MINIMAL SLIT REGIONS AND LINEAR OPERATOR METHOD

By Kôtaro Oikawa

1. Let Ω be a plane region containing the point at infinity. Let $\mathfrak{F}_\Omega$ be the family of all the univalent functions f on Ω having the expansion

$$(1) \qquad f(z)=z+\frac{c}{z}+\cdots$$

about ∞. The function maximizing (minimizing) $\operatorname{Re} c$ in $\mathfrak{F}_\Omega$ exists and is determined uniquely, which we denote by $\varphi_\Omega(\psi_\Omega$, resp.).

The image region $\psi_\Omega(\Omega)$ $(\psi_\Omega(\Omega))$ is a horizontal (vertical) parallel slit plane. Conversely, however, an arbitrary horizontal (vertical) parallel slit plane can not be, in general, the image of an Ω under $\varphi_\Omega(\psi_\Omega)$; in fact the measure of $\varphi_\Omega(\Omega)^c$ and $\psi_\Omega(\Omega)^c$ vanish. Accordingly, with Koebe, we introduce the following:

Definition. A horizontal (vertical) parallel slit plane Δ is said to be *minimal* if $\Delta=\varphi_\Omega(\Omega)$ $(\Delta=\psi_\Omega(\Omega)$, resp.) for an Ω containing ∞.

The minimality of slit regions is characterized by moduli of quadrilaterals (Grötzsch [2]) or extremal length (Jenkins [3]). From the point of view of the latter a number of interesting properties are derived in Suita's paper in these Reports [8].

The linear operator method due to Sario [6] (see also Chapter III of the book by Ahlfors-Sario [1]) gives us another approach to φ_Ω and ψ_Ω. From this a characterization of minimality is derived, which is rather similar to the original one due to Koebe [4]. It is the purpose of the present paper to show how to use this method to prove alternatively a part of Suita's results mentioned above.

2. We begin with reviewing the definition of the normal linear operators L_0 and L_1 in Ahlfors-Sario [1].

Let W be an open Riemann surface, let V be a regularly imbedded non-compact subregion with compact relative boundary α. For any real analytic function f on α, consider the problem of constructing the function u such that

$$(2) \qquad \text{harmonic on } V\cup\alpha, \qquad u=f \text{ on } \alpha.$$

If V is the interior of a compact bordered surface we can assign the behavior of u on $\beta=(\text{border of } V)-\alpha$ so that u may be determined uniquely. For our purpose the following two are necessary:

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(L_0): $du^*=0$ along β,

(L_1): $du=0$ along β, $\int du^*=0$ for each contour of β;

here the correspondence $f\rightarrow u$ is expressed by the notations in the left.

Note that the present L_1 is the $(P)L_1$ in Ahlfors-Sario's book with respect to the canonical partition P. (See [1, p. 160].)

If V is arbitrary we may define L_0 and L_1 as the limit through an exhaustion. We can define them also as follows:

DEFINITION. $L_0 f$ is defined as the u determined uniquely by the condition (2), $D_V(u)<\infty$, and

$$\int_V (du)(dv)^* = \int_\alpha v du^* \tag{3}$$

for every harmonic function v on $\bar{V}$ with $D_V(v)<\infty$. $L_1 f$ is defined as the u determined uniquely by the condition (2), $D_V(u)<\infty$, $\int_\gamma du^*=0$ for every dividing cycle γ which does not separate components of α, and

$$\int_V (du)\omega = \int_\alpha f\omega \tag{4}$$

for every harmonic differential ω on $V\cup\alpha$ such that $||\omega||_V<\infty$ and $\int_\gamma \omega=0$ for every γ mentioned above.

We remark the following:

(i) If V is the interior of a compact bordered surface, this definition coincides with the previous.

(ii) In (3), the harmonicity of v may be replaced by the following: v is of $C^{(1)}$ on $\bar{V}$. In (4) the harmonicity of ω may be replaced by the following: ω is of $C^{(1)}$ and closed on $\bar{V}$.

(iii) If $V'\subset V$ then

$$L_{0V'}(L_{0V}f)=L_{0V}f, \qquad L_{1V'}(L_{1V}f)=L_{1V}f$$

on V' for any f on α; here the subscripts V' and V express the region where the operators are considered.

(iv) Conversely, let $V_1,\cdots,V_n\subset V$ be mutually disjoint and such that $V-\cup_{k=1}^n V_k$ is relatively compact. Given f on α, suppose a u on V satisfy (2) and

$$u=L_{0V_k}u \qquad (u=L_{1V_k}u)$$

on V_k, $k=1,\cdots,n$. Then $u=L_0 f$ ($u=L_1 f$, resp.) on V.

3. We find in Ahlfors-Sario [1, p. 176ff] that φ_Ω and ψ_Ω are characterized as functions regular on $\Omega-\{\infty\}$, having expansion (1) about ∞, and such that

$$L_0(\mathrm{Re}\,\varphi_\Omega)=\mathrm{Re}\,\varphi_\Omega, \qquad L_1(\mathrm{Re}\,\psi_\Omega)=\mathrm{Re}\,\psi_\Omega \tag{5}$$

on $\partial\Omega$; this means the validity of (5) on $V_1,\cdots,V_n$ with compact $\Omega-\cup_{k=1}^n V_k$, which is independent of the choice of V_k because of the above remarks (iii) and (iv). Therefore

THEOREM 1. *A region Δ in the $z=x+iy$-plane with $\infty\in\Delta$ is a minimal horizontal (vertical) parallel slit plane if and only if*

$$L_0x=x \qquad (L_1x=x, \text{ resp.})$$

on $\partial\Delta$.

