ON AN ESTIMATE FOR SEMI-LINEAR ELLIPTIC DIFFERENTIAL EQUATIONS OF THE SECOND ORDER WITH DINI-CONTINUOUS COEFFICIENTS

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§1. Introduction.

In this paper, we are concerned with the a priori estimate for derivatives of solutions of the semi-linear elliptic differential equation

$$(1.1) \qquad \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} = f\left(x, u, \frac{\partial u}{\partial x}\right).^{1)}$$

Concerning the estimate of this sort, Nagumo obtained a result on the assumption that the coefficients $a_{ij}(x)$ satisfy the Lipschitz condition ([4],[2]) pp. 211–215, Theorem 2), and thereafter Simoda [5] and the author [3] improved Nagumo's result on the assumption that the coefficients $a_{ij}(x)$ satisfy the Hölder condition.

As, however, it is desirable from a theoretical point of view, that we have the a priori estimate under the weakest possible condition on the continuity of the coefficients $a_{ij}(x)$, we shall form, in this paper, an a priori estimate of the same type as obtained in the above-cited papers, provided that the coefficients $a_{ij}(x)$ satisfy the Dini condition.

The Dini condition which we impose on the coefficients $a_{ij}(x)$, is more restrictive than usual, but it seems to be considerably general.

Our method of proof in this paper is analoguous to one in the previous paper [3], which was composed of Nagumo's one and Cordes' modified results [2]. Therefore, the parts of the proof which can be carried out in the same way as in the previous paper, will often be omitted.

In §2, we shall give two definitions concerning Dini functions, and prove two lemmas in regard to the properties of Dini functions given in these two definitions.

In §3, we state the main result of this paper, whose proof is left to §5. A set of lemmas will be made in §4, and two other results will be proved in §6.

§2. Dini functions $\varphi(t)$.

In this section, we make some preliminary remarks on Dini functions $\varphi(t)$ which define the modulus of continuity of the coefficients $a_{ij}(x)$ in the differential

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1) $\partial u/\partial x$ denotes the n-dimensional real vector $(\partial u/\partial x_1, \partial u/\partial x_2, \cdots, \partial u/\partial x_n)$.

2) The numbers in the brackets refer to the list of references at the end of this paper.

operator $\sum_{i,j=1}^{n} a_{ij}(x)\partial^2/\partial x_i \partial x_j$.

For the sake of convenience we first give the following definitions.

DEFINITION 1. We say that a real-valued function $\varphi(t)$ satisfies *the condition* $(D(\delta))$ $(0<\delta<+\infty)$, if

(i) $\varphi(t)$ is continuous and non-negative in an interval I: $0\leqq t\leqq\delta$;

(ii) $\varphi(t)$ is monotone increasing in I;[3]

(iii) we have

$$\int_0^\delta \frac{\varphi(t)}{t}\,dt<+\infty;$$

(iv) $\varphi(t)$ is concave below in I in the weak sense.[4]

DEFINITION 2. We say that a real-valued function $\hat{\varphi}(t)$ satisfies *the condition* $(D'(\delta))$ $(0<\delta<+\infty)$, if

(i') $\hat{\varphi}(t)$ is continuous and non-negative in an interval I: $0\leqq t\leqq\delta$, and twice continuously differentiable in the interval I': $0<t\leqq\delta$;

(ii') $d\hat{\varphi}(t)/dt\geqq 0$ in I';

(iii') we have the same as the condition (iii) of $(D(\delta))$;

(iv') $d^2\hat{\varphi}(t)/dt^2\leqq 0$ in I'.

As a Dini function we shall take, in this paper, a function satisfying the condition $(D(\delta))$, but we shall need a function satisfying the condition $(D'(\delta))$ in lemmas.

For any functions $\varphi(t)$, $\hat{\varphi}(t)$ satisfying the conditions $(D(\delta))$ or $(D'(\delta))$, we put in general

$$\Phi(t)=\int_0^t \frac{\varphi(t)}{t}\,dt, \qquad \hat{\Phi}(t)=\int_0^t \frac{\hat{\varphi}(t)}{t}\,dt.$$

It is obvious that a function $\varphi(t)$ (or $\hat{\varphi}(t)$) satisfying the condition $(D(\delta))$ (or $(D'(\delta))$) tends to zero as $t\to+0$, and the corresponding function $\Phi(t)$ (or $\hat{\Phi}(t)$) does so.

We shall now prove the following:

LEMMA 1. *Let $\varphi(t)$ be a function satisfying the condition $(D(\delta))$. Then, for any positive number $\eta>0$, there exists a function $\hat{\varphi}(t)$ satisfying the condition $(D'(\delta))$, such that*

$$\varphi(t)\leqq\hat{\varphi}(t)\leqq\varphi(t)+\eta,$$
$$\Phi(t)\leqq\hat{\Phi}(t)\leqq\Phi(t)+\eta$$

in the interval I: $0\leqq t\leqq\delta$.

Proof. If $\varphi(t)=0$ for some value t in the interval I': $0<t\leqq\delta$, then, by virtue

3) That is, the inequality $\varphi(t_1)\leqq\varphi(t_2)$ holds for any $t_1, t_2 \in I$, such that $t_1<t_2$.

4) That is, the inequality $\varphi(t_1)+\varphi(t_2)\leqq 2\varphi((t_1+t_2)/2)$ holds for any $t_1, t_2 \in I$.

of the conditions (ii), (iv) of $(D(\delta))$, we see $\varphi(t)\equiv 0$ in the interval I, and hence, in this case we have only to take $\hat{\varphi}(t)\equiv\varphi(t)\equiv 0$. We can therefore assume $\varphi(t)>0$ in the interval I'.

Let η' be a positive number such that $\eta'\leqq\eta/\delta$. Then the function

$$\tilde{\varphi}(t)=\varphi(t)+\eta' t$$

satisfies the condition $(D(\delta))$ and the inequalities

$$\varphi(t)\leqq\tilde{\varphi}(t)\leqq\varphi(t)+\eta \quad \text{in} \quad I;$$

$$\varphi(t)<\tilde{\varphi}(t) \qquad\qquad \text{in} \quad I',$$

and moreover this function $\tilde{\varphi}(t)$ is strictly monotone increasing with respect to t in I.[5] Consequently, by putting $t_0=\delta$, we can determine uniquely a decreasing sequence of positive numbers $\{t_k\}$ so that $\tilde{\varphi}(t_{k+1})=\varphi(t_k)$ $(k=0, 1, 2, \cdots)$.

