CERTAIN ALMOST CONTACT HYPERSURFACES IN EUCLIDEAN SPACES

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Introduction.

An odd-dimensional differentiable manifold M^{2n+1} is said to have an almost contact structure or to be an almost contact manifold if the structural group of its tangent bundle is reducible to the product of a unitary group with the 1-dimensional identity group [3].[1] Recently Sasaki and Hatakeyama [4, 5] proved that an almost contact structure is equivalent to the existence of a set of tensor fields ϕ, ξ, η of the type (1, 1), (1, 0) and (0, 1) satisfying the following five conditions:

$$\xi^i\eta_i = 1, \tag{0.1}$$

$$\operatorname{rank}(\phi_j{}^i) = 2n, \tag{0.2}$$

$$\phi_j{}^i\xi^j = 0, \tag{0.3}$$

$$\phi_j{}^i\eta_i = 0, \tag{0.4}$$

$$\phi_j{}^i\phi_k{}^j = -\delta_k^i+\eta_k\xi^i. \tag{0.5}$$

This permits us to study almost contact structures by use of the tensor calculus. They also proved that we can introduce an associated Riemannian metric tensor which satisfies both of the relations

$$g_{ji}\xi^i = \eta_j, \tag{0.6}$$

$$g_{rs}\phi_i{}^r\phi_j{}^s = g_{ij} - \eta_i\eta_j. \tag{0.7}$$

We call an almost contact metric structure an almost contact structure with this associated Riemannian metric.

On the other hand, Tashiro [7] proved that any orientable differentiable hypersurface in an almost Hermitian manifold admits an almost contact structure and that the Riemannian metric induced on the hypersurface is an associated metric of the almost contact structure.

Thus, an even-dimensional Euclidean space E^{2n} being regarded as a flat Kaehlerian manifold, any differentiable hypersurface of E^{2n} has an induced almost contact metric structure. The purpose of this paper is to study certain almost contact hypersurfaces and to show that, in a Euclidean space E^{2n}, only E^{2n-1}, S^{2n-1} and $E^r\times S^{2n-r-1}$ can admit induced normal (see [6]) almost contact metric structure.

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1) The numbers in the brackets refer to the bibliography at the end of the paper.

In §1, we give first of all some formulas concerning hypersurfaces in an almost Hermitian manifold. In §2, we consider Nijenhuis tensors of the induced almost contact structure and give some lemmas for the later use. Hypersurfaces in Euclidean spaces E^{2n} are discussed in §3, and in this section we prove some fundamental identities. After these preliminaries we shall, in §4, prove that in a normal almost contact hypersurface of E^{2n}, the second fundamental tensor can have only two characteristic roots and that they are both constant except for a certain case. From this fact, we can deduce a classification of the normal contact hypersurfaces by the method similar to that used by Yano and Sumitomo [11] for hypersurfaces in a Cayley space. Finally in §5, we show that we can induce in $S^r \times E^s$ a structure which has been recently introduced by Yano [10].

1. Induced almost contact structure of a hypersurface in an almost Hermitian manifold.

Let us consider a real $2n$-dimensional almost Hermitian manifold M^{2n} with local coordinate systems $\{X^\kappa\}$ and let $(F_\mu{}^\lambda, G_{\mu\lambda})$ be the almost Hermitian structure, that is, $F_\mu{}^\lambda$ be the almost complex structure defined on M^{2n} and $G_{\mu\lambda}$ be the Riemannian metric tensor satisfying $G_{\kappa\lambda}=G_{\mu\nu}F_\kappa{}^\mu F_\lambda{}^\nu$. A differentiable hypersurface M^{2n-1} of M^{2n} may be represented parametrically by the equation $X^\lambda=X^\lambda(x^i)$.[2)] If we put

$$B_j{}^\kappa=\partial_i X^\kappa \qquad (\partial_i=\partial/\partial x^i), \tag{1.1}$$

they span a tangent plane of M^{2n-1} at each point, and induced Riemannian metric g_{ji} in M^{2n-1} is given by

$$g_{ji}=G_{\mu\lambda}B_j{}^\mu B_i{}^\lambda. \tag{1.2}$$

Assuming that our hypersurface is orientable, we choose the unit normal vector C^κ to the hypersurface and put

