ON INFINITESIMAL TRANSFORMATIONS OF ALMOST-KÄHLERIAN SPACE AND K-SPACE

By SUMIO SAWAKI

1. Introduction.

In the present paper, we shall consider mainly infinitesimal conformal and projective transformations of almost-Kählerian space, K-space and, as their special case, Kählerian space. Under what conditions do these transformations become isometric? Even though there are many papers about this problem, it seems to the author that there exist few about non-compact spaces. Recently Couty [1] proved that in an almost-Kähler-Einstein space with positive scalar curvature, an infinitesimal projective transformation is necessarily an isometry and Tashiro [7] proved that in a Kählerian space with non-vanishing constant curvatue scalar, an infinitesimal conformal transformation is necessarily an isometry. In this paper, we shall deal with the same problem but throughout this paper we do not assume that the space is compact. In § 2 we shall state some properties of almost-Kählerian space and K-space for later use. In § 3 we shall obtain sufficient conditions for an infinitesimal conformal transformation to be an isometry and especially a condition corresponding to Couty's result on a projective transformation and give a decomposition of an infinitesimal conformal transformation in an Einstein K-space. In § 4 we shall deal with the same problem of an infinitesimal projective transformation. A remark on the result obtained by Tachibana [4] about an infinitesimal analytic conformal transformation in a K-space will be given in the last § 5.

2. Almost-Kählerian space and K-space.

Let X_{2n} be a $2n$-dim. almost-complex space[1] and $\varphi_j{}^i$ its almost-complex structure, then by definition we have

$$\varphi_j{}^s\varphi_s{}^i=-\delta_j^i. \tag{2.1}$$

An almost-complex space with a positive definite Riemannian metric g_{ji} satisfying

$$g_{rs}\varphi_j{}^r\varphi_i{}^s=g_{ji} \tag{2.2}$$

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1) For example, see Yano [8]. Indices run over 1, 2, $\cdots$, $2n$.

is called an almost-Hermitian space. From (2.2) it follows that $\varphi_{ji}\equiv g_{ri}\varphi_j{}^r$ is skew-symmetric.

If an almost-Hermitian space satisfies

$$\nabla_j\varphi_{ih}+\nabla_i\varphi_{hj}+\nabla_h\varphi_{ji}=0 \tag{2.3}$$

where ∇_j denotes the operator with respect to Riemannian connection, then it is called an almost-Kählerian space and if it satisfies

$$\nabla_j\varphi_{ih}+\nabla_i\varphi_{jh}=0 \tag{2.4}$$

then it is called a K-space. In an almost-Hermitian space, if $\nabla_j\varphi_{ih}=0$, then it is called a Kählerian space.

Now, first of all, we shall assume we are in an almost-Kählerian space and transvecting (2.3) with $\varphi^{ji}\equiv g^{rj}\varphi_r{}^i$, we have

$$2\varphi^{ji}\nabla_j\varphi_{ih}+\varphi^{ji}\nabla_h\varphi_{ji}=0 \quad \text{or} \quad 2\varphi_{ih}\nabla_j\varphi^{ij}+\varphi^{ji}\nabla_h\varphi_{ji}=0.$$

From the last equation we have $2\varphi_{ih}\nabla_j\varphi^{ij}=0$ and therefore

$$\nabla_j\varphi_r{}^j=0, \tag{2.5}$$

because by (2.1) $\varphi^{ji}\nabla_h\varphi_{ji}=0$.

Operating $\nabla^k\nabla_k\equiv g^{rk}\nabla_r\nabla_k$ to (2.1), we have

$$\nabla^k\nabla_k(\varphi_j{}^s\varphi_s{}^i)=(\nabla^k\nabla_k\varphi_j{}^s)\varphi_s{}^i+2(\nabla_k\varphi_j{}^s)\nabla^k\varphi_s{}^i+\varphi_j{}^s\nabla^k\nabla_k\varphi_s{}^i=0. \tag{2.6}$$

