SOME THEOREMS ON TIME CHANGE AND KILLING OF MARKOV PROCESSES

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§1. Introduction and definitions.

It is known that a Markov process is transformed to another Markov process by its continuous non-negative additive functional φ_t through time change or killing.[1] On the other hand, φ_t is determined by an excessive function $u(a)=\mathbf{M}_a[\varphi_{\zeta-0}]$. Moreover, if the Green measure of the process is expressed by $g(a, b)m(db)$ and if the process satisfies some additional conditions, then u has the Riesz representation: $u(a)=\int g(a, b)n(db)$ with some measure n. These results are found in the works of Hunt [4], Volkonskii [13] and Meyer [9] under a general setup and in McKean-Tanaka [7] in a concrete case. We want to study what meaning the measure n or m has for the process obtained through time change or killing. In the course of the study we need various generalizations of the resolvent equation and we are compelled to give a unified form in their treatments which is given in §2. In §3 we state construction theorems of processes by time change and killing and give some lemmas concerning (sub) invariant measures. Further, it is proved that the terminal measure[1] of the killed process is represented by K_0^1 (defined in §2) and that a measure n is the terminal measure of the killed process with initial measure n if and only if n is an invariant measure of the process obtained through time change. In §4, G_α^λ and K_α^λ, defined in §2, are represented using a kernel function $g_\alpha^\lambda(a, b)$ under some regularity conditions for the Green kernel $g_\alpha(a, b)$. In §5 we prove that the Riesz measure n is a subinvariant measure of the process obtained through time change by the corresponding additive functional and give some sufficient conditions for the measure to be invariant. And also the meanings of n for killed process are discussed. In order to obtain a necessary and sufficient condition for the measure n to be invariant, we need some considerations on the adjoint process of the process obtained through time change or killing, which is given in §6. The necessary and sufficient condition is stated in §7. The adjoints of the processes are also treated in [4], [8], [11] and [13].

We use the notations and terminologies of Dynkin's book [2] unless specifically mentioned. Concerning a (temporally homogeneous) *Markov process* $X=(x_t, \zeta, \mathcal{M}_t, \mathbf{P}_a, \theta_t)$ with *state space* $(E, \mathcal{B})$, we denominate, for brevity, the following conditions:

M_1. E is a locally compact Hausdorff space with a countable base and $\mathcal{B}$ is the smallest σ-algebra containing all the open subsets of E.

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1) Terminologies and notations are found in the latter half of this section.

M_2. $\mathbf{P}_a[\zeta>0]=1$, for $a \in E$.

M_3. For all ω, $x_t(\omega)$ is right-continuous in $t \in [0, \zeta)$.

M_4. X has the strict Markov property.

M_5. If $\tau_n(\omega) \uparrow \tau(\omega) < \zeta(\omega)$ for all $\omega \in B$, where τ_n are random variables independent of the future (Markov times), then $\mathbf{P}_a[x_{\tau_n} \to x_\tau | B]=1$[2] for all $a \in E$.

M_6. $\mathcal{M}_{t+0}=\mathcal{M}_t$.

M_7. $\bar{\mathcal{M}}_t=\mathcal{M}_t$.[3]

M_8. For all ω, $x_t(\omega)$ has limit from the left in $t \in (0, \zeta)$.

Define the *first hitting time* to a set A, as

$$\sigma_A(\omega)=\begin{cases} \inf\{t\colon 0 \leqq t < \zeta(\omega) \text{ and } x_t(\omega) \in A\} & \text{if such } t \text{ exists,} \\ \zeta(\omega) & \text{if such } t \text{ does not exist.} \end{cases}$$

We call X to be *conservative* if $\mathbf{P}_a[\zeta=\infty]=1$ for all $a \in E$, and to be *recurrent* if $\mathbf{P}_a[\sigma_A<\zeta]=1$ for all $a \in E$ and all non-empty open sets A.

