$

if (3. 1) is the case, or

$$\frac{(a_1-a_2)^2}{a_1{}^4 a_2{}^4}\beta_1{}^4 \neq 0$$

if (3. 2) is the case. Hence in both cases we have an equation

$$D(z)=A'(e^H-\gamma_1)(e^H-\delta_1)(e^H-\gamma_1')(e^H-\delta_1'),$$

where A', γ_1, δ_1, γ_1' and δ_1' are all non-zero constants. In fact, we have

$$A'\gamma_1\delta_1\gamma_1'\delta_1'=(a_2-a_3)^2(a_3-a_4)^2(a_4-a_2)^2 \neq 0$$

if (3. 1) is the case, or

$$A'\gamma_1\delta_1\gamma_1'\delta_1'=a_3{}^2 a_4{}^2(a_3-a_4)^2 \neq 0$$

if (3. 2) is the case. From (2. 6) we have

$$-27g^2(f_2{}^3-f_3{}^3g)^2=A'(e^H-\gamma_1)(e^H-\delta_1)(e^H-\gamma_1')(e^H-\delta_1').$$

Since $\gamma_1\delta_1\gamma_1'\delta_1' \neq 0$, by considering simple zero points of the function $e^H-\gamma$ $(\gamma \neq 0)$, we have

(4. 1) $$D(z)=A'(e^H-\gamma_1)^2(e^H-\delta_1)^2, \qquad A'\neq 0,\ \gamma_1\delta_1 \neq 0,\ \gamma_1 \neq \delta_1.$$

We substitute (3. 1) and (3. 2) into the quadratic equation (2. 8) respectively, then remarking (2. 9) and (4. 1), in both cases we have the equations

(4. 2) $$f_2{}^3g=A(e^H-\gamma)(e^H-\delta)(e^H-\eta), \qquad A\neq 0,$$

(4. 3) $$f_3{}^3g^2=A(e^H-\gamma')(e^H-\delta')(e^H-\eta'), \qquad A\neq 0.$$

From (4. 2) we see that γ, δ and η are not zero simultaneously. Hence we may assume that $\gamma \neq 0$.

First, we assume that $\gamma \neq \delta$, $\gamma \neq \eta$. Since a simple zero point z_1 of $e^H-\gamma$ is a simple zero point of the right hand side term of (4. 2), z_1 is a simple zero point of $g(z)$. Hence from (4. 3) we have $\gamma=\gamma'$ or $\gamma=\delta'$ or $\gamma=\eta'$, say $\gamma=\gamma'$. If we put $\gamma \neq \delta'$, $\gamma \neq \eta'$, then z_1 is a simple zero point of the right hand side term of (4. 3), however, z_1 is not a simple zero point of the left hand side term of (4. 3). This is a contradiction. Hence we have $\gamma=\delta'$ or $\gamma=\eta'$, say $\gamma=\eta'$. Then (4. 2) and (4. 3) reduce to

$$f_2{}^3g=A(e^H-\gamma)(e^H-\delta)(e^H-\eta),$$

$$f_3{}^3g^2=A(e^H-\gamma)^2(e^H-\delta'),$$

where of course $\gamma \neq \delta'$. And we have $\delta' \neq 0$. In fact if $\delta'=0$, then we have

$$f_2{}^3g-f_3{}^3g^2=A(e^H-\gamma)((\gamma-\delta-\eta)e^H+\delta\eta).$$

Since $\gamma \neq 0$, from (4. 1) and (2. 6) we have $\delta\eta \neq 0$. By eliminating $g(z)$ we arrive at an absurdity relation

$$f_3{}^3A(e^H-\delta)^2(e^H-\eta)^2=f_2{}^6e^H.$$

Hence we obtain the disired $\delta'\neq 0$.

If $\delta=0$ and $\eta=0$, then we have an absurdity relation

$$f_3{}^3Ae^{4H}=f_2{}^6(e^H-\delta').$$

If $\delta\neq 0$ and $\eta=0$, we have

$$\frac{f_3{}^3}{f_2{}^6}Ae^{2H}(e^H-\delta)^2=e^H-\delta'.$$

The right hand side term has simple zeros, but the left hand side term has no simple zero. This is absured. If $\delta=0$ and $\eta\neq 0$, we similarly have a contradiction. Therefore we obtain $\delta\eta\neq 0$.