$$\phi_j{}^i=B_j{}^\lambda F_\lambda{}^\kappa B^i{}_\kappa, \tag{1.3}$$

$$\eta_j=B_j{}^\mu F_\mu{}^\lambda C_\lambda=B_j{}^\mu F_{\mu\lambda}C^\lambda, \tag{1.4}$$

where we have put

$$B^j{}_\kappa=G_{\lambda\kappa}B_i{}^\lambda g^{ji}, \tag{1.5}$$

$C_\lambda=G_{\lambda\kappa}C^\kappa$ and $F_{\lambda\mu}=F_\lambda{}^\kappa G_{\kappa\mu}$.

Then it is known that the aggregate $(\phi_j{}^i, g^{ir}\eta_r, \eta_j, g_{ji})$ defines an almost contact metric structure in the hypersurface. Denote by ∇_j the covariant differentiation with respect to the Christoffel symbol formed from the induced Riemannian metric g_{ji} and consider the covariant derivation of η_j and $\phi_{ji}=g_{ir}\phi_j{}^r$ for the later use. At first, remember the following formulas:

$$\nabla_j B_i{}^\kappa=H_{ji}C^\kappa, \tag{1.6}$$

2) In this paper Greek indices run on the range 1, 2, ⋯, $2n$ and small Latin indices run on the range 1, 2, ⋯, $2n-1$.

$$\nabla_j C_\kappa = -H_{jk} B^k{}_\kappa. \tag{1.7}$$

The left hand side of these equations are so-called Bortolotti-van der Waerden covariant derivatives and H_{ji} is the second fundamental tensor of the hypersurface M^{2n-1}. By means of these formulas and anti-symmetric property of $F_{\lambda\mu}$, we have

$$\nabla_j \eta_i = B_i{}^\mu B_j{}^\nu \nabla_\nu F_\mu{}^\lambda C_\lambda - B_i{}^\mu F_\mu{}^\lambda B^r{}_\lambda H_{jr},$$

$$\nabla_j \phi_{ih} = H_{ji} C^\kappa F_{\kappa\lambda} B_h{}^\lambda + B_i{}^\mu B_j{}^\nu \nabla_\nu F_{\mu\lambda} B_h{}^\lambda + B_i{}^\mu F_\mu{}^\lambda H_{jh} C^\lambda.$$

Substituting (1. 4) and (1. 5) into the above equations, we get

$$\nabla_j \eta_i = B_i{}^\mu B_j{}^\nu \nabla_\nu F_{\mu\lambda} C^\lambda - \phi_i{}^r H_{jr}, \tag{1.8}$$

$$\nabla_j \phi_{ih} = B_i{}^\mu B_j{}^\kappa \nabla_\kappa F_{\mu\lambda} B_h{}^\lambda + \eta_i H_{jh} - \eta_h H_{ji}. \tag{1.9}$$

If the almost Hermitian manifold M^{2n} is a Kaehlerian manifold, that is, it satisfies $\nabla_\nu F_{\mu\lambda}=0$, (1. 8) and (1. 9) change their forms as

$$\nabla_j \eta_i = -\phi_i{}^r H_{jr}, \tag{1.10}$$

$$\nabla_j \phi_{ih} = \eta_i H_{jh} - \eta_h H_{ji}. \tag{1.11}$$

2. Nijenhuis tensors of the induced almost contact structure.

In an almost contact manifold, the following four tensors are fundamental:

$$N_{kj}{}^i = \phi_k{}^r(\nabla_r \phi_j{}^i - \nabla_j \phi_r{}^i) - \phi_j{}^r(\nabla_r \phi_k{}^i - \nabla_k \phi_r{}^i) + \nabla_j \eta^i \eta_k - \nabla_k \eta^i \eta_j, \tag{2.1}$$

$$N_j{}^i = \eta^r(\nabla_r \phi_j{}^i - \nabla_j \phi_r{}^i) - \phi_j{}^r \nabla_r \eta^i, \tag{2.2}$$

$$N_{ji} = \phi_i{}^r(\nabla_j \eta_r - \nabla_r \eta_j) - \phi_j{}^r(\nabla_i \eta_r - \nabla_r \eta_i), \tag{2.3}$$

$$N_j = \eta^r(\nabla_r \eta_j - \nabla_j \eta_r). \tag{2.4}$$

These tensors are introduced by Sasaki and Hatakeyama [5] and called the Nijenhuis tensors of the almost contact structure. They studied the relations between these four tensors and have obtained the