We introduce definitions on some special measures. Let m and n be σ-finite measures on E. We say that m is an *invariant measure* for X, if and only if

$$\mathbf{P}_m[x_t \in A]=m(A)^{4)} \tag{1. 1}$$

holds for all $A \in \mathcal{B}$ and $t>0$. If the left-hand side in (1. 1) is not greater than the right, m is called to be a *subinvariant measure* for X. We call n the *Green measure* (of order zero) *of* (X, m) if and only if

$$\mathbf{M}_m\left[\int_0^\zeta \chi_A(x_t)dt\right]=n(A)^{5)} \tag{1. 2}$$

holds for all $A \in \mathcal{B}$, and the *terminal measure of* (X, m) if and only if $\mathbf{P}_m[\zeta<\infty$ and $x_{\zeta-0}$ does not exist$]=0$ and

$$\mathbf{P}_m[\zeta<\infty \text{ and } x_{\zeta-0} \in A]=n(A) \tag{1. 3}$$

holds for all $A \in \mathcal{B}$.

We call $\varphi_t(\omega)$ $(\omega \in \Omega_t)$ *continuous non-negative additive functional of* X *of order* α if and only if the followings are satisfied:

A_1. $\varphi_s(\omega)+e^{-\alpha s}\theta_s\varphi_t(\omega)=\varphi_{s+t}(\omega)$, for $\omega \in \Omega_{s+t}$;

A_2. φ_t is $\mathcal{R}_t$-measurable;[6]

A_3. $0 \leqq \varphi_t(\omega) < \infty$, for $\omega \in \Omega_t$;

A_4. $\mathbf{P}_a[\varphi_0=0]=1$, for $a \in E$;

A_5. $\varphi_t(\omega)$ is continuous in t.

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2) We write $\mathbf{P}_a[A|B]=1$ if and only if $\mathbf{P}_a[B \setminus A]=0$.

3) $\bar{\mathcal{M}}_t$ is the family of B such that, for every finite measure m, there exist B_1 and $B_2 \in \mathcal{M}_t$ satisfying $B_1 \subseteq B \subseteq B_2$ and $\mathbf{P}_m[B_2 \setminus B_1]=0$.

4) $\mathbf{P}_m[B]=\int_E \mathbf{P}_a[B]m(da)$ and $\mathbf{M}_m[f]=\int_E \mathbf{M}_a[f]m(da)$.

5) χ_A is the indicator function of set A.

6) We put $\mathcal{R}_t=\bar{\mathcal{N}}_{t+0} \cap \mathcal{N}^*$.

If $\alpha=0$, we omit the word "of order 0".

Further we introduce some notations:

$B(E)$=the set of bounded $\mathfrak{B}$-measurable functions;
$C(E)$=the set of bounded continuous functions;
$B_0(E)$=the set of $f\in B(E)$ with compact supports;
$C_0(E)=C(E)\cap B_0(E)$;
$B^+(E)$=the set of non-negative $f\in B(E)$;

$C^+(E)$, $B_0^+(E)$ and $C_0^+(E)$ are the meets with $B^+(E)$ of $C(E)$, $B_0(E)$ and $C_0(E)$, respectively. $\|f\|$ means $\sup_{a\in E}|f(a)|$.

Let E and $\hat{E}$ be subsets of a larger space and let X and $\hat{X}$ be two Markov processes with state spaces E and $\hat{E}$, respectively. Then, we say that X and $\hat{X}$ are mutually *adjoint* with respect to a σ-finite measure m, if and only if

$$m(E\backslash\hat{E})=m(\hat{E}\backslash E)=0, \tag{1.4}$$

and

$$\int_{E\cap\hat{E}}\mathbf{M}_a[f(x_t)]g(a)m(da)=\int_{E\cap\hat{E}}f(a)\hat{\mathbf{M}}_a[g(\hat{x}_t)]m(da)^{7)} \tag{1.5}$$

for all $f\in B_0(E)$, $g\in B_0(\hat{E})$ and $t>0$.