Considering the simple zeros of $e^H-\gamma(\gamma\neq 0)$, we can see that $\delta=\eta$ and $\delta=\delta'$. Hence we attain

$$f_2{}^3g=A(e^H-\gamma)(e^H-\delta)^2,$$

$$f_3{}^3g^2=A(e^H-\gamma)^2(e^H-\delta),\qquad A\neq 0,\ \gamma\delta\neq 0,\ \gamma\neq\delta.$$

Next, we may assume that $\gamma=\delta$, $\gamma\neq\eta$, since we have not $\gamma=\delta=\eta$ from (4.2). In this case by considering the simple zeros of the function $e^H-\gamma$ ($\gamma\neq 0$) similarly, we can see that $\eta=\delta'=\eta'\neq 0$ in (4.2) and (4.3). Therefore we also attain the form (4.4). From the above discussion we can conclude the following result:

Let R be a regularly branched three-sheeted covering Riemann surface defined by the equation (2.1). If $P(R)=6$, then there exist entire functions $f(z)$, $H(z)$ of z such that

$$f^3g=(e^H-\gamma)(e^H-\delta)^2,\qquad \gamma\neq\delta,\ \gamma\delta\neq 0, \tag{4.5}$$

where $H(z)$ is a non-constant function with $H(0)=0$ and γ and δ are constants. Conversely if $g(z)$ in the equation (2.1) is defined by (4.5) with $f(z)$ and $H(z)$ having the described properties, then $P(R)=6$.

In fact a function f_0, which is regular on R with $P(f_0)=6$, is given by

$$f_0=-\frac{\omega}{\delta-\gamma}\left[(1-\omega)e^H+\omega\delta-\gamma+(1-\omega)\sqrt[3]{(e^H-\gamma)(e^H-\delta)^2}+\frac{1-\omega}{e^H-\delta}\left(\sqrt[3]{(e^H-\gamma)(e^H-\delta)^2}\right)^2\right],$$

because that f_0 has the form (2.5) and its six Picard's exceptional valules are 0, 1, $-\omega(\sqrt[3]{\gamma/\delta}-\omega)/(\sqrt[3]{\gamma/\delta}-1)$, $-\omega^2(\sqrt[3]{\gamma/\delta}-1)/(\omega\sqrt[3]{\gamma/\delta}-1)$, $-\omega^2(\omega\sqrt[3]{\gamma/\delta}-1)/(\omega^2\sqrt[3]{\gamma/\delta}-1)$ and ∞.

This is our desired characterization of R with $P(R)=6$. Thus we have completely proved our theorem 1.

§5. Now we shall prove theorem 2, that is, there is no three-sheeted covering

Riemann surface defined by the equation (2.1) with $P(R)=5$.

We assume that there exists such a Riemann surface. Then there is a meromorphic function $f\in\mathfrak{M}(R)$ with $P(f)=5$. Further we may assume that its five Picard's exceptional values are 0, a_1, a_2, a_3 and ∞. Then f becomes a three-valued entire algebroid function of z which is regular on R and satisfies (2.2) and (2.3). By Rémoundos' reasoning [4] it is sufficient to consider the following four cases:

$$\text{(i)}\quad \begin{pmatrix} F(z,0) \\ F(z,a_1) \\ F(z,a_2) \\ F(z,a_3) \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \beta_1 e^{H_1} \\ \beta_2 e^{H_2} \end{pmatrix}, \qquad \text{(ii)}\quad \begin{pmatrix} \beta_1 e^{H_1} \\ c_1 \\ c_2 \\ \beta_2 e^{H_2} \end{pmatrix}, \qquad \text{(iii)}\quad \begin{pmatrix} c_1 \\ \beta_1 e^{H_1} \\ \beta_2 e^{H_2} \\ \beta_3 e^{H_3} \end{pmatrix}, \qquad \text{(iv)}\quad \begin{pmatrix} \beta_1 e^{H_1} \\ c_1 \\ \beta_2 e^{H_2} \\ \beta_3 e^{H_3} \end{pmatrix},$$

where c_1, c_2, β_1, β_2 and β_3 are non-zero constants, and H_1, H_2 and H_3 are non-constant entire functions of z with $H_1(0)=H_2(0)=H_3(0)=0$.

Case (i). We have

$$\begin{cases} -S_3=c_1, \\ a_1^3-a_1^2S_1+a_1S_2-S_3=c_2, \\ a_2^3-a_2^2S_1+a_2S_2-S_3=\beta_1 e^{H_1}, \\ a_3^3-a_3^3S_1+a_3S_2-S_3=\beta_2 e^{H_2}. \end{cases}$$