On the other hand, let $R_{kji}{}^h$ and $R_{ji}\equiv R_{lji}{}^l$ be Riemannian curvature tensor and Ricci tensor respectively, then by the Ricci's identity and (2.5), we get

$$\varphi_j{}^r\nabla_s\nabla_r\varphi_i{}^s=\varphi_j{}^r(\nabla_r\nabla_s\varphi_i{}^s-R_{sri}{}^l\varphi_l{}^s+R_{srl}{}^s\varphi_i{}^l)=-\varphi_j{}^r\varphi_l{}^sR_{sri}{}^l+\varphi_j{}^r\varphi_i{}^lR_{rl} \tag{2.7}$$

from which by the Bianchi's identity we get

$$\varphi_j{}^r\nabla^s\nabla_r\varphi_{is}=\varphi_j{}^r\varphi_i{}^lR_{rl}-R^*{}_{ji} \tag{2.8}$$

where

$$R^*{}_{ji}\equiv(1/2)\varphi^{sl}R_{slri}\varphi_j{}^r.$$

Similarly we have

$$\varphi_j{}^r\nabla^s\nabla_i\varphi_{sr}=R_{ji}-R^*{}_{ji}. \tag{2.9}$$

Forming next the sum (2.8)+(2.9), we have

$$\varphi_j{}^r\nabla^s(\nabla_r\varphi_{is}+\nabla_i\varphi_{sr})=\varphi_j{}^r\varphi_i{}^lR_{rl}+R_{ji}-2R^*{}_{ji}$$

or using (2.3)

(2.10) $$-\varphi_j{}^r \nabla^s \nabla_s \varphi_{ri} = \varphi_j{}^r \varphi_i{}^l R_{rl} + R_{ji} - 2R^*{}_{ji}.$$

Substituting (2.10) into (2.6), we have

(2.11) $$2(\nabla_k \varphi_j{}^s) \nabla^k \varphi_{si} = \varphi_i{}^r \varphi_j{}^l R_{rl} + R_{ij} - 2R^*{}_{ij} + \varphi_j{}^r \varphi_i{}^l R_{rl} + R_{ji} - 2R^*{}_{ji}$$

and therefore for any vector field v, from (2.11) we have

(2.12) $$\nabla_k \varphi_j{}^s (\nabla^k \varphi_{si}) v^j v^i = \varphi_j{}^r \varphi_i{}^l R_{rl} v^j v^i + R_{ji} v^j v^i - 2R^*{}_{ji} v^j v^i \leqq 0.$$

For an almost-Kähler-Einstein space:

$$R_{ji} = \frac{R}{2n} g_{ji}$$

where $R \equiv g^{ji} R_{ji}$, (2.12) becomes

(2.13) $$(R_{ji} - R^*{}_{ji}) v^j v^i \leqq 0.$$

Thus we have the following

LEMMA. *In an almost-Kählerian space, the inequality*

$$(2R^*{}_{ji} - R_{ji} - \varphi_j{}^r \varphi_i{}^l R_{rl}) v^j v^i \geqq 0$$

is valid for any vector field v and in an almost-Kähler-Einstein space

$$(R^*{}_{ji} - R_{ji}) v^j v^i \geqq 0$$

is valid for any vector field v.

In the next place, let us assume we are in a K-space. In a K-space, we know the following identities obtained by Tachibana [5]:

(2.14) $$R^*{}_{ji} = R^*{}_{ij},$$

(2.15) $R_{ji} - R^*{}_{ji} = (\nabla_j \varphi_{rs}) \nabla_i \varphi^{rs}$, $\quad (R_{ji} - R^*{}_{ji}) v^j v^i \geqq 0$ for any vector field v,

(2.16) $$R - R^* = \text{constant} \geqq 0,$$

where $R^* \equiv g^{ji} R^*{}_{ji}$,

(2.17) $\nabla^h N(v)_h = 0$ for any vector field v,

where $N(v)_h \equiv \varphi_h{}^t (\nabla_t \varphi_{rs}) \nabla^r v^s$.