LEMMA 2. 1. *If any one of N_{ji}, $N_j{}^i$ and $N_{ji}{}^h$ vanishes, then N_j vanishes. If the tensor $N_{ji}{}^h$ vanishes, then other tensors N_j, N_{ji} and $N_j{}^i$ vanish.*

The almost contact structure with vanishing $N_{ji}{}^h$ is called a normal almost contact structure and the manifold with such a structure is called to be normal almost contact manifold. In this paper, we call a hypersurface with the induced (normal) almost contact structure as an (a normal) almost contact hypersurface. Now, in what follows, we assume that M^{2n} be a Kaehlerian manifold and we only consider an almost contact hypersurface in such a manifold. In particular, if an almost contact hypersurface is a totally geodesic hypersurface, (1. 10), (1. 11) and (2. 1) show us that the hypersurface has vanishing Nijenhuis tensors. Thus, we have the

THEOREM 2. 2. *In a totally geodesic hypersurface of a Kaehlerian manifold, the induced almost contact structure is normal.*

Substituting (1. 10) and (1. 11) into (2. 1), (2. 2), (2. 3) and (2. 4), we get

$$N_{kj}{}^{i}=\eta_j(\phi_k{}^r H_r{}^i+\phi^{ir}H_{rk})-\eta_k(\phi_j{}^r H_r{}^i+\phi^{ir}H_{rj}), \tag{2. 5}$$

$$N_j{}^i=\eta_j H_r{}^i\eta^r-H_j{}^i+\phi_j{}^r\phi^{is}H_{rs}, \tag{2. 6}$$

$$N_{ji}=H_{ri}\eta^r\eta_j-H_{rj}\eta^r\eta_i, \tag{2. 7}$$

$$N_j=-\phi_j{}^r H_{rs}\eta^s. \tag{2. 8}$$

From these relations we can obtain some lemmas and we enumerate them here.

LEMMA 2. 3. *In order that the tensor N_j vanish, it is necessary and sufficient that the vector η^i defines a principal direction of the second fundamental tensor H_{ji} of the hypersurface, i.e. it satisfies that*

$$H_{ji}\eta^i=\alpha\cdot\eta_j, \tag{2. 9}$$

for a suitable function α.

This lemma follows from (0. 2) and (2. 8) immediately and by virtue of Lemma 2. 1, 2. 3 and (2. 7), we have

LEMMA 2. 4. *In the almost contact hypersurface N_{ji} vanishes if and only if N_j vanishes.*

Next we prove the

THEOREM 2. 5. *Let M^{2n} be a Kaehlerian manifold. In order that the induced almost contact structure of a hypersurface in M^{2n} be normal it is necessary and sufficient that the tensor $N_j{}^i$ vanishes identically.*

Proof. We have only to prove the sufficiency of the condition. Transvecting (2. 5) with $\phi_l{}^k$ and making use of (2. 6), we have

$$N_{kj}{}^i\phi_l{}^k=\eta_j(\eta_l H_r{}^i\eta^r-H_l{}^i+\phi_l{}^r\phi^{is}H_{sr})=\eta_j N_l{}^i. \tag{2. 10}$$

On the other hand, after some calculations, we get

$$N_{kj}{}^i\eta^k = N_k{}^i\phi_j{}^k. \tag{2. 11}$$

Thus, from our assumption, we have $N_{kj}{}^i(\phi_l{}^k+\eta_l\eta^k)=0$. The matrix $(\phi_l{}^k+\eta_l\eta^k)$ being non-singular, we get $N_{kj}{}^i=0$. Q.E.D.