Calculating as similarly as in § 3 and using the impossibility of Borel's identity, we obtain

$$H_1\equiv H_2\equiv H, \qquad (c_1+a_1a_2a_3)(a_1-a_2)(a_1-a_3)-c_2a_2a_3=0, \qquad \beta_2=\frac{a_3(a_1-a_3)}{a_2(a_1-a_2)}\beta_1;$$

$$\text{(5.1)}\qquad \begin{cases} S_1=\dfrac{1}{a_2(a_1-a_2)}\beta_1 e^{H}+A_1, \\ S_2=\dfrac{a_1}{a_2(a_1-a_2)}\beta_1 e^{H}+B_1, \\ S_3=-c_1, \end{cases}$$

where

$$A_1=-\frac{c_1}{a_2a_3}+a_2+a_3, \qquad B_1=-\frac{a_2+a_3}{a_2a_3}c_1+a_2a_3.$$

Then the discriminant $D(z)$ is a polynomial of degree 4 of e^H. If the constant term[1] of $D(z)$ is not zero, then the same reasoning in § 4 holds and we can conclude the

1) Here we say "constant term" when we take $D(z)$ for a polynomial of e^H. From now on we use the term "constant term" in this sense.

existence of a function $f \in \mathfrak{M}(R)$ with $P(f)=6$. This is absurd. Hence the constant term of $D(z)$ is zero. Then if the constant term of

$$S(z) \equiv -\left(S_3(z)+\frac{2}{27}S_1(z)^3-\frac{1}{3}S_1(z)S_2(z)\right)$$

in (2.8) is not zero, then $\gamma\delta\eta \neq 0$, $\gamma'\delta'\eta' \neq 0$ in (4.2) and (4.3), and again we can say that there exists a function $f \in \mathfrak{M}(R)$ with $P(f)=6$, which is absurd. Hence the constant term of $S(z)$ is also zero. Since the constant term of $D(z)$ is zero, we have

$$-4A_1^3S_3+A_1^2B_1^2+18A_1B_1S_3-27S_3^2-4B_1^3=0.$$

Since the constant term of $S(z)$ is zero, we have

$$S_3+(2/27)A_1^3-(1/3)A_1B_1=0.$$

From these we obtain $B_1=A_1^2/3$ and $S_3=A_1^3/27$. Substituting these and (5.1) into (2.2), we have

$$\begin{aligned} F(z,f)&=f^3-\left(\frac{\beta_1 e^H}{a_2(a_1-a_2)}+A_1\right)f^2+\left(\frac{a_1\beta_1 e^H}{a_2(a_1-a_2)}+\frac{A_1^2}{3}\right)f-\frac{A_1^3}{27} \\ &=\left(f-\frac{A_1}{3}\right)^3-\frac{\beta_1 e^H}{a_2(a_1-a_2)}(f^2-a_1 f). \end{aligned}$$

From this we have four exceptional values $A_1/3$, 0, a_1, ∞ and these are all exceptional values of f. Thus we have $P(f) \leqq 4$. This is a contradiction. Therefore the case (i) does not occur.

Case (ii). Similarly we have

$$H_1 \equiv H_2 \equiv H, \qquad a_1a_2(a_1-a_2)(a_1-a_3)(a_2-a_3)-c_1a_2(a_2-a_3)+c_2a_1(a_1-a_3)=0,$$

$$\beta_2=\frac{(a_1-a_3)(a_2-a_3)}{a_1a_2}\beta_1;$$

$$\text{(5.2)} \qquad \begin{cases} S_1=-\dfrac{1}{a_1a_3}\beta_1 e^H+A_2, \\ S_2=-\dfrac{a_1+a_2}{a_1a_2}\beta_1 e^H+B_2, \\ S_3=-\beta_1 e^H, \end{cases}$$

where

$$A_2=-\frac{c_1}{a_1(a_1-a_3)}+a_1+a_3, \qquad B_2=-\frac{a_3c_1}{a_1(a_1-a_3)}+a_1a_3.$$

Then the coefficient of e^{4H} in $D(z)$ is $(a_1-a_2)^2\beta_1^4/a_1^4a_2^4 \neq 0$. Hence we similarly have a function $f \in \mathfrak{M}(R)$ with $P(f)=6$, otherwise $P(f) \leqq 4$. This contradicts $P(f)=5$.

Therefore the case (ii) does not occur.

Case (iii). We have

$$H_1 \equiv H_2 \equiv H_3 \equiv H, \qquad c_1 + a_1 a_2 a_3 = 0, \qquad a_2 a_3 (a_2 - a_3)\beta_1 - a_1 a_3 (a_1 - a_3)\beta_2 + a_1 a_2 (a_1 - a_2)\beta_3 = 0;$$

$$(5.3) \qquad \begin{cases} S_1 = -\dfrac{a_2\beta_1 - a_1\beta_2}{a_1 a_2 (a_1 - a_2)} e^H + a_1 + a_2 + a_3, \\ S_2 = -\dfrac{a_2^2\beta_1 - a_1^2\beta_2}{a_1 a_2 (a_1 - a_2)} e^H + a_1 a_2 + a_2 a_3 + a_3 a_1, \\ S_3 = a_1 a_2 a_3. \end{cases}$$

Comparing (5.3) with (3.1), we can see that the constant term of $D(z)$ is not zero.