In general,

(2.18) $$\nabla^j R_{ji} = \frac{1}{2} \nabla_i R$$

is a well known identity and the present author [3] proved in a K-space

$$\nabla^j R^*{}_{ji}=\frac{1}{2}\nabla_i R^*. \tag{2.19}$$

3. Infinitesimal conformal transformations.

Let v^i be an infinitesimal conformal transformation in an almost-Hermitian space X_{2n}, then by definition there exists a scalar function ρ satisfying

$$\nabla_j v_i+\nabla_i v_j=2\rho g_{ji} \tag{3.1}$$

and as is well known, we have

$$\nabla_j\nabla_i v^h+R_{rji}{}^h v^r=\rho_j\delta_i^h+\rho_i\delta_j^h-\rho^h g_{ji},\qquad \rho_i\equiv\nabla_i\rho,\qquad \rho=\frac{1}{2n}\nabla_r v^r, \tag{3.2}$$

$$\nabla^r\nabla_r v^i+R_r{}^i v^r+\frac{n-1}{n}\nabla^i\nabla_r v^r=0. \tag{3.3}$$

Multiplying (3.2) by $\varphi_l{}^j\varphi_h{}^i$, we have

$$\varphi_l{}^j\varphi_h{}^i\nabla_j\nabla_i v^h+2R^*{}_{lr}v^r=-2\rho_l \quad \text{i.e.}$$

$$\varphi_l{}^j\nabla_j(\varphi_h{}^i\nabla_i v^h)+\varphi_l{}^j(\nabla_j\varphi_{hi})\nabla^h v^i+2R^*{}_{lr}v^r=-2\rho_l. \tag{3.4}$$

First of all, let X_{2n} be an almost-Kähler-Einstein space with non-vanishing scalar curvature, then it is well known that v^i is decomposed into

$$v^i=p^i+k\eta^i,\qquad k=\text{constant} \tag{3.5}$$

where p^i is a Killing vector and η^i is a gradient vector defining an infinitesimal conformal transformation [2]. Consequently, for η^i by (3.3) we have

$$\nabla^r\nabla_r\eta^i+R_r{}^i\eta^r+\frac{n-1}{n}\nabla^i\nabla_r\eta^r=0. \tag{3.6}$$

Since $\nabla_r\nabla^r\eta^i=\nabla_r\nabla^i\eta^r=\nabla^i\nabla_r\eta^r+R_r{}^i\eta^r$, (3.6) turns to

$$\frac{2n-1}{n}\nabla_i\nabla_r\eta^r+2R_{ir}\eta^r=0. \tag{3.7}$$

Let the scalar function for η^i be λ, then from (3.1) and (3.4), we have

$$\frac{1}{2n}\nabla_r\eta^r=\lambda,\qquad R^*{}_{ir}\eta^r=-\lambda_i,\qquad \lambda_i\equiv\nabla_i\lambda$$

respectively and therefore combining these two equations, we have

(3.8)
$$\frac{1}{2n}\nabla_i\nabla_r\eta^r+R^*{}_{ir}\eta^r=0.$$

Eliminating $\nabla_i\nabla_r\eta^r$ from (3.7) and (3.8), we get

$$(2n-1)R^*{}_{ir}\eta^r-R_{ir}\eta^r=0$$

from which it follows that

(3.9)
$$(2n-1)(R^*{}_{ir}-R_{ir})\eta^i\eta^r+2(n-1)R_{ir}\eta^i\eta^r=0$$

where $R_{ir}=(R/2n)g_{ir}$.

But in this place, according to Lemma in § 2, $(R^*{}_{ir}-R_{ir})\eta^i\eta^r\geqq 0$ and therefore if $R>0$, then from (3.9) we have

$$\frac{2(n-1)}{2n}R\eta_r\eta^r=0,$$

hence, for $n>1$, $\eta^r=0$. Thus from (3.5) we have the following

THEOREM 1.[2] *In an almost-Kähler-Einstein space* ($n>1$) *with positive scalar curvature, an infinitesimal conformal transformation is necessarily an isometry.*

Secondly, assume that X_{2n} is a K-space with constant curvature scalar (i. e. $\nabla_iR=0$). From (3.3), we can easily deduce

$$\frac{2n-1}{n}\nabla^r\nabla_r\nabla_iv^i+2\nabla_i(R_r{}^iv^r)=0$$

or making use of $\rho=\dfrac{1}{2n}\nabla_iv^i$

(3.10)
$$\nabla^r\rho_r+\frac{1}{2n-1}\nabla_i(R_r{}^iv^r)=0.$$

Operating ∇^l to (3.4) and using (2.17) and $\nabla^r\varphi_r{}^j=0$, we have

(3.11)
$$\nabla^l(R^*{}_{lr}v^r)=-\nabla^l\rho_l.^{3)}$$

Accordingly, from (3.10) and (3.11), we have

(3.12)
$$\frac{1}{2n-1}\nabla_i(R_r{}^iv^r)-\nabla_i(R^*{}_r{}^iv^r)=0.$$

But since, by (2.16), (2.18) and (2.19), we find

2) For a compact Kähler-Einstein space, see Yano [8], p. 277.