If we draw the curves $y=\varphi(t)$, $y=\tilde{\varphi}(t)$ on the (t, y)-plane, and denote the points $(t_k, \tilde{\varphi}(t_k))$ by P_k, then the points P_k lie on the curve $y=\tilde{\varphi}(t)$, and by connecting each couple of two points (P_k, P_{k+1}) by the segment $\overline{P_kP_{k+1}}$, we have a concave polygon $y=\pi(t)$ with an infinite number of vertices $\{P_k; k=0, 1, \cdots\}$, such that $\varphi(t)<\pi(t)\leqq\tilde{\varphi}(t)$ in I' and $\varphi(0)=\pi(0)=\tilde{\varphi}(0)=0$.

As there are no points of the curve $y=\varphi(t)$ in a sufficiently small neighborhood U_k of each vertex P_k, we obtain a desired curve $y=\hat{\varphi}(t)$ by modifying the polygon $y=\pi(t)$ in the neighborhoods U_k.

LEMMA 2. *Let $\hat{\varphi}(t)$ be a function satisfying the condition $(D'(\delta))$ and let $\hat{\psi}(t)$ be a function defined by the expression*

$$\hat{\psi}(t)=-t^2\frac{d^2\hat{\varphi}(t)}{dt^2}+t\frac{d\hat{\varphi}(t)}{dt}+\hat{\varphi}(t).$$

Then $\hat{\psi}(t)/t$ is improperly integrable over the interval I: $0\leqq t\leqq\delta$, and we have

$$\hat{\Psi}(t)\equiv\int_0^t\frac{\hat{\psi}(t)}{t}dt=-t\frac{d\hat{\varphi}(t)}{dt}+2\hat{\varphi}(t)+\hat{\Phi}(t),$$

where

$$\hat{\Phi}(t)=\int_0^t\frac{\hat{\varphi}(t)}{t}dt.$$

Proof. Since it holds that

$$\hat{\varphi}(\tau)-\hat{\varphi}(\tau/2)=\int_{\tau/2}^{\tau}\frac{d\hat{\varphi}(t)}{dt}dt\geqq\frac{d\hat{\varphi}(t)}{dt}\bigg|_{t=\tau}\int_{\tau/2}^{\tau}dt=\frac{1}{2}\left\{t\frac{d\hat{\varphi}(t)}{dt}\right\}\bigg|_{t=\tau}\ ^{6)}$$

for $0<\tau\leqq\delta$, we see

5) That is, $\tilde{\varphi}(t_1)<\tilde{\varphi}(t_2)$ for any $t_1, t_2\in I$ such that $t_1<t_2$.

6) For a function $g(t)$, the notation $g(t)|_{t=\tau}$ means the value of $g(t)$ at $t=\tau$.

$$\lim_{\tau\to+0}\left\{t\frac{d\hat{\varphi}(t)}{dt}\right\}\Big|_{t=\tau}=0.$$

We have therefore on integrating by parts,

$$\int_0^t t\frac{d^2\hat{\varphi}(t)}{dt^2}dt=\lim_{\tau\to+0}\left\{\left[t\frac{d\hat{\varphi}(t)}{dt}\right]_\tau^t-[\hat{\varphi}(t)]_\tau^t\right\}=t\frac{d\hat{\varphi}(t)}{dt}-\hat{\varphi}(t),$$

by which this lemma can be easily proved.

§ 3. Statement of main result.

We use the notations $\partial_i u$ for $\partial u/\partial x_i$ and $\partial_i\partial_j u$ for $\partial^2 u/\partial x_i\partial x_j$. x and $\partial_x u$ denote the n-dimensional real vectors $(x_1, x_2, \cdots, x_n)$ and $(\partial_1 u, \partial_2 u, \cdots, \partial_n u)$ respectively. We use $\partial_i' u$ for $\partial u/\partial x_i'$ and the notations of the same kind for the others too.

We define $|x^{(1)}-x^{(2)}|$ and $|\partial_x u|$ as follows:

$$|x^{(1)}-x^{(2)}|=\left\{\sum_{i=1}^n(x_i^{(1)}-x_i^{(2)})^2\right\}^{1/2}\quad\text{and}\quad|\partial_x u|=\left\{\sum_{i=1}^n(\partial_i u)^2\right\}^{1/2}.$$

In the present section, we state a theorem about the a priori estimate for derivatives of solutions of the semi-linear elliptic differential equation

$$L(u)\equiv\sum_{i,j=1}^n a_{ij}(x)\partial_i\partial_j u=f(x, u, \partial_x u). \tag{3. 1}$$

Let D be a bounded domain with the boundary $\dot{D}$ in the n-dimensional Euclidean space and let d be the diameter of D.

Let $(a_{ij}(x))$ be a symmetric matrix satisfying the uniform ellipticity condition with two positive constants $\underline{A}$, $\overline{A}$, $(\underline{A}\leqq\overline{A})$:

$$\underline{A}|\xi|^2\leqq\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j\leqq\overline{A}|\xi|^2 \tag{3. 2}$$

for any real vector $\xi=(\xi_1, \xi_2, \cdots, \xi_n)$ and for any point $x\in D$. Furthermore, the coefficients $a_{ij}(x)$ are supposed to satisfy the condition:

$$\left\{\sum_{i,j=1}^n(a_{ij}(x^{(1)})-a_{ij}(x^{(2)}))^2\right\}^{1/2}\leqq\varphi(|x^{(1)}-x^{(2)}|) \tag{3. 3}$$

for any $x^{(1)}$, $x^{(2)}\in D$ such that $|x^{(1)}-x^{(2)}|\leqq\delta$, where $\varphi(t)$ is a function satisfying the condition $(D(\delta))$ given in § 2.

Let $\mathfrak{D}$ be a $(2n+1)$-dimensional domain defined by

$$\mathfrak{D}=\{(x, u, p);\ x\in D, |u|\leqq M, |p|<+\infty\},$$

where M is a positive quantity and p denotes the n-dimensional real vector $(p_1, p_2, \cdots, p_n)$.

We suppose that the function $f(x, u, p)$ is defined in the domain $\mathfrak{D}$, and satisfies the condition of growth order with respect to p:

$$|f(x, u, p)| \leqq B|p|^2 + \Gamma \tag{3. 4}$$

for any $(x, u, p) \in \mathfrak{D}$, where B and Γ are positive constants.

THEOREM 1. *Suppose that the above-mentioned hypotheses are fulfilled. Let $u(x)$ be a solution of the equation* (3. 1), *belonging to $C^2[D]$*[7] *and satisfying $|u(x)| \leqq M$, and let N be the oscillation of $u(x)$ in the domain D.*

If the condition

$$8K(n)\frac{BN}{\underline{A}} < 1 \quad \left(K(n) = \frac{n+1}{2}B\left(\frac{1}{2}, \frac{n+1}{2}\right)^{-2}\right) \tag{3. 5}$$

is fulfilled, then we have

$$|\partial_x u(a)| \leqq C^{(1)}\rho(a)^{-1} \operatorname*{Osc}_{|x-a| \leqq \rho(a)} \{u(x)\} + C^{(2)}\rho(a) \tag{3. 6}$$ [8]

for any point $a \in D$, where $\rho(a) = \operatorname{dist}(a, \dot{D})$, and $C^{(1)}$ and $C^{(2)}$ are positive constants depending only on $\underline{A}$, $\bar{A}$, B, Γ, N, n, d, δ and on the function $\varphi(t)$.