If $a_2\beta_1 - a_1\beta_2 \neq 0$ and $a_2^2\beta_1 - a_1^2\beta_2 \neq 0$, then the coefficient of e^{4H} in $D(z)$ is not zero. Hence the same reasoning in § 4 holds from the above remark, and we can conclude the existence of a function $f \in \mathfrak{M}(R)$ with $P(f) = 6$. This is absurd.

If $a_2\beta_1 - a_1\beta_2 = 0$ or $a_2^2\beta_1 - a_1^2\beta_2 = 0$, then $D(z)$ is a polynomial of degree 3 of e^H, because that $a_2\beta_1 - a_1\beta_2$ and $a_2^2\beta_1 - a_1^2\beta_2$ are not zero simultaneously. And from (2.6) we have the equation

$$-27g^2(f_2^3 - f_3^3 g)^2 = A'(e^H - \gamma_1)(e^H - \delta_1)(e^H - \eta_1), \qquad A' \neq 0, \ \gamma_1\delta_1\eta_1 \neq 0.$$

From this and considering simple zero points of the function $e^H - \gamma$ $(\gamma \neq 0)$, we have a contradiction. Therefore the case (iii) does not occur.

Case (iv). We have

$$H_1 \equiv H_2 \equiv H_3 \equiv H, \qquad c_1 = a_1(a_1 - a_2)(a_1 - a_3),$$

$$(a_1 - a_2)(a_1 - a_3)(a_2 - a_3)\beta_1 + a_1 a_3 (a_1 - a_3)\beta_2 - a_1 a_2 (a_1 - a_2)\beta_3 = 0;$$

$$(5.4) \qquad \begin{cases} S_1 = -\dfrac{(a_1 - a_2)\beta_1 - a_1\beta_2}{a_1 a_2 (a_1 - a_2)} e^H + a_2 + a_3, \\ S_2 = -\dfrac{(a_1^2 - a_2^2)\beta_1 - a_1^2\beta_2}{a_1 a_2 (a_1 - a_2)} e^H + a_2 a_3, \\ S_3 = -\beta_1 e^H. \end{cases}$$

Comparing (5.4) with (3.2), we can see that the constant term of $D(z)$ is not zero. The coefficient of e^{4H} in $D(z)$ is

$$\frac{((a_1 - a_2)\beta_1 - a_1\beta_2)^2((a_1 - a_2)^2\beta_1 - a_1^2\beta_2)^2}{a_1^4 a_2^4 (a_1 - a_2)^4}.$$

If $(a_1 - a_2)\beta_1 - a_1\beta_2 \neq 0$ and $(a_1 - a_2)^2\beta_1 - a_1^2\beta_2 \neq 0$, then similarly we have a contradiction.

If $(a_1 - a_2)\beta_1 - a_1\beta_2 = 0$, then $(a_1^2 - a_2^2)\beta_1 - a_1^2\beta_2 = a_2(a_1 - a_2)\beta_1 \neq 0$ and hence the coefficient of e^{3H} in $D(z)$ is not zero, which is similarly a contradiction.

If $(a_1-a_2)^2\beta_1-a_1^2\beta_2=0$, then $(a_1-a_2)\beta_1-a_1\beta_2=a_2(a_1-a_2)\beta_1/a_1\neq 0$, $(a_1^2-a_2^2)\beta_1-a_1^2\beta_2=2a_2(a_1-a_2)\beta_1\neq 0$ and hence from (2.6) and the similar discussion as in § 4, we can conclude the existence of a function $f\in\mathfrak{M}(R)$ with $P(f)=6$. This is absurd. Therefore the case (iv) does not occur.

By the above discussion in (i), (ii), (iii) and (iv) we have completely proved our theorem 2.

§ 6. From theorem 1, theorem 2 and Ozawa's lemma every Riemann surface defined by the equation (2.1) with $g(z)$, which is an entire function of z having no zero other than an infinite number of simple zeros or having no zero other than an infinite number of double zeros, always satisfies $P(R)\leqq 4$. And an example R with $P(R)=4$ is easily given. In fact let R be a Riemann surface defined by the equation (2.1) with $g(z)=e^z+1$. From the above remark we have $P(R)\leqq 4$. The function $f=\sqrt[3]{e^z+1}$ belongs to $\mathfrak{M}(R)$ and $P(f)=4$. Therefore $P(R)=4$.

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