Suppose that the induced almost contact structure of the hypersurface is normal. Then, Theorem 2. 5 and (2. 6) imply that

$$H_{ji}=\eta_j H_{ri}\eta^r+\phi_j{}^r\phi_i{}^s H_{rs},$$

which, together with (2. 9), implies that

(2. 12) $$H_{ji}=\alpha\cdot\eta_j\cdot\eta_i+\phi_j{}^r\phi_i{}^s H_{rs}.$$

Transvecting this with $\phi_k{}^j$ and taking account of Lemma 2. 3, we have

(2. 13) $$H_{ji}\phi_k{}^j=-\phi_i{}^j H_{jk},$$

from which we have

$$\nabla_j\eta_i+\nabla_i\eta_j=0$$

because of (1. 10). Thus we have

LEMMA 2. 6. *If the induced almost contact structure in M^{2n-1} be normal, the vector η^i is a Killing vector.*

Differentiating (0. 1) covariantly, we have from this Lemma

(2. 14) $$\eta^r\nabla_r\eta_i=0,\quad \eta^r\nabla_i\eta_r=0.$$

3. Some properties of a hypersurface of a Euclidean space.

In this section, we assume that the manifold M^{2n} be a Euclidean space E^{2n} with the natural Kaehlerian structure. Then the hypersurface satisfies the following Gauss and Codazzi equations:

(3. 1) $$R_{kjih}=H_{ji}H_{kh}-H_{ki}H_{jh},$$

(3. 2) $$\nabla_k H_{ji}-\nabla_j H_{ki}=0,$$

where R_{kjih} is the curvature tensor of the hypersurface.

Suppose that the induced almost contact structure in M^{2n-1} be normal. Then we have

(3. 3) $$R_{rijk}\eta^r=\alpha(\eta_k H_{ji}-\eta_j H_{ki}),$$

by virtue of Lemma 2. 1, 2. 2 together with (3. 1).

On the other hand, by Lemma 2. 6, we have seen that the vector η^i is a Killing vector, and consequently it is an infinitesimal affine transformation. Therefore, it satisfies

(3. 4) $$\underset{\eta}{\mathcal{L}}\left\{ {h \atop ji} \right\}=\nabla_j\nabla_i\eta^h+R_{rji}{}^h\eta^r=0,$$

where $\underset{\eta}{\mathcal{L}}$ means the operator of Lie derivation[3] with respect to the vector η^i. Comparing (3. 3) and (3. 4), we have

(3. 5) $$\nabla_j\nabla_i\eta_h=-\alpha(\eta_h H_{ji}-\eta_i H_{jh})$$

from which by contraction with η^h

(3. 6) $$\eta^r\nabla_j\nabla_i\eta_r=-\alpha(H_{ji}-\alpha\eta_j\cdot\eta_i)$$

3) See Yano [9].

because of (2. 9). Differentiating (0. 1) covariantly and taking account of Lemma 2. 5, we can see that the last equation can be rewritten as

$$\nabla_j\eta^r\nabla_i\eta_r=\alpha(H_{ji}-\alpha\eta_j\cdot\eta_i). \tag{3. 7}$$

Substituting (1. 10) into (3. 7) and making use of (2. 9), we have

$$H_{jr}H_i^{\ r}=\alpha H_{ji}. \tag{3. 8}$$

Now, we shall prove the

LEMMA 3. 1. *Let M^{2n-1} be a normal almost contact hypersurface in a Euclidean space E^{2n}. Then one of the following two conditions must be satisfied:*

1) *The scalar function α in* (2. 9) *is constant;*
2) *The hypersurface is locally developable to E^{2n-1}.*

Proof. Suppose that the scalar α is not constant. Then applying the operator ∇_j to (2. 9), we have

$$\nabla_jH_{rk}\eta^r+H_{rk}\nabla_j\eta^r=\nabla_j\alpha\cdot\eta_k+\alpha\nabla_j\eta_k. \tag{3. 9}$$

Making similar equation to (3. 9) under interchanging of the indices j and k and taking account of Lemma 2. 6, we have

$$H_{rk}\nabla_j\eta^r-H_{rj}\nabla_k\eta^r=\nabla_j\alpha\cdot\eta_k-\nabla_k\alpha\cdot\eta_j+2\alpha\nabla_j\eta_k,$$

where we have used the Codazzi equation (3. 2). Contracting the last equation with η^k and making use of (2. 14), we get

$$\nabla_j\alpha=\beta\cdot\eta_j \qquad (\beta=\eta^r\nabla_r\alpha), \tag{3. 10}$$

from which

$$\nabla_i\nabla_j\alpha=\nabla_i\beta\cdot\eta_j+\beta\nabla_i\eta_j. \tag{3. 11}$$