It is evident that the condition is equivalent with

$$L_1y=y \qquad (L_0y=y, \text{ resp.}). \tag{7}$$

On regarding the definition of L_0 we see that the validity of $L_0x=x$ on a V is equivalent with the following: $\iint_V(\partial v/\partial x)dxdy=\int_\alpha vdy$. Consequently a region Δ with $\infty\in\Delta$ is a minimal horizontal parallel slit plane if and only if

$$\iint_\Delta\frac{\partial h}{\partial x}dxdy=0$$

for every h which is of $C^{(1)}$ in Δ, vanishes identically in a neighborhood of ∞, and has finite $D_\Delta(h)$. This is nothing but the original characterization of minimality due to Koebe [4].

From Theorem 1 and remarks (iii), (iv) of 2°, we obtain the following which is Theorem 12 of Suita [8]:

THEOREM 2. *Let $\infty\in\Delta_k$ $(k=1,\cdots,n)$ have mutually disjoint Δ_k^C, and let $\Delta=\cap_{k=1}^n\Delta_k$. Then Δ is a minimal horizontal (vertical) parallel slit plane if and only if so are all the Δ_k.*

4. Circular and radial slit planes are characterized by L_0 and L_1 in the similar way. Slit disks and annuli are the same if the outer (and inner) periphery is assumed to be *isolated from other part of the boundary.* For example

Let Δ be a circular slit annulus with inner and outer radius $0<Q'$ and $Q<\infty$, respectively. Let $(|z|=Q')$ and $(|z|=Q)$ be isolated from $E=\Delta^C\cap\{z|Q'<|z|<Q\}$. Then Δ is a minimal circular slit annulus if and only if $L_1(\log|z|)=\log|z|$ on E.

The change of the independent variable in (4) implies the following, which is contained in Theorem 11 of Suita [8]:

THEOREM 3. *Let a circular slit annulus Δ and its slits E be as above. Let Δ' be a horizontal parallel slit plane such that $E'=\Delta'^C$ is contained in the interior of a vertical parallel strip with width 2π. Suppose that E is the image of E' under the mapping $z\to\exp iz$. Then Δ is minimal if and only if Δ' is minimal.*

5. Characterizing minimal circular slit annuli by extremal length is easier than that of parallel slit plane. The former is found in, e.g., Reich-Warschawski [5] (for slit disk, though) or Sakai [7], and the latter is in Jenkins [3] as we have mentioned.

The former is as the following:

Let Δ be as in 4^0. Let Γ be the family of all the closed rectifiable curves in Δ separating the inner and outer peripheries. Then Δ is minimal if and only if $\log(Q/Q')=2\pi/\lambda(\Gamma)$.

The following is derived from this:

THEOREM 4. *Let Δ be a plane region containing ∞. Let R be a rectangle whose interior contains Δ^C and sides are parallel to the coordinate axes. Let a and b be respectively the width and the height of R. Let Γ be the family of all the rectifiable curves in $R\cap\Delta$ joining the both vertical sides of R.* (i) *If Δ is minimal, then $\lambda(\Gamma)=a/b$ for any R;* (ii) *If there exists an R with $\lambda(\Gamma)=a/b$, then Δ is minimal.*

Concerning (ii), Jenkins [3] assumed the validity of $\lambda(\Gamma)=a/b$ for all sufficiently large square R. The present form the characterization by moduli of quadrilaterals is stated without proof by Grötzsch [2, p. 188]. The above is Theorem 8 of Suita [8].

Proof. (i) With the aid of linear transformation, we may assume in advance that $a=2\pi$. Map R by $\zeta=\text{const}\cdot\exp iz$ onto $1<|\zeta|<\exp b$ and let the image of Δ^C be $\tilde{E}$. By Theorem 3 $\tilde{\Delta}=(1<|\zeta|<\exp b)-\tilde{E}$ is minimal, so that $b=2\pi/\lambda(\tilde{\Gamma})$, where $\tilde{\Gamma}$ is the family of all the closed curves in $\tilde{\Delta}$ separating the inner and outer peripheries. From the general theory of extremal length, it is easy to obtain $2\pi/b\leqq\lambda(\Gamma)$, $\lambda(\Gamma)\leqq\lambda(\tilde{\Gamma})$. Thus $\lambda(\Gamma)=2\pi/b$.

(ii) We may assume in advance that $a=\pi$. Let $\hat{R}$ and $\hat{E}$ be obtained from R and E, respectivey, by the reflection across the right vertical side of R. Let $\hat{\Gamma}$ be the family of curves obtained from Γ by the same reflection. Map $\hat{R}$ by $\zeta=\text{const}\cdot\exp iz$ onto $1<|\zeta|<\exp b$ and let the image of $\hat{E}$ be $\tilde{E}$. Consider $\tilde{\Delta}$ and $\tilde{\Gamma}$ as before. From the general theory, we have $2\pi/b\leqq\lambda(\tilde{\Gamma})$, $\lambda(\tilde{\Gamma})\leqq\lambda(\hat{\Gamma})$, $\lambda(\hat{\Gamma})=2\lambda(\Gamma)$. Thus, by the assumption, $b=2\pi/\lambda(\tilde{\Gamma})$, and, therefore, $\tilde{\Delta}$ is minimal. By Theorem 3 $\hat{E}^c$ is minimal, so that, by Theorem 2, Δ is minimal.

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