3) See Tachibana [6].

$$0=\nabla^r R_{ri}=\frac{1}{2}\nabla_i R=\frac{1}{2}\nabla_i R^*=\nabla^r R^*_{ri},$$

from (3.12) it follows

$$\frac{1}{2n-1}R^{ir}\nabla_i v_r-R^{*ir}\nabla_i v_r=0. \tag{3.13}$$

On the other hand, transvecting (3.1) with symmetric tensors R^{ji} and R^{*ji}, we have

$$R^{ji}\nabla_j v_i=\rho R, \qquad R^{*ji}\nabla_j v_i=\rho R^* \tag{3.14}$$

respectively, so from (3.13) we obtain

$$\left(\frac{1}{2n-1}R-R^*\right)\rho=0. \tag{3.15}$$

Consequently if $(1/(2n-1))R-R^*\neq 0$, then we have $\rho=0$. Thus, from (3.1), we have the following

THEOREM 2. *In a K-space with constant curvature scalar, if* $(1/(2n-1))R-R^*\neq 0$, *then an infinitesimal conformal transformation is necessarily an isometry.*

According to this theorem and (2.16), we have

COROLLARY. *In a K-space* $(n>1)$ *with constant curvature scalar, if* $R<0$ *or* $R^*<0$ *or* $R>0$ *and* $R^*=0$, *etc., then an infinitesimal conformal transformation is necessarily an isometry.*

When the space is a Kählerian space, $R=R^*$ and hence from Theorem 2 we have

COROLLARY. (Tashiro [7]) *In a Kählerian space* $(n>1)$ *with non vanishing constant curvature scalar, an infinitesimal conformal transformation is necessarily an isometry.*

Again returning to a K-space, if we consider a homothetic motion, then by definition we have

$$\nabla_j v_i+\nabla_i v_j=2cg_{ji}, \qquad c=\text{constant} \tag{3.16}$$

from which it follows

$$\rho=\frac{1}{2n}\nabla_i v^i=c. \tag{3.17}$$

Transvecting (3.16) with R^{ji} and R^{*ji}, we have

(3.18) $$R^{ji}\nabla_j v_i = cR, \qquad R^{*ji}\nabla_j v_i = cR^*$$

respectively.

Making use of (3.17) and (3.18), from (3.10) it follows

(3.19) $$cR + v^r \nabla_i R_r{}^i = 0$$

and from (3.11)

(3.20) $$cR^* + v^r \nabla_i R^*{}_r{}^i = 0.$$

But, by virtue of (2.16), (2.18) and (2.19), we have

(3.21) $$\nabla_i R_r{}^i - \nabla_i R^*{}_r{}^i = \frac{1}{2}(\nabla_r R - \nabla_r R^*) = 0$$

and then forming the difference (3.19)−(3.20), we have

(3.22) $$c(R - R^*) = 0.$$

Thus, in a K-space such that $R \neq R^*$ (which will be called a proper K-space), from (3.22) we can deduce $c=0$ and therefore from (3.16) we have the following

THEOREM 3. *In a proper K-space, a homothetic motion is necessarily a motion.*

Finally, suppose that we are in an Einstein K-space with non-vanishing scalar curvature, then v^i is decomposed in the form

(3.23) $$v^i = p^i + k\eta^i, \qquad k = \text{constant}$$

and let the scalar function for η^i be λ, then from (3.1) we have

(3.24) $$\nabla_j \eta_i = \lambda g_{ji}.$$

Consequently, putting

(3.25) $$q^i = -k\varphi_r{}^i \eta^r$$

and operating ∇^j to (3.25), we have

$$\nabla^j q^i = -k(\nabla^j \varphi_r{}^i)\eta^r - k\varphi_r{}^i \nabla^j \eta^r$$

or substituting (3.24) into this equation

(3.26) $$\nabla^j q^i = -k(\nabla^j \varphi_r{}^i)\eta^r - k\lambda\varphi^{ji}$$

from which it follows

(3.27) $$\nabla^j q^i + \nabla^i q^j = 0$$

i. e. q^i is a Killing vector. Since, from (3.25) we get $k\eta^i = \varphi_r{}^i q^r$, we can state

THEOREM 4. *In an Einstein K-space with non-vanishing scalar curvature, any vector v^i defining an infinitesimal conformal transformation is decomposed in the form*

$$v^i = p^i + \varphi_r{}^i q^r$$

where p^i and q^i are both Killing vectors and $\varphi_r{}^i q^r$ is a gradient vector. The decomposition stated above is unique.