Obviously, (1.5) is equivalent to

$$\int_{E\cap\hat{E}}G_\alpha^0f(a)g(a)m(da)=\int_{E\cap\hat{E}}f(a)\hat{G}_\alpha^0g(a)m(da) \tag{1.6}$$

for all $f\in B_0(E)$, $g\in B_0(\hat{E})$ and $\alpha>0$ (cf. [11]), where

$$G_\alpha^0f(a)=\mathbf{M}_a\left[\int_0^\zeta e^{-\alpha t}f(x_t)dt\right], \tag{1.7}$$

$$\hat{G}_\alpha^0g(a)=\hat{\mathbf{M}}_a\left[\int_0^{\hat{\zeta}} e^{-\alpha t}g(\hat{x}_t)dt\right]. \tag{1.8}$$

Now we fix a Markov process $X=(x_t, \zeta, \mathfrak{M}_t, \mathbf{P}_a, \theta_t)$ with state space $(E, \mathfrak{B})$ having the properties $M_1\sim M_7$, and go on throughout this paper.

§2. A generalization of the resolvent equation.

Let $\varphi_t(\omega)$ and $\psi_t(\omega)$ be continuous non-negative additive functionals of X, and put, for measurable f,

$$U_\alpha^\lambda f(a)=\mathbf{M}_a\left[\int_0^\zeta e^{-\alpha\varphi_t-\lambda\psi_t}f(x_t)d\varphi_t\right] \tag{2.1}$$

and

7) Convention: $\mathbf{M}_a[f(x_\sigma)]=\mathbf{M}_a[f(x_\sigma)\colon \sigma<\zeta]$, where $\mathbf{M}_a[F(\omega)\colon A]=\int_A F(\omega)\mathbf{P}_a[d\omega]$. The semi-group of the process X is defined by $T_tf(a)=\mathbf{M}_a[f(x_t)]$.

$$V_\alpha^\lambda f(a)=\mathbf{M}_a\left[\int_0^\zeta e^{-\alpha\psi_t-\lambda\varphi_t}f(x_t)d\psi_t\right], \tag{2. 2}$$

when the right-hand sides are defined. Clearly, $U_\alpha^\lambda f$ and $V_\alpha^\lambda f$ are finite and bounded if $\alpha>0$, $\lambda\geqq 0$ and $f\in B(E)$. They satisfy the equation (2. 3) below, which is a generalization of the so-called resolvent equation.

THEOREM 2. 1. *For any* α, $\beta>0$, λ, $\mu\geqq 0$ *and* $f\in B(E)$, *it holds that*

$$U_\alpha^\lambda f-U_\beta^\mu f+(\alpha-\beta)U_\alpha^\lambda U_\beta^\mu f+(\lambda-\mu)V_\lambda^\alpha U_\beta^\mu f=0. \tag{2. 3}$$

Before the proof, we prepare

LEMMA 2. 1.[8] *Let* $p(t)$ *be a continuous non-decreasing mapping of* $[0, t_0]$ *onto* $[0, p_0]$ (t_0 *and* p_0 *may possibly be* ∞). *Put* $c(t)=\sup\{s\colon s<t_0$ *and* $p(s)\leqq t\}$ *and* $c_0=\sup\{s\colon s<t_0$ *and* $p(s)<\infty\}$. *Then,*

$$\int_{[0,c_0)} f(t, p(t))dp(t)=\int_{[0,p_0)} f(c(t), t)dt \tag{2. 4}$$

holds for each non-negative measurable function $f(t, s)$.

Proof. It is sufficient to prove (2. 4) for the function of the form $f(t, s)=\chi_{[0,a)}(t)\chi_{[0,b]}(s)$ where $0<a<c_0$ and $0<b<p_0$. We have, obviously,

$$\int_{[0,c_0)} \chi_{[0,a)}(t)\chi_{[0,b]}(p(t))dp(t)=p(t_1)$$

where $t_1=a\wedge\sup\{t\colon p(t)\leqq b\}=a\wedge c(b)$,[9] and

$$\int_{[0,p_0)} \chi_{[0,a)}(c(t))\chi_{[0,b]}(t)dt=t_2$$

where $t_2=b\wedge\sup\{t\colon c(t)<a\}$. Provided that $0\leqq t<t_0$, $p(t)\leqq s$ if and only if $t\leqq c(s)$. Hence $t_2=b\wedge\sup\{t\colon t<p(a)\}=b\wedge p(a)$. If $c(b)<a$, then we have $b<p(a)$ and $p(t_1)=p(c(b))=b=t_2$. On the other hand, in case $c(b)\geqq a$, we have $b\geqq p(a)$ and $p(t_1)=p(a)=t_2$. In both cases (2. 4) is verified.