The proof of this theorem is left to § 5, and especially the dependence of the constants $C^{(1)}$, $C^{(2)}$ on $\underline{A}$, $\bar{A}$, B, Γ, N, n, d, δ and $\varphi(t)$ will be made clear at the end of § 5.

§ 4. Lemmas.

We use a letter r for $|x| \equiv \{\sum_{i=1}^n x_i^2\}^{1/2}$. $S[r_0]$ denotes a closed sphere $r \leqq r_0$, and $C^\nu[r \leqq r_0]$ denotes the family of all functions which are ν times continuously differentiable in $S[r_0]$, where r_0 is a positive real number.

LEMMA 3. *Let $u(x)$ be a function belonging to $C^2[r \leqq r_0]$ and let $\hat{\varphi}(t)$ be a function satisfying the condition $(D'(r_0))$. Then we have*

$$\begin{aligned} &\left\{\int_{r \leqq r_0} \sum_{i,j=1}^n (\partial_i \partial_j u)^2 \hat{\varphi}(r) r^{2-n}(r_0^2 - r^2)^2 dx\right\}^{1/2} \\ &- \left\{\int_{r \leqq r_0} (\Delta u)^2 \hat{\varphi}(r) r^{2-n}(r_0^2 - r^2)^2 dx\right\}^{1/2} \\ &\leqq c r_0^2 \left\{\int_{r \leqq r_0} \sum_{i=1}^n (\partial_i u)^2 \hat{\varphi}(r) r^{-n} dx\right\}^{1/2}, \end{aligned} \tag{4. 1}$$

7) $C^2[D]$ denotes the family of all functions which are twice continuously differentiable in D.

8) The notation $\mathrm{Osc}_{|x-a| \leqq \rho(a)}\{u(x)\}$ denotes the oscillation of $u(x)$ in the sphere $|x-a| \leqq \rho(a)$ possessing the point a as the center,

where c is a positive constant depending only on n, but not on r_0, and $\hat{\varphi}(r)$ is a function defined by

$$\hat{\varphi}(r)=-r^2\frac{d^2\varphi(r)}{dr^2}+r\frac{d\varphi(r)}{dr}+\varphi(r) \tag{4. 2}$$

Proof. Suppose, in the first place, that $u(x)$ belongs to $C^3[r\leqq r_0]$, and multiply the identity

$$\sum_{i,j=1}^{n}(\partial_i\partial_j u)^2-(\Delta u)^2=\Delta\left\{\sum_{i=1}^{n}(\partial_i u)^2\right\}-\sum_{i,j=1}^{n}\partial_i\partial_j\{(\partial_i u)(\partial_j u)\}$$

by the factor $k(r)\equiv\varphi(r)r^{2-n}(r_0{}^2-r^2)^2$, and integrate the result over the sphere $S[r_0]$. Then, on integrating by parts we obtain

$$\begin{aligned}J&\equiv\int_{r\leqq r_0}\sum_{i,j=1}^{n}(\partial_i\partial_j u)^2k(r)dx-\int_{r\leqq r_0}(\Delta u)^2k(r)dx\\&=\int_{r\leqq r_0}\left\{\sum_{i=1}^{n}(\partial_i u)^2\Delta k(r)-\sum_{i,j=1}^{n}(\partial_i u)(\partial_j u)\partial_i\partial_j k(r)\right\}dx.\end{aligned}\tag{4. 3}$$

Furthermore, by the limiting process, it can be shown that the above relation (4. 3) holds for any $u(x)\in C^2[r\leqq r_0]$.

Now, since

$$\partial_i\partial_j k(r)=\frac{x_ix_j}{r^2}\left\{\frac{d^2k(r)}{dr^2}-\frac{1}{r}\frac{dk(r)}{dr}\right\}+\frac{\delta_{ij}}{r}\frac{dk(r)}{dr}$$

and $\sum_{i=1}^{n}x_i\partial_i u=r\partial_r u\equiv r\partial u/\partial r$, we have

$$\sum_{i,j=1}^{n}(\partial_i u)(\partial_j u)\partial_i\partial_j k(r)=\left\{\frac{d^2k(r)}{dr^2}-\frac{1}{r}\frac{dk(r)}{dr}\right\}(\partial_r u)^2+\frac{1}{r}\frac{dk(r)}{dr}\sum_{i=1}^{n}(\partial_i u)^2. \tag{4. 4}$$

Hence, by the expression (4. 4) and the formula

$$\Delta k(r)=\frac{d^2k(r)}{dr^2}+\frac{n-1}{r}\frac{dk(r)}{dr},$$

it is easily seen that the integrand I_0 in the right-hand side of the relation (4. 3) has the form

$$I_0=I_1\sum_{i=1}^{n}(\partial_i u)^2-I_2(\partial_r u)^2,$$

where

$$I_1=\frac{d^2k(r)}{dr^2}+\frac{n-2}{r}\frac{dk(r)}{dr},\quad I_2=\frac{d^2k(r)}{dr^2}-\frac{1}{r}\frac{dk(r)}{dr}.$$

Moreover, by making use of the inequality $\sum_{i=1}^{n}(\partial_i u)^2\geqq(\partial_r u)^2$, we get

$$|I_0| \leqq (|I_1|+|I_2|) \sum_{i=1}^{n} (\partial_i u)^2.$$

We next evaluate the two terms I_1 and I_2.

Since

$$\frac{dk(r)}{dr} = \frac{d\hat{\varphi}(r)}{dr} r^{2-n}(r_0^2-r^2)^2 + \hat{\varphi}(r)\{-(n-2)r_0^2+(n-6)r^2\} r^{1-n}(r_0^2-r^2),$$

$$\frac{d^2k(r)}{dr^2} = \frac{d^2\hat{\varphi}(r)}{dr^2} r^{2-n}(r_0^2-r^2)^2 + 2\frac{d\hat{\varphi}(r)}{dr}\{-(n-2)r_0^2+(n-6)r^2\} r^{1-n}(r_0^2-r^2)$$

$$+\hat{\varphi}(r)\{(n-1)(n-2)r_0^4-(n^2-11n+22)r_0^2r^2+(n-5)(n-6)r^4\} r^{-n},$$

we obtain

$$I_1 = \left\{P_1(r_0, r) r^2 \frac{d^2\hat{\varphi}(r)}{dr^2} + P_2(r_0, r) r \frac{d\hat{\varphi}(r)}{dr} + P_3(r_0, r)\hat{\varphi}(r)\right\} r^{-n},$$