4. Infinitesimal projective transformations.

Let v^i be an infinitesimal projective transformation in an almost-Hermitian space X_{2n}, then by definition there exists a vector ρ_i such that

$$\nabla_j \nabla_i v^h + R_{rji}{}^h v^r = \rho_j \delta_i^h + \rho_i \delta_j^h \tag{4.1}$$

from which we have

$$\nabla^r \nabla_r v^i + R_r{}^i v^r = 2\rho^i, \qquad \rho^i = \frac{1}{2n+1} \nabla^i \nabla_r v^r. \tag{4.2}$$

By the same method in § 3, we have the following equation corresponding to (3.4):

$$\varphi_l{}^j \nabla_j (\varphi_h{}^i \nabla_i v^h) + \varphi_l{}^j (\nabla_j \varphi_{hi}) \nabla^h v^i + 2R^*{}_{lr} v^r = -\rho_l. \tag{4.3}$$

Here, as is well known, if X_{2n} is an Einstein space with non vanishing scalar curvature, then v^i is decomposed into

$$v^i = p^i + k\eta^i, \qquad k = \text{constant} \tag{4.4}$$

where p^i is a Killing vector and η^i is a gradient vector defining an infinitesimal projective transformation [8].

Thus, for η^i by (4.2) we have

$$\nabla^r \nabla_r \eta^i + R_r{}^i \eta^r = \frac{2}{2n+1} \nabla^i \nabla_r \eta^r$$

from which it follows

$$\frac{2n-1}{2n+1} \nabla_i \nabla_r \eta^r + 2R_i{}^r \eta_r = 0 \tag{4.5}$$

and from (4.3)

$$2R^*{}_{ir} \eta^r + \frac{1}{2n+1} \nabla_i \nabla_r \eta^r = 0. \tag{4.6}$$

Eliminating $\nabla_i \nabla_r \eta^r$ from (4.5) and (4.6), we have

$$(2n-1) R^*{}_{ir} \eta^r - R_{ir} \eta^r = 0. \tag{4.7}$$

Multiplying (4.7) by η^i and rewriting, we get

$$(2n-1)(R^*_{ji}-R_{ji})\eta^j\eta^i+2(n-1)R_{ji}\eta^j\eta^i=0. \tag{4.8}$$

In this place, when X_{2n} is an almost-Kähler-Einstein space, by Lemma in §2, $(R^*_{ji}-R_{ji})\eta^j\eta^i\geqq 0$. Accordingly, if $R>0$, then from (4.8) we find $((n-1)/n)R\eta^i\eta_i=0$ and therefore $\eta^i=0$.

Hence we have the following

THEOREM 5. (Couty [1]) *In an almost-Kähler-Einstein space* $(n>1)$ *with positive scalar curvature, an infinitesimal projective transformation is necessarily an isometry.*

And if X_{2n} is an Einstein K-space, then from (2.15) $(R^*_{ji}-R_{ji})\eta^j\eta^i\leqq 0$ and therefore if $R<0$, then from (4.8) we have $((n-1)/n)R\eta^i\eta_i=0$.

Hence we have the following

THEOREM 6. *In an Einstein K-space* $(n>1)$ *with negative scalar curvature, an infinitesimal projective transformation is necessarily an isometry.*

As a special case, when X_{2n} is a Kähler-Einstein space, taking account of $R_{ji}=R^*_{ji}$, from (4.8) we have

COROLLARY.[4] *In a Kähler-Einstein space* $(n>1)$ *with non vanishing scalar curvature, an infinitesimal projective transformation is necessarily an isometry.*

5. Infinitesimal analytic conformal transformations in a K-space.

In an almost-Hermitian space X_{2n}, if v^i satisfies

$$\underset{v}{\mathcal{L}}\varphi_j{}^i\equiv v^r\nabla_r\varphi_j{}^i-\varphi_j{}^r\nabla_r v^i+\varphi_r{}^i\nabla_j v^r=0 \text{ [5]} \tag{5.1}$$

where $\underset{v}{\mathcal{L}}$ is the operator of Lie derivative, then v^i is called a contravariant almost-analytic vector. When a vector v^i defining an infinitesimal conformal transformation is almost-analytic, we shall say that v^i is an infinitesimal analytic conformal transformation.