REMARK. If $0\leqq t<p_0$, then $p(c(t))=t$ and $c(t)$ is right-continuous and strictly increasing. These are already used in the above proof.

Proof of Theorem 2. 1. Put

$$\tau_t(\omega)=\sup\{s\colon s<\zeta(\omega) \text{ and } \varphi_s(\omega)\leqq t\} \tag{2. 5}$$

and

$$\sigma_t(\omega)=\sup\{s\colon s<\zeta(\omega) \text{ and } \psi_s(\omega)\leqq t\}. \tag{2. 6}$$

8) This is an extension of Lemma 7. 1 of Meyer [9].

9) $t\wedge s=\min\{t, s\}$.

Since $\{\tau_t<s+\varepsilon<\zeta\}=\{t<\varphi_{s+\varepsilon},\ s+\varepsilon<\zeta\}\in\mathfrak{R}_{s+\varepsilon}$, we have $\{\tau_t\leqq s<\zeta\}\in\mathfrak{R}_{s+0}\subseteqq\overline{\mathfrak{M}}_{s+0}=\mathfrak{M}_s$ and τ_t is a random variable independent of the future. And so is σ_t. By virtue of Lemma 2. 1, the strict Markov property and Fubini's theorem, we have

$$
\begin{aligned}
U_\alpha^\lambda U_\beta^\mu f(a) &= \mathbf{M}_a\left[\int_0^\zeta e^{-\alpha\varphi_t-\lambda\psi_t}\,U_\beta^\mu f(x_t)d\varphi_t\right]\\
&= \mathbf{M}_a\left[\int_0^{\varphi_{\zeta-0}} e^{-\alpha t-\lambda\psi_{\tau_t}}\,U_\beta^\mu f(x_{\tau_t})dt\right]\\
&= \mathbf{M}_a\left[\int_0^\infty e^{-\alpha t-\lambda\psi_{\tau_t}}\chi_{\{t<\varphi_{\zeta-0}\}}\,dt\mathbf{M}_{x_{\tau_t}}\left[\int_0^\zeta e^{-\beta\varphi_s-\mu\psi_s}f(x_s)d\varphi_s\right]\right]\\
&= \mathbf{M}_a\left[\int_0^\infty e^{-\alpha t-\lambda\psi_{\tau_t}}\,\chi_{\{\tau_t<\zeta\}}\,dt\int_0^{\zeta-\tau_t} e^{-\beta\theta_{\tau_t}\varphi_s-\mu\theta_{\tau_t}\psi_s}f(x_{\tau_t+s})d_s\theta_{\tau_t}\varphi_s\right]\\
&= \mathbf{M}_a\left[\int_0^\zeta e^{-\alpha\varphi_t-\lambda\psi_t}\,d\varphi_t\int_0^{\zeta-t} e^{-\beta\theta_t\varphi_s-\mu\theta_t\psi_s}f(x_{t+s})d_s\theta_t\varphi_s\right]\\
&= \mathbf{M}_a\left[\int_0^\zeta e^{-(\alpha-\beta)\varphi_t-(\lambda-\mu)\psi_t}\,d\varphi_t\int_t^\zeta e^{-\beta\varphi_s-\mu\psi_s}f(x_s)d\varphi_s\right]\\
&= \mathbf{M}_a\left[\int_0^\zeta e^{-\beta\varphi_s-\mu\psi_s}\,f(x_s)d\varphi_s\int_0^s e^{-(\alpha-\beta)\varphi_t-(\lambda-\mu)\psi_t}\,d\varphi_t\right].
\end{aligned}
$$

If $\lambda>0$, similar argument yields

$$
\begin{aligned}
V_\lambda^\alpha U_\beta^\mu f(a) &= \mathbf{M}_a\left[\int_0^\zeta e^{-\alpha\varphi_t-\lambda\psi_t}\,U_\beta^\mu f(x_t)d\psi_t\right]\\
&= \mathbf{M}_a\left[\int_0^\zeta e^{-\beta\varphi_s-\mu\psi_s}f(x_s)d\varphi_s\int_0^s e^{-(\alpha-\beta)\varphi_t-(\lambda-\mu)\psi_t}\,d\psi_t\right].
\end{aligned}
$$