$$I_2 = \left\{P_4(r_0, r) r^2 \frac{d^2\hat{\varphi}(r)}{dr^2} + P_5(r_0, r) r \frac{d\hat{\varphi}(r)}{dr} + P_6(r_0, r)\hat{\varphi}(r)\right\} r^{-n},$$

where

$$P_1(r_0, r) = (r_0^2-r^2)^2,\ P_2(r_0, r) = \{-(n-2)r_0^2+(n-10)r^2\}(r_0^2-r^2)^2,$$

$$P_3(r_0, r) = (n-2)r_0^4+(n^2-n-4)r_0^2r^2-3(n-6)r^4,$$

$$P_4(r_0, r) = (r_0^2-r^2)^2,\ P_5(r_0, r) = \{-(2n-3)r_0^2+(2n-11)r^2\}(r_0^2-r^2),$$

$$P_6(r_0, r) = n(n-2)r_0^4-(n-2)(n-7)r_0^2r^2+(n-4)(n-6)r^4.$$

Consequently, the inequalities

$$\hat{\varphi}(r) \geqq 0, \quad \frac{d\hat{\varphi}(r)}{dr} \geqq 0, \quad \frac{d^2\hat{\varphi}(r)}{dr^2} \leqq 0$$

yield

$$|I_0| \leqq \left\{\sum_{\nu=1}^{6} |P_\nu(r_0, r)|\right\}\left\{\sum_{i=1}^{n} (\partial_i u)^2\right\}\hat{\varphi}(r) r^{-n},$$

where $\hat{\varphi}(r)$ is the function defined by the expression (4. 2).

If we put

$$c' = r_0^{-4} \operatorname*{Max}_{0 \leqq r \leqq r_0} \left\{\sum_{\nu=1}^{6} |P_\nu(r_0, r)|\right\} = \operatorname*{Max}_{0 \leqq t \leqq 1} \left\{\sum_{\nu=1}^{6} |P_\nu(1, t)|\right\},$$

then c' is a positive constant depending only on n, but not on r_0, and we get

$$J \leqq c' r_0^4 \int_{r \leqq r_0} \sum_{i=1}^{n} (\partial_i u)^2 \hat{\varphi}(r) r^{-n} dx,$$

from which the desired inequality (4. 1) follows with $c = \sqrt{c'}$.

LEMMA 4. *Let $a_{ij}(x)$ be continuous functions defined in the sphere $S[r_0]$ and satisfying the condition*

$$\left\{\sum_{i,j=1}^{n}(a_{ij}(x)-\delta_{ij})^2\right\}^{1/2}\leqq\varepsilon<1$$

for any point $x\in S[r_0]$ and for a positive constant ε not depending on x.

If $\hat{\varphi}(t)$ is a function satisfying the condition $(D'(r_0))$ and if $u(x)\in C^2[r\leqq r_0]$, then we have

$$\left\{\int_{r\leqq r_0}\sum_{i,j=1}^{n}(\partial_i\partial_j u)^2\hat{\varphi}(r)r^{2-n}(r_0^2-r^2)^2dx\right\}^{1/2}$$

$$\leqq\frac{1}{1-\varepsilon}\left\{\int_{r\leqq r_0}\left(\sum_{i,j=1}^{n}a_{ij}(x)\partial_i\partial_j u\right)^2\hat{\varphi}(r)r^{2-n}(r_0^2-r^2)^2dx\right\}^{1/2}$$

$$+\frac{c}{1-\varepsilon}r_0^2\left\{\int_{r\leqq r_0}\sum_{i=1}^{n}(\partial_i u)^2\hat{\varphi}(r)r^{-n}\,dx\right\}^{1/2},$$

where c and $\hat{\varphi}(r)$ are the same as in Lemma 3.

By using Lemma 3 and by adopting the function $\hat{\varphi}(r)r^{2-n}$ instead of $r^{2+\alpha-n}$, we can easily prove this lemma as Lemma 2 in the previous paper [3].

LEMMA 5. *Let $a_{ij}(x)$ and $u(x)$ be the same as in Lemma 4. If $\varphi(t)$ be a function satisfying the condition $(D(r_0))$ and if we put*

$$\Psi(t)=2\varphi(t)+\Phi(t),\qquad \Phi(t)=\int_0^t\frac{\varphi(t)}{t}dt,$$

then we have

$$\left\{\int_{r\leqq r_0}\sum_{i,j=1}^{n}(\partial_i\partial_j u)^2\varphi(r)r^{2-n}(r_0^2-r^2)^2dx\right\}^{1/2} \tag{4.5}$$

$$\leqq\frac{\sqrt{\omega_n}r_0^2}{1-\varepsilon}\left[r_0\{\varphi(r_0)\}^{1/2}\operatorname*{Max}_{r\leqq r_0}\left|\sum_{i,j=1}^{n}a_{ij}(x)\partial_i\partial_j u(x)\right|+c\{\Psi(r_0)\}^{1/2}\operatorname*{Max}_{r\leqq r_0}|\partial_x u(x)|\right],$$

where c is the same as in Lemma 3, and ω_n denotes the surface area of the n-dimensional unit sphere.[9]

Proof. Let η be a positive number. Then, by virtue of Lemma 1 in § 2, there exists a function $\hat{\varphi}(t)$ satisfying the condition $(D'(r_0))$, such that

$$\begin{aligned}&\varphi(t)\leqq\hat{\varphi}(t)\leqq\varphi(t)+\eta,\\ &\Phi(t)\leqq\hat{\Phi}(t)\leqq\Phi(t)+\eta\end{aligned} \tag{4.6}$$

9) That is, $\omega_n=2\pi^{n/2}/\Gamma(n/2)$.

in the interval $0\leqq t\leqq r_0$. Therefore, we get by Lemmas 2 and 4,

$$J_1\equiv\left\{\int_{r\leqq r_0}\sum_{i,j=1}^{n}(\partial_i\partial_j u)\varphi(r)r^{2-n}(r_0{}^2-r^2)^2dx\right\}^{1/2}$$

$$\leqq\left\{\int_{r\leqq r_0}\sum_{i,j=1}^{n}(\partial_i\partial_j u)\hat{\varphi}(r)r^{2-n}(r_0{}^2-r^2)^2dx\right\}^{1/2}$$

$$\leqq\frac{1}{1-\varepsilon}\left[\left\{\int_{r\leqq r_0}\left(\sum_{i,j=1}^{n}a_{ij}(x)\partial_i\partial_j u\right)^2\hat{\varphi}(r)r^{2-n}(r_0{}^2-r^2)^2dx\right\}^{1/2}\right.$$

$$\left.+cr_0{}^2\left\{\int_{r\leqq r_0}\sum_{i=1}^{n}(\partial_i u)^2\hat{\varphi}(r)r^{-n}\,dx\right\}^{1,2}\right]$$

$$\leqq\frac{\sqrt{\omega_n}r_0{}^2}{1-\varepsilon}\left[r_0\{\hat{\varphi}(r_0)\}^{1/2}\operatorname*{Max}_{r\leqq r_0}\left|\sum_{i,j=1}^{n}a_{ij}(x)\partial_i\partial_j u(x)\right|+c\{\hat{\Psi}(r_0)\}^{1/2}\operatorname*{Max}_{r\leqq r_0}|\partial_x u(x)|\right],$$

where

$$\hat{\Psi}(r_0)=\left\{-t\frac{d\hat{\varphi}(t)}{dt}+2\hat{\varphi}(t)+\hat{\Phi}(t)\right\}\Big|_{t=r_0}.$$