Now, we consider an infinitesimal analytic conformal transformation of a K-space.

Multiplying (5.1) by $\nabla_k\varphi_i{}^j$, we have

$$v^r(\nabla_r\varphi_j{}^i)\nabla_k\varphi_i{}^j-\varphi_j{}^r(\nabla_k\varphi_i{}^j)\nabla_r v^i+\varphi_r{}^t(\nabla_k\varphi_i{}^j)\nabla_j v^r=0$$

or by (2.1)

4) For a compact Kähler-Einstein space, see Yano [8], p. 273.

5) See Tachibana [4].

(5.2) $$v^r(\nabla_r\varphi_j{}^i)\nabla_k\varphi_i{}^j-2\varphi_j{}^r(\nabla_k\varphi_i{}^j)\nabla_r v^i=0.$$

In this place, by (2.1) and (2.4), we see

$$\varphi_j{}^r(\nabla_k\varphi_i{}^j)\nabla_r v^i=-\varphi_j{}^r(\nabla_i\varphi_k{}^j)\nabla_r v^i=\varphi_k{}^j(\nabla_i\varphi_j{}^r)\nabla_r v^i=\varphi_k{}^j(\nabla_j\varphi_{ri})\nabla^r v^i=N(v)_k$$

so (5.2) turns to

$$v^r(\nabla_r\varphi_{ji})\nabla_k\varphi^{ji}+2N(v)_k=0$$

or by (2.15)

(5.3) $$v^r(R_{kr}-R^*{}_{kr})+2N(v)_k=0.$$

Operating ∇^k to (5.3) and using (2.17), we have

(5.4) $$\nabla^k v^r(R_{kr}-R^*{}_{kr})+v^r(\nabla^k R_{kr}-\nabla^k R^*{}_{kr})=0$$

but since by (3.21) we have

$$\nabla^k R_{kr}-\nabla^k R^*{}_{kr}=0,$$

(5.4) becomes

(5.5) $$\nabla^k v^r(R_{kr}-R^*{}_{kr})=0.$$

Moreover, v^i being an infinitesimal conformal transformation, then making use of (3.14), (5.5) turns to

(5.6) $$\rho(R-R^*)=0$$

from which it follows that $\rho=0$ if $R\neq R^*$.

Thus we have the following

THEOREM 7.[6] *In a proper K-space, an infinitesimal analytic conformal transformation is necessarily an isometry.*

BIBLIOGRAPHY

[1] COUTY, R., Transformations projectives des variétés presque kählériennes. C. R. Acad. Sci. Paris **254** (1962), 4132–4135.

[2] LICHNEROWICZ, A., Some problems on transformations of Riemannian and Kählerian manifolds. Mimeographed Notes, Princeton (1956).

[3] SAWAKI, S., On the Matsushima's theorem in a compact Einstein K-space. Tôhoku Math. Jour. **13** (1961), 455–465.

[4] TACHIBANA, S., On almost-analytic vectors in almost-Kählerian manifolds. Tôhoku Math. Jour. **11** (1959), 247–265.

6) For a compact K-space, see Tachibana [5].

[5] TACHIBANA, S., On almost-analytic vectors in certain almost-Hermitian manifolds. Tôhoku Math. Jour. **11** (1959). 351–363.
[6] TACHIBANA, S., On infinitesimal conformal and projective transformations of compact K-spaces. Tôhoku Math. Jour. **13** (1961), 386–392.
[7] TASHIRO, Y., On conformal and projective transformations in Kählerian manifolds. Tôhoku Math. Jour. **14** (1962), 317–320.
[8] YANO, K., The theory of Lie derivatives and its applications. Amsterdam. (1957)

NIIGATA UNIVERSITY.