Accordingly,

$$
\begin{aligned}
&(\alpha-\beta)U_\alpha^\lambda U_\beta^\mu f(a)+(\lambda-\mu)V_\lambda^\alpha U_\beta^\mu f(a)\\
&= \mathbf{M}_a\left[\int_0^\zeta e^{-\beta\varphi_s-\mu\psi_s}\,f(x_s)d\varphi_s\int_0^s e^{-(\alpha-\beta)\varphi_t-(\lambda-\mu)\psi_t}\,((\alpha-\beta)d\varphi_t+(\lambda-\mu)d\psi_t)\right]\\
&= \mathbf{M}_a\left[\int_0^\zeta e^{-\beta\varphi_s-\mu\psi_s}\,f(x_s)d\varphi_s\,(1-e^{-(\alpha-\beta)\varphi_s-(\lambda-\mu)\psi_s})\right]\\
&= U_\beta^\mu f(a)-U_\alpha^\lambda f(a),
\end{aligned}
$$

thus (2. 3) holds for $\lambda>0$. Without loss of generalities, we may suppose $f\geqq 0$. Then, $V_\lambda^\alpha U_\beta^\mu f\uparrow V_0^\alpha U_\beta^\mu f$ and $U_\alpha^\lambda f-U_\beta^\mu f+(\alpha-\beta)U_\alpha^\lambda U_\beta^\mu f\to U_\alpha^0 f-U_\beta^\mu f+(\alpha-\beta)U_\alpha^0 U_\beta^\mu f$, as $\lambda\downarrow 0$. Hence, (2. 3) holds for $\lambda=0$, too, and the proof is complete.

COROLLARY. *The following commutativity holds*:

(2. 7) $$U_\alpha^\lambda U_\beta^\lambda = U_\beta^\lambda U_\alpha^\lambda,$$

(2. 8) $$V_\lambda^\alpha U_\alpha^\mu = V_\mu^\alpha U_\alpha^\lambda.$$

A slight extension of Theorem 2. 1 is

THEOREM 2. 2. *If* $U_0^{\alpha_0}1(a)$ *is bounded for some* $\alpha_0 \geqq 0$, *then* $U_0^{\alpha}f$ *is finite and bounded for arbitrary* $\alpha>0$ *and* $f \in B(E)$ *and, furthermore,* (2. 3) *remains true for all* $f \in B(E)$ *and all* $\alpha, \beta, \lambda, \mu \geqq 0$ *such that* $\alpha+\lambda>0$ *and* $\beta+\mu>0$.

Proof. By Theorem 2. 1, we have

$$U_\beta^\alpha 1 = U_\beta^{\alpha_0} 1 - (\alpha-\alpha_0) V_\alpha^\beta U_\beta^{\alpha_0} 1 \tag{2. 9}$$

for $\alpha \geqq 0$ and $\beta>0$. Make β tend to zero, then (2. 9) turns out to

$$U_0^\alpha 1 = U_0^{\alpha_0} 1 - (\alpha-\alpha_0) V_\alpha^0 U_0^{\alpha_0} 1$$

because $U_0^{\alpha_0}1$ is bounded. Hence the right-hand side of the above is bounded if $\alpha>0$, proving the first half of the theorem. In case $\alpha=0$ or $\beta=0$, (2. 3) is obtained as the limit from the same formula with positive α and β, making use of the dominated convergence theorem. Thus the proof is complete.

It must be noted that the above two theorems are valid for processes satisfying M_2, M_4, and $\bar{\mathfrak{N}}_{t+0} \subseteqq \mathfrak{M}_t$.