Since $td\hat{\varphi}/dt\geqq 0$ in the interval $0<t\leqq r_0$, the function

$$\tilde{\Psi}(t)=2\hat{\varphi}(t)+\hat{\Phi}(t)$$

satisfies the inequality

$$\hat{\Psi}(t)\leqq\tilde{\Psi}(t)$$

in the interval $0<t\leqq r_0$, and further, it follows from the inequality (4. 6), that

$$\Psi(t)\leqq\tilde{\Psi}(t)\leqq\Psi(t)+3\eta$$

in the interval $0\leqq t\leqq r_0$.

Thus, we obtain

$$J_1\leqq\frac{\sqrt{\omega_n}r_0{}^2}{1-\varepsilon}\left[r_0\{\varphi(r_0)+\eta\}^{1/2}\operatorname*{Max}_{r\leqq r_0}\left|\sum_{i,j=1}^{n}a_{ij}(x)\partial_i\partial_j u(x)\right|+c\{\Psi(r_0)+3\eta\}^{1/2}\operatorname*{Max}_{r\leqq r_0}|\partial_x u(x)|\right]$$

and making $\eta\to 0$, we have the inequality (4. 5).

LEMMA 6. *Let $G(x,\xi)$ be Green's function of Laplace's equation $\Delta u=0$ with respect to the sphere $S[r_0]$, and let $f(x)$ be a bounded measurable function in $S[r_0]$: $(|f(x)|\leqq M)$. Then we have the estimate*

$$\left|\omega_n{}^{-1}\int_{|\xi|\leqq r_0}\partial_x G(0,\xi)f(\xi)d\xi\right|\leqq c_n M r_0,$$

where ω_n denotes the surface area of the n-dimensional unit sphere, and

$$c_n=\frac{2}{n+1}B(n), \qquad B(n)=B\left(\frac{1}{2}, \frac{n+1}{2}\right)^{-1}.$$

For the proof of this lemma, see Lemma 3 in a paper of Akō [1], p. 384.

§5. Proof of Theorem 1.

Let a be a point of D and let Σ_κ be a closed sphere defined by

$$\Sigma_\kappa=\{x;\ |x-a|\leqq\kappa\rho(a)\} \qquad (0<\kappa<1),$$

whose boundary is denoted by $\dot{\Sigma}_\kappa$.

If we put

$$\mu_\kappa=\operatorname*{Max}_{x\in\Sigma_\kappa}\{|\partial_x u(x)|\rho_\kappa(x)\},\ \rho_\kappa(x)=\text{dist}\,(x, \dot{\Sigma}_\kappa),$$

then there exists a point $x^{(0)}$ in the interior of Σ_κ, such that

$$|\partial_x u(x^{(0)})|\rho_\kappa(x^{(0)})=\mu_\kappa.$$

Let $x'=Tx$ be a linear transformation of variables, by which we have

$$\sum_{i,j=1}^n a_{ij}(x^{(0)})\partial_i\partial_j u(x)=\sum_{i=1}^n \partial_i'\partial_i' u'(x'), \qquad u'(x')=u(T^{-1}x'),$$

and further put

$$f(x, u, \partial_x u)=f'(x', u', \partial_{x'}u'),$$

$$\sum_{i,j=1}^n a_{ij}(x)\partial_i\partial_j u(x)=\sum_{i,j=1}^n b_{ij}(x')\partial_i'\partial_j' u'(x').$$

Now, let λ be a positive constant, and let $S_{(\lambda)}$ and $S_{(2\lambda)}$ be two concentric closed spheres with radii $\lambda\rho_\kappa(x^{(0)})$ and $2\lambda\rho_\kappa(x^{(0)})$ and having the point $x^{(0)\prime}=Tx^{(0)}$ as the common center.

If the constant λ is so small that

$$\lambda\leqq\frac{1}{8\sqrt{A}}<\frac{\sqrt{2}-1}{2\sqrt{2}\sqrt{A}}<\frac{1}{2\sqrt{A}}, \tag{5.1}$$

then, as in the proofs in the papers [3], [4],[10] we obtain

$$S_{(\lambda)}\subset S_{(2\lambda)}\subset T(\Sigma_\kappa),$$

and

$$|\partial_x u(x)|\leqq\sqrt{2}\rho_\kappa(x^{(0)})^{-1}\mu_\kappa, \tag{5.2}$$

10) See [3], pp. 62–63; [4], pp. 212–213.

(5. 3) $$|f'(x', u'(x'), \partial_{x'}u'(x'))| = |f(x, u(x), \partial_x u(x))| \leqq 2B\rho_\kappa(x^{(0)})^{-2}\mu_\kappa^2 + \Gamma$$

for any $x' = Tx \in S_{(2\lambda)} \subset T(\Sigma_\kappa)$.

Furthermore, let $G(x', \xi)$ be Green's function of the equation $\varDelta' u' \equiv \sum_{i=1}^n \partial_i'\partial_i' u' = 0$ with respect to the sphere $S_{(\lambda)}$ and let $h'(x')$ be the harmonic function defined in $S_{(\lambda)}$ and taking the boundary values $u'(x')$ on the boundary $\dot{S}_{(\lambda)}$ of the sphere $S_{(\lambda)}$. Then the following inequality can be obtained also in the same manner as in the proofs in the papers [3], [4]:

(5. 4) $$|\partial_{x'}u'(x^{(0)\prime})| \leqq (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}),$$

where

$$(\mathrm{I}) = |\partial_{x'}h'(x^{(0)\prime})|,$$

$$(\mathrm{II}) = \left|\omega_n^{-1}\int_{S_{(\lambda)}} \partial_{x'}G(x^{(0)\prime}, \xi) f'(\xi, u'(\xi), \partial_\xi u'(\xi)) d\xi\right|,$$

$$(\mathrm{III}) = \left|\omega_n^{-1}\int_{S_{(\lambda)}} \partial_{x'}G(x^{(0)\prime}, \xi) \sum_{i,j=1}^n (\delta_{ij} - b_{ij}(\xi))\partial_i'\partial_j' u'(\xi) d\xi\right|,$$

and ω_n denotes the surface area of the n-dimensional unit sphere.