Since $\nabla_j\alpha$ is a gradient vector and η_j is a unit Killing vector, we get by contraction with $\nabla^i\eta^j$

$$\beta\cdot\nabla_i\eta_j\nabla^i\eta^j=0. \tag{3. 12}$$

On the other hand our assumption and (3. 10) show us that $\beta\neq0$. Thus, the Riemannian metric being positive definite, we have $\nabla_i\eta_j=0$ or from (1. 10)

$$H_{jr}\phi_i^{\ r}=0, \tag{3. 13}$$

from which

$$H_{ji}=\alpha\cdot\eta_j\eta_i. \tag{3. 14}$$

Substituting (3. 14) into (3. 1), we have $R_{kjih}=0$. This completes the proof.

4. Normal almost contact hypersurfaces in Euclidean space.

In this section, we assume that the hypersurface of E^{2n} satisfies the first case of Lemma 3. 1 and that the hypersurface is an analytic hypersurface.[4] Then, for some constant $c(\neq 0)$, we have

$$H_r{}^j H_i{}^r = c H_i{}^j \tag{4.1}$$

by virtue of (3. 8).

Let λ be a characteristic root of the matrix $H_j{}^j$ and v^i the corresponding eigenvector to the root, then we have

$$H_i{}^j v^i = \lambda v^j, \tag{4.2}$$

from which contracting with $H_j{}^k$

$$H_j{}^k H_i{}^j v^i = \lambda H_j{}^k v^j$$

or

$$c\lambda v^k = \lambda^2 v^k$$

because of (4. 1) and (4. 2). Thus we have $\lambda(\lambda - c) = 0$, which means that the characteristic roots of the matrix $H_j{}^i$ are c or 0. Therefore, we can see that they have constant multiplicities. If neither of the roots has zero multiplicity, the hypersurface M^{2n-1} admits two principal curvature c and 0.[5]

Now, denote by L and M the distributions spanned by the vectors corresponding to c and 0 respectively, and let u^i and v^i be two arbitrary vectors belonging to L. Then we have

$$H_j{}^i u^j = c u^i, \qquad H_j{}^i v^j = c v^i$$

from which

$$(\nabla_k H_j{}^i) v^k u^j + H_j{}^i (v^k \nabla_k u^j) = c v^k \nabla_k u^i,$$

$$(\nabla_k H_j{}^i) u^k v^j + H_j{}^i (u^k \nabla_k v^j) = c u^k \nabla_k v^i,$$

and consequently

$$H_j{}^i (u^k \nabla_k v^j - v^k \nabla_k u^j) = c(u^k \nabla_k v^i - v^k \nabla_k u^i)$$

by virtue of the equation of Codazzi. The last equation tells us that the vector $u^k \nabla_k v^j - v^k \nabla_k u^j = [u, v]^j$ belongs to L and this proves that the distribution L is integrable. Similarly the distribution M is also integrable. From the definition of L and M, the integral manifolds of L are totally umbilical and those of M are totally geodesic. Thus, if we denote the multiplicities of the characteristic roots c and 0 by r and s respectively, the hypersurface M^{2n-1} must be everywhere locally isometric with $S^r \times E^s$, and $r + s = 2n - 1$.

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4) A hypersurface is said to be an analytic hypersurface if $X^\kappa(x^i)$ be analytic functions of x^i.

5) E. Cartan proved that in a Euclidean space, a hypersurface whose principal curvatures are all constants is an isoparametric hypersurface, and such hypersurfaces were studied by him [1, 2]. Making use of his results, we can also prove Theorem 4. 2.

Now, we shall prove the following

LEMMA 4.1. *The multiplicity of the root c must be an odd number.*

Proof. Since characteristic roots of the matrix $(H_j{}^i)$ are c and 0, with respect to a suitable frame, it has components of the form at each point of M^{2n-1}

$$(H_j{}^i)=\left(\begin{array}{ccc|c} \multicolumn{3}{c|}{\overbrace{\qquad\qquad}^{r}} & \\ c & & 0 & 0 \\ & \ddots & & \\ 0 & & c & \\ \hline \multicolumn{3}{c|}{0} & 0 \end{array}\right), \qquad (g_{ji})=\begin{pmatrix} 1 & & & \\ & 1 & & 0 \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}, \tag{4.3}$$

from which

$$(\nabla^j\eta_i)=\begin{pmatrix} -c\phi_1{}^1 & \cdots & \cdots & -c\phi_m{}^1 \\ \cdots & \cdots & \cdots & \cdots \\ -c\phi_1{}^r & \cdots & \cdots & -c\phi_m{}^r \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad m=2n-1, \tag{4.4}$$

because of (1.10).