For later use, we introduce two operators K_α^λ and G_α^λ, which are special cases of U_α^λ and V_α^λ;

$$K_\alpha^\lambda f(a) = \mathbf{M}_a\left[\int_0^\zeta e^{-\alpha\varphi_t - \lambda t} f(x_t)\, d\varphi_t\right] \tag{2. 10}$$

and

$$G_\alpha^\lambda f(a) = \mathbf{M}_a\left[\int_0^\zeta e^{-\alpha t - \lambda\varphi_t} f(x_t)\, dt\right]. \tag{2. 11}$$

Then, as corollaries of Theorem 2. 1, we have two formulae:

$$K_\alpha^\lambda - K_\beta^\mu + (\alpha-\beta) K_\alpha^\lambda K_\beta^\mu + (\lambda-\mu) G_\lambda^\alpha K_\beta^\mu = 0, \tag{2. 12}$$

$$G_\alpha^\lambda - G_\beta^\mu + (\alpha-\beta) G_\alpha^\lambda G_\beta^\mu + (\lambda-\mu) K_\lambda^\alpha G_\beta^\mu = 0, \tag{2. 13}$$

which are reduced to the resolvent equations of K_α^λ and G_α^λ when $\lambda=\mu$. The processes having K_α^λ (or G_α^λ) as their resolvent operators are described in the next section.

§ 3. Time change and killing.

As is well known, a continuous non-negative additive functional serves as time change function to construct a new Markov process. In fact, put $\tilde{\zeta} = \varphi_{\zeta-0}$, $\tilde{x}_t(\omega) = x_{\tau_t}(\omega)$ $(0 \leqq t < \tilde{\zeta}(\omega))$, and $F = \{a\colon \mathbf{P}_a[\tau_0>0]=0\}$, where τ_t is defined by (2. 5).

LEMMA 3. 1. *F is nearly Borel measurable* (*in the sense of* [4]) *and* $\mathbf{P}_a[\tilde{x}_t \in F$ *for* $0 \leqq t < \tilde{\zeta} \mid \tilde{\zeta}>0] = 1$ *holds for any* $a \in E$.

Proof. Put $u^\alpha(a) = \mathbf{M}_a[e^{-\alpha\tau_0}]$, then $u^\alpha(a)=1$ for $a \in F$ and $u^\alpha(a)<1$ for $a \in E \setminus F$.

Since $u^{\alpha}(a)$ is α-excessive,[10] $u^{\alpha}(x_t)$ is right-continuous with $\mathbf{P}_a$-measure 1 by Hunt [4] and Doob [1], and we have

$$\mathbf{P}_a[u^{\alpha}(x_{\tau_t}) \text{ is right-continuous in } t<\tilde{\zeta} \mid \tilde{\zeta}>0]=1.$$

On the other hand,

$$\mathbf{P}_a[u^{\alpha}(x_{\tau_r})=1 \text{ for all rational } r\in[0, \tilde{\zeta}) \mid \tilde{\zeta}>0]=1.$$

Hence, $\mathbf{P}_a[u^{\alpha}(x_{\tau_t})=1$ for all $t\in[0, \tilde{\zeta}) \mid \tilde{\zeta}>0]=1$. Nearly Borel measurability of $u^{\alpha}(a)$ follows from excessivity by [4], and so is F.

Put $\tilde{\Omega}=\{\omega\colon \tilde{x}_t(\omega)\in F$ for all $t\in[0, \tilde{\zeta})\}$, then the restriction $X^0=(x_t^0, \zeta^0, \mathfrak{M}_t^0, \mathbf{P}_a^0, \theta_t^0)$ of X to $\tilde{\Omega}$ is again a Markov process satisfying $M_1\sim M_7$. Let $\tilde{\mathfrak{M}}_t$, $\tilde{\mathbf{P}}_a\,(a\in F)$ and $\tilde{\theta}_t$ be $\mathfrak{M}_{\tau_t}^0$, $\mathbf{P}_a^0$, and $\theta_{\tau_t}^0$, respectively, and let $\bar{\mathfrak{B}}$ be the intersection of $\bar{\mathfrak{B}}(m)$ ranging over all finite measures m, where $\bar{\mathfrak{B}}(m)$ is the completion of $\mathfrak{B}$ with respect to m. Then we have

THEOREM 3.1. *$\tilde{X}=(\tilde{x}_t, \tilde{\zeta}, \tilde{\mathfrak{M}}_t, \tilde{\mathbf{P}}_a, \tilde{\theta}_t)$ is a Markov process with state space $(F, \bar{\mathfrak{B}