Moreover, we see

(5. 5) $$(\mathrm{I}) \leqq \frac{B(n)}{\lambda\rho_\kappa(x^{(0)})} \operatorname*{Osc}_{S_{(\lambda)}} \{u'(x')\},$$

and

(5. 6) $$(\mathrm{II}) \leqq \frac{2B(n)}{n+1} \{2B\rho_\kappa(x^{(0)})^{-1}\mu_\kappa^2 + \Gamma\rho_\kappa(x^{(0)})\}\lambda,$$ [11]

where $B(n) = B(1/2, (n+1)/2)^{-1}$.

This time, we attack the estimate of the third term (III) in the right-hand side of the inequality (5. 4).

It follows from the property of the transformation $x' = Tx$ and the condition (3. 3), that

(5. 7) $$\left\{\sum_{i,j=1}^n (b_{ij}(x') - \delta_{ij})^2\right\}^{1/2} \leqq \frac{n}{\underline{A}}\left\{\sum_{i,j=1}^n (a_{ij}(x) - a_{ij}(x^{(0)}))^2\right\}^{1/2}$$
$$\leqq \frac{n}{\underline{A}}\varphi(|x - x^{(0)}|) \leqq \frac{n}{\underline{A}}\varphi(\sqrt{\overline{A}}|x' - x^{(0)\prime}|)$$

if $\sqrt{\overline{A}}|x' - x^{(0)\prime}| \leqq \delta$.

Consequently, by making use of the estimate

11) In order to form this estimate, we use Lemma 6.

$$|\partial_{x'}G(x^{(0)\prime}, \xi)| \leqq |\xi - x^{(0)\prime}|^{1-n}$$

and by putting $r=|\xi-x^{(0)\prime}|$ and $r_1=\lambda\rho_\kappa(x^{(0)})$, we obtain

$$\begin{aligned}(\text{III}) \quad &\leqq \frac{1}{\omega_n}\int_{S(\lambda)} |\partial_{x'}G(x^{(0)\prime}, \xi)| \left\{\sum_{i,j=1}^n (\delta_{ij}-b_{ij}(\xi))^2\right\}^{1/2} \left\{\sum_{i,j=1}^n (\partial_i'\partial_j'u'(\xi))^2\right\}^{1/2} d\xi \\ (5.8) \quad & \\ &\leqq \frac{n}{\omega_n \underline{A}}\int_{r\leqq r_1} \left\{\sum_{i,j=1}^n (\partial_i'\partial_j'u'(\xi))^2\right\}^{1/2} \varphi(\sqrt{\bar{A}}r) r^{1-n} d\xi,\end{aligned}$$

where the constant λ is taken so small that $\lambda\sqrt{\bar{A}}\rho_\kappa(x^{(0)}) \leqq \delta$.

Moreover, the inequality

$$(5.9) \qquad 2\rho_\kappa(x^{(0)}) \leqq 2\rho(a) \leqq d$$

shows that the inequality (5.8) has the meaning if only we take λ so small that $0<\lambda\leqq 2\delta/\sqrt{\bar{A}}d$.

Now, since

$$\begin{aligned} J_2 &\equiv \int_{r\leqq r_1} \left\{\sum_{i,j=1}^n (\partial_i'\partial_j'u'(\xi))^2\right\}^{1/2} \varphi(\sqrt{\bar{A}}r) r^{1-n} d\xi \\ &\leqq \left\{\int_{r\leqq r_1} \varphi(\sqrt{\bar{A}}r) r^{-n} d\xi\right\}^{1/2} \left\{\int_{r\leqq r_1} \sum_{i,j=1}^n (\partial_i'\partial_j'u'(\xi))^2 \varphi(\sqrt{\bar{A}}r) r^{2-n}\, d\xi\right\}^{1/2} \\ &= \left\{\omega_n \Phi(\sqrt{\bar{A}}r_1)\right\}^{1/2} \left\{\int_{r\leqq r_1} \sum_{i,j=1}^n (\partial_i'\partial_j'u'(\xi))^2 \varphi(\sqrt{\bar{A}}r) r^{2-n} d\xi\right\}^{1/2}, \end{aligned}$$

we get easily

$$J_2 \leqq \frac{\sqrt{\omega_n}}{(r_0^2-r_1^2)} \left\{\Phi(\sqrt{\bar{A}}r_1)\right\}^{1/2} \left\{\int_{r\leqq r_0} \sum_{i,j=1}^n (\partial_i'\partial_j'u'(\xi))^2 \varphi(\sqrt{\bar{A}}r) r^{2-n} (r_0^2-r^2)^2 d\xi\right\}^{1/2}$$

by putting $r_0=2r_1=2\lambda\rho_\kappa(x^{(0)})$ and by taking λ so small that

$$(5.10) \qquad 0<\lambda\leqq \frac{\delta}{\sqrt{\bar{A}}\,d}.$$

Hence, by observing

$$\Phi(\sqrt{\bar{A}}r_1) = \Phi\left(\sqrt{\bar{A}}\frac{r_0}{2}\right) \leqq \Phi(\sqrt{\bar{A}}r_0),$$

we have

$$(\text{III}) \quad \leqq K' r_0^{-2} \left\{\Phi(\sqrt{\bar{A}}r_0)\right\}^{1/2} \left\{\int_{r\leqq r_0} \sum_{i,j=1}^n (\partial_i'\partial_j'u'(\xi))^2 \varphi(\sqrt{\bar{A}}r) r^{2-n} (r_0^2-r^2)^2 d\xi\right\}^{1/2},$$

where $K'=4n/3\sqrt{\omega_n}\underline{A}$.

We next modify the above estimate by making use of Lemma 5 in §4.