Moreover in §2, we have already seen that $\eta_j\nabla^j\eta_i=0$ and that η^i belongs to L. Hence, we have

$$\operatorname{rank}(\nabla^j\eta_i)\leqq r-1, \tag{4.7}$$

which holds for any frame, especially for natural frame.

Now, suppose that there exists in L a vector u^i satisfying

$$\nabla^j\eta_i u^i=0, \tag{4.8}$$

and that u^i and η^i are linearly independent. Then, without loss of generality, we can put

$$\eta_i u^i=0. \tag{4.9}$$

Transvecting (4.8) with $\nabla^k\eta_j$ and making use of (3.7) and (4.9), we get

$$H_i{}^k u^i=0, \tag{4.10}$$

which means that the vector u^i belongs to M. Thus we know that, in L, there exists no other vector than η^i which satisfies (4.8) and so

$$\operatorname{rank}(\nabla^j\eta_i)\geqq r-1.$$

Hence from this inequality and (4.7), we have

$$\operatorname{rank}(\nabla^j\eta_i)=r-1, \tag{4.11}$$

On the other hand, the rank of the Riemannian metric being $2n-1$, the matrix $(\nabla^j\eta_i)$ has the same rank of the matrix $(\nabla_j\eta_i)$. Since the matrix $(\nabla_j\eta_i)$ is anti-symmetric, it has an even rank. Thus we have

$$r-1=\text{rank}\,(\nabla_j\eta_i)=\text{even},$$

which implies that r is an odd number. This completes the proof of the lemma.

Summing up all the discussions in this section and regarding Lemma 3. 1, we have

THEOREM 4. 2. *A normal almost contact hypersurface M^{2n-1} in Euclidean space is locally isometric with one of the following*:

$$S^{2n-1}, \qquad E^{2n-1}, \qquad S^r\times E^s$$

where r is an odd number and $r+s=2n-1$.

An almost contact manifold which satisfies, for some constant c',

$$\phi_{ji}=c'(\partial_j\eta_i-\partial_i\eta_j)=c'(\nabla_j\eta_i-\nabla_i\eta_j) \tag{4. 12}$$

is said to be a contact metric manifold. If the structure is normal, this manifold is called to be a normal contact metric manifold or a Sasakian manifold.

Now, suppose that our hypersurface admits the induced normal contact metric structure. Then, η^i being a Killing vector, we have

$$\phi_{ji}=2c'\nabla_j\eta_i, \tag{4. 13}$$

from which

$$\text{rank}\,(\nabla_j\eta_i)=\text{rank}\,(\phi_{ji})=2n-2$$

because of (0. 2). On the other hand, from (4. 11) we get

$$\text{rank}\,(\nabla_j\eta_i)=r-1.$$

Thus we have $r=2n-1$, from which we see that the matrix $(H_j{}^i)$ is non-singular and consequently $H_{ji}=cg_{ji}$ by virtue of (4. 1).

COROLLARY 4. 3.[6] *A normal contact hypersurface M^{2n-1} in Euclidean space E^{2n} is a totally umbilical hypersurface and consequently a portion of a sphere.*

5. Another structure in $S^r\times E^s$.

In this section, we shall introduce a certain structure in $S^r\times E^s$ and give some properties of the structure. Let $f_j{}^i$ be a (1, 1) tensor defined by

$$f_j{}^i=\frac{1}{c}\nabla^i\eta_j=-\frac{1}{c}\phi_j{}^rH_r{}^i,\text{[7]} \tag{5. 1}$$

6) Tashiro and Tachibana [8].