The function $\varphi(\sqrt{\bar{A}}r)$ satisfies the condition $(D(r_0))$, and it is seen by virtue of the inequalities (5. 1), (5. 7) and (5. 10), that if the constant λ is taken so small that

$$\lambda \leqq \operatorname{Min}\left\{\frac{1}{8\sqrt{\bar{A}}}, \frac{\delta}{\sqrt{\bar{A}}\,d}\right\} \quad \text{and} \quad \frac{n}{\underline{A}}\varphi(\sqrt{\bar{A}}d\lambda) \leqq \frac{1}{2}, \tag{5. 11}$$

we can choose 1/2 as ε in Lemmas 4 and 5 in §4. Therefore, taking λ in this manner, we obtain by Lemma 5,

$$\begin{aligned}\text{(III)} &\leqq 2\sqrt{\omega_n}K'\{\Phi(\sqrt{\bar{A}}r_0)\}^{1/2} \\ &\cdot\left[r_0\{\varphi(\sqrt{\bar{A}}r_0)\}^{1/2}\operatorname*{Max}_{r\leqq r_0}\left|\sum_{i,j=1}^{n} b_{ij}(\xi)\partial_i{}'\partial_j{}'u'(\xi)\right| + c\{\Psi(\sqrt{\bar{A}}r_0\}^{1/2}\operatorname*{Max}_{r\leqq r_0}|\partial_{x'}u'(\xi)|\right], \\ &\leqq 2\sqrt{\omega_n}K'\Psi(\sqrt{\bar{A}}r_0)\left\{r_0\operatorname*{Max}_{r\leqq r_0}\left|\sum_{i,j=1} b_{ij}(\xi)\partial_i{}'\partial_j{}'u'(\xi)\right| + c\operatorname*{Max}_{r\leqq r_0}|\partial_{x'}u'(\xi)|\right\},\end{aligned}$$

where

$$\Psi(\sqrt{\bar{A}}r_0) = \{2\varphi(t) + \Phi(t)\}|_{t=\sqrt{\bar{A}}r_0} \tag{5. 12}$$

and c is the positive constant given in Lemma 3, which depends only on n.

On the other hand, it follows from the estimates (5. 2) and (5. 3), that

$$|\partial_{x'}u'(x')| \leqq \sqrt{\bar{A}}|\partial_x u(x)| \leqq \sqrt{2\bar{A}}\rho_\kappa(x^{(0)})^{-1}\mu_\kappa,$$

$$\begin{aligned}\left|\sum_{i,j=1}^{n} b_{ij}(x')\partial_i{}'\partial_j{}'u'(x')\right| &= |f'(x', u'(x'), \partial_{x'}u'(x')| \\ &\leqq 2B\rho_\kappa(x^{(0)})^{-2}\mu_\kappa{}^2 + \Gamma\end{aligned}$$

for any $x' = Tx \in S_{(2\lambda)}$, and hence, by putting

$$K = 4n/\underline{A}(>2\sqrt{2\omega_n}K'),$$

we have

$$\text{(III)} \leqq K\Psi(\sqrt{\bar{A}}d\lambda)[2\{2B\rho_\kappa(x^{(0)})^{-1}\mu_\kappa{}^2 + \Gamma\rho_\kappa(x^{(0)})\}\lambda + c\sqrt{\bar{A}}\rho_\kappa(x^{(0)})^{-1}\mu_\kappa], \tag{5. 13}$$

which is the last form of the estimate of the third term (III) in the right-hand side of the inequality (5. 4).

Now, since

$$|\partial_{x'}u'(x^{(0)\prime})| \geqq \sqrt{\underline{A}}\,|\partial_x u(x^{(0)})| = \sqrt{\underline{A}}\rho_\kappa(x^{(0)})^{-1}\mu_\kappa, \tag{5. 14}$$

it follows from the inequalities (5. 4), (5. 5), (5. 6) and (5. 13), that

$$\begin{aligned}&\sqrt{\underline{A}}\rho_\kappa(x^{(0)})^{-1}\mu_\kappa \\ &\leqq \frac{B(n)}{\lambda\rho_\kappa(x^{(0)})}\operatorname*{Osc}_{S_{(\lambda)}}\{u'(x')\}\end{aligned}$$

$$+\frac{2B(n)}{n+1}\{2B\rho_\kappa(x^{(0)})^{-1}\mu_\kappa^2+\Gamma\rho_\kappa(x^{(0)})\}\lambda$$

$$+K\Psi(\sqrt{\bar{A}}d\lambda)[2\{2B\rho_\kappa(x^{(0)})^{-1}\mu_\kappa^2+\Gamma\rho_\kappa(x^{(0)})\}\lambda+c\sqrt{\bar{A}}\rho_\kappa(x^{(0)})^{-1}\mu_\kappa],$$

and consequently we obtain

$$\lambda C_0\mu_\kappa^2-\{1-\Psi(\sqrt{\bar{A}}d\lambda)C_1\}\mu_\kappa+\frac{1}{\lambda}C_2\geqq 0, \tag{5.15}$$

where

$$C_0=\frac{4B}{\sqrt{\underline{A}}}\left\{\frac{B(n)}{n+1}+K\Psi(\sqrt{\bar{A}}d\lambda)\right\},\qquad C_1=Kc\sqrt{\frac{\bar{A}}{\underline{A}}},$$

$$C_2=\frac{B(n)}{\sqrt{\underline{A}}}\underset{|x-a|\leqq\rho(a)}{\mathrm{Osc}}\{u(x)\}+\frac{2}{\sqrt{\underline{A}}}\lambda^2\left\{\frac{B(n)}{n+1}+K\Psi(\sqrt{\bar{A}}d\lambda)\right\}\rho(a)^2\Gamma.$$

We remark here that, for the validity of the inequality (5. 15), it is sufficient to take λ so small that the inequality (5. 11) holds.

Furthermore, since

$$C_0C_2=\frac{4B}{\sqrt{\underline{A}}}\left\{\frac{B(n)}{n+1}+K\Psi(\sqrt{\bar{A}}d\lambda)\right\}\left[\frac{B(n)}{\sqrt{\underline{A}}}\underset{|x-a|\leqq\rho(a)}{\mathrm{Osc}}\{u(x)\}\right.$$
$$\left.+\frac{2}{\sqrt{\underline{A}}}\lambda^2\left\{\frac{B(n)}{n+1}+K\Psi(\sqrt{\bar{A}}d\lambda)\right\}\rho(a)^2\Gamma\right],$$

by virtue of the condition (3. 5), the constant λ can be taken so small that

$$\{1-\Psi(\sqrt{\bar{A}}d\lambda)C_1\}^2>4(\lambda C_0)(\lambda^{-1}C_2), \tag{5.16}$$

$$1-\Psi(\sqrt{\bar{A}}d\lambda)C_1\geqq\frac{1}{2}. \tag{5.17}$$

If we determine λ in this manner, then an algebraic equation in X:

$$\lambda C_0X^2-\{1-\Psi(\sqrt{\bar{A}}d\lambda)C_1\}X+\frac{1}{\lambda}C_2=0$$

possesses two distinct real positive roots R_1, $R_2(0<R_1<R_2)$, and we have either of the two following inequalities:

$$\mu_\kappa\leqq R_1\qquad\text{or}\qquad R_2\leqq\mu_\kappa.$$

We see however that μ_κ depends continuously on κ and $\lim_{\kappa\to+0}\mu_\kappa=0$, and therefore it holds that

$$\mu_\kappa\leqq R_1.$$

Thus, by making $\kappa\to1$, we get

$$|\partial_x u(a)| \leqq R_1 \rho(a)^{-1},$$

and by observing

$$R_1 \leqq \frac{4C_0C_2}{2\lambda C_0\{1-\Psi(\sqrt{\bar{A}}d\lambda)C_1\}} \leqq \frac{4C_2}{\lambda},$$

we have the desired estimate

$$|\partial_x u(a)| \leqq C^{(1)} \rho(a)^{-1} \operatorname*{Osc}_{|x-a| \leqq \rho(a)} \{u(x)\} + C^{(2)} \rho(a)$$

where $C^{(1)}$ and $C^{(2)}$ are positive constants depending only on $\bar{A}$, $\underline{A}$, B, Γ, N, n, d, δ and the function $\varphi(t)$.

Exactly writing, we obtain

$$C^{(1)} = \frac{4B(n)}{\lambda\sqrt{\underline{A}}}, \qquad C^{(2)} = \frac{8\lambda}{\sqrt{\underline{A}}}\left\{\frac{B(n)}{n+1} + K\Psi(\sqrt{\bar{A}}d\lambda)\right\}\Gamma,$$

where $B(n) = B(1/2, (n+1)/2)^{-1}$, $K = 4n/\underline{A}$ and

(5. 18) $$\Psi(t) = 2\varphi(t) + \int_0^t \frac{\varphi(t)}{t} dt.$$

The dependence of the constants $C^{(1)}$ and $C^{(2)}$ on the function $\varphi(t)$ is seen from the definition (5. 18) of the function $\Psi(t)$ and from the restriction that we must choose λ so small that the inequalities (5. 11), (5. 16) and (5. 17) hold.

§6. Other results.

In cases where the growth order of the function $f(x, u, p)$ with respect to p is less than in Theorem 1, we have the following results.

THEOREM 2. *If we suppose the condition*

(6. 1) $$|f(x, u, p)| \leqq B'|p| + \Gamma,$$

instead of the condition (3. 4) *in Theorem* 1, *then the condition* (3. 5) *may be omitted and we have*

(6. 2) $$|\partial_x u(a)| \leqq C^{(1)} \rho(a)^{-1} \operatorname*{Osc}_{|x-a| \leqq \rho(a)} \{u(x)\} + C^{(2)} \rho(a)$$

for any point $a \in D$, where $C^{(1)}$ and $C^{(2)}$ are positive constants depending only on $\underline{A}$, $\bar{A}$, B', Γ, n, d, δ and the function $\varphi(t)$.

Proof. In this case, we obtain, instead of (5. 6) and (5. 13),

(6. 3) $$(\mathrm{II}) \leqq \frac{2B(n)}{n+1}\{2B'\mu_\kappa + \Gamma\rho_\kappa(x^{(0)})\}\lambda,$$

(6. 4) $$(\mathrm{III})\leqq K\Psi(\sqrt{\bar{A}}d\lambda)[2\{2B'\mu_\kappa+\rho_\kappa(x^{(0)})\Gamma\}\lambda+c\sqrt{\bar{A}}\rho_\kappa(x^{(0)})^{-1}\mu_\kappa],$$

and therefore it follows from (5. 4), (5. 5), (5. 14), (6. 3) and (6. 4), that

$$\begin{aligned}\sqrt{\underline{A}}\rho_\kappa(x^{(0)})^{-1}\mu_\kappa \\ &\leqq \frac{B(n)}{\lambda\rho_\kappa(x^{(0)})}\operatorname*{Osc}_{S(\lambda)}\{u'(x')\}+\frac{B(n)}{n+1}\{2B'\mu_\kappa+\Gamma\rho_\kappa(x^{(0)})\}\lambda \\ &\quad+K\Psi(\sqrt{\bar{A}}d\lambda)[2\{2B'\mu_\kappa+\rho_\kappa(x^{(0)})\Gamma\}\lambda+c\sqrt{\bar{A}}\rho_\kappa(x^{(0)})^{-1}\mu_\kappa].\end{aligned}$$

We thus have

(6. 5) $$[1-\{\lambda C_1+\Psi(\sqrt{\bar{A}}d\lambda)C_1'\}]\mu_\kappa\leqq\frac{1}{\lambda}C_2,$$

where

$$C_1=\frac{B'd}{\sqrt{\underline{A}}}\left\{\frac{B(n)}{n+1}+2K\Psi(\sqrt{\bar{A}}d\lambda)\right\},\qquad C_1'=Kc\sqrt{\frac{\bar{A}}{\underline{A}}}$$

$$C_2=\frac{1}{\sqrt{\underline{A}}}\left[B(n)\operatorname*{Osc}_{|a-a|\leqq\rho(a)}\{u(x)\}+\lambda^2\left\{\frac{B(n)}{n+1}+2K\Psi(\sqrt{\bar{A}}d\lambda)\right\}\rho(a)^2\Gamma\right].$$

By taking λ so small that

(6. 6) $$\lambda\leqq\mathrm{Min}\left\{\frac{1}{8\sqrt{\bar{A}}},\ \frac{\delta}{\sqrt{\bar{A}}d}\right\},$$
$$\varphi(\sqrt{\bar{A}}d\lambda)\leqq\frac{\underline{A}}{2n},\quad \lambda C_1+\Psi(\sqrt{\bar{A}}d\lambda)C_1'\leqq\frac{1}{2},$$

we obtain

(6. 7) $$\mu_\kappa\leqq\frac{2}{\lambda}C_2,$$

from which the estimate (6. 2) follows along the same lines as in the proof of Theorem 1.

THEOREM 3. *If we suppose the condition*

(6. 8) $$|f(x,u,p)|\leqq\Gamma$$

instead of the condition (3. 4) *in Theorem* 1, *then the condition* (3. 5) *may be omitted and we have*

(6. 9) $$|\partial_x u(a)|\leqq K^{(1)}\rho(a)^{-1}\operatorname*{Osc}_{|x-a|\leqq\rho(a)}\{u(x)\}+K^{(2)}\rho(a)$$

for any point $a\in D$, *where* $K^{(1)}$ *and* $K^{(2)}$ *are positive constants depending only on* $\underline{A}$, $\bar{A}$, Γ, n, d, δ *and on the function* $\varphi(t)$.

Proof. In this case, we get also the inequality (6. 5) and we see that $C_1=0$, and C_1', C_2 are the same as in the proof of Theorem 2.

Hence, by choosing λ so small that

$$\lambda \leqq \mathrm{Min}\left\{\frac{1}{8\sqrt{\bar{A}}}, \frac{\delta}{\sqrt{\bar{A}}d}\right\},$$

(6. 10)

$$\varphi(\sqrt{\bar{A}}d\lambda) \leqq \frac{\underline{A}}{2n}, \qquad \Psi(\sqrt{\bar{A}}d\lambda) \leqq \frac{\sqrt{\underline{A}^3}}{8nc\sqrt{\bar{A}}},$$

we have the inequality (6. 7), from which the estimate (6. 9) follows along the same lines as in the proof of Theorem 1.

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