7) See §1, (1. 10).

then, we have from (2. 13) and (4. 1)

$$f_k{}^j f_j{}^i = \frac{1}{c^2}\phi_k{}^r H_r{}^j \phi_j{}^s H_s{}^i = -\frac{1}{c} H_k{}^i + \eta^i \cdot \eta_k,$$

from which

$$f_l{}^k f_k{}^j f_j{}^i = -\frac{1}{c^2}\phi_l{}^r H_r{}^k H_k{}^i = -\frac{1}{c}\phi_l{}^r H_r{}^i = -f_l{}^i \tag{5. 2}$$

or by a matrix form

$$f^3 + f = 0. \tag{5. 3}$$

Since the rank of f is $r-1$, our $S^r \times E^s$ admits an f-structure[8] of rank $r-1$.

THEOREM 5. 1. *In normal almost contact hypersurface $S^r \times E^s$ the tensor f defines an f-structure of rank $r-1$.*

The Nijenhuis tensor $N(f)_{ji}{}^h$ and the Haantjes tensor $H_{ji}{}^h$ of the structure f are given by

$$N(f)_{ji}{}^h = f_j{}^l \nabla_l f_i{}^h - f_i{}^l \nabla_l f_j{}^h - (\nabla_j f_i{}^l - \nabla_i f_j{}^l) f_l{}^h, \tag{5. 4}$$

$$H_{ji}{}^h = N(f)_{ji}{}^h - m_j{}^t N(f)_{ti}{}^h + m_i{}^t N(f)_{tj}{}^h + m_j{}^t m_i{}^s N(f)_{ts}{}^h \tag{5. 5}$$

respectively, where we have put $m = f^2 + 1$ or

$$m_j{}^i = f_j{}^r f_r{}^i + \delta_j{}^i. \tag{5. 6}$$

From (5. 5) and (5. 6) we have

$$H_{ji}{}^h = f_j{}^r f_r{}^t f_i{}^l f_l{}^s N(f)_{ts}{}^h. \tag{5. 7}$$

Calculating these two tensors by use of (5. 1), we get

$$N(f)_{ji}{}^h = \frac{1}{c^2}[\nabla^l \eta_j \nabla_l \nabla^h \eta_i - \nabla^l \eta_i \nabla_l \nabla^h \eta_j - (\nabla_j \nabla^l \eta_i - \nabla_i \nabla^l \eta_j) \nabla^h \eta_l],$$

from which

$$N(f)_{ji}{}^h = \frac{1}{c}[-\phi_j{}^t H_t{}^l (\eta^h H_{li} - \eta_i H_l{}^h) + \phi_i{}^t H_t{}^l (\eta^h H_{lj} - \eta_j H_l{}^h) + \phi_l{}^t H_t{}^h (\eta_j H_i{}^l - \eta_i H_j{}^l)].$$

Making use of (2. 13) and (4. 1), we can deduce that

$$N(f)_{ji}{}^h = 2\phi_i{}^t H_{tj} \eta^h = -(\nabla_j \eta_i - \nabla_i \eta_j)\eta^h \tag{5. 8}$$

which implies that $fN(f) = 0$.

On the other hand, we have from (5. 1) and (5. 2)

$$H_{ji}{}^h = \frac{1}{c^4}\nabla^r \eta_j \nabla^t \eta_r \nabla^l \eta_i \nabla^s \eta_l N(f)_{ts}{}^h$$

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8) Yano [10].

from which

$$H_{ji}{}^{h}=\frac{1}{c^2}(H_j{}^t-c\eta^t\eta_j)(H_i{}^s-c\eta^s\eta_i)N(f)_{ts}{}^h$$

because of (3. 7).

If we substitute (5. 8) into the above equation, it follows that

$$H_{ji}{}^{h}=\frac{2}{c^2}\phi_s{}^l H_{lt}\eta^h(H_j{}^tH_i{}^s-cH_j{}^t\eta^s\eta_i-cH_i{}^s\eta^t\eta_j+c^2\eta^t\eta^s\eta_j\eta_i)$$

which implies that

$$H_{ji}{}^{h}=2\phi_i{}^s H_{sj}\eta^h \tag{5. 9}$$

by virtue of (2. 9), (2. 13) and (4. 1). Thus we have

THEOREM 5. 2. *In our $S^r\times E^s$ the Haantjes tensor of the f-strucure is identical with the Nijenhuis tensor of f and satisfies*

$$fH=fN(f)=0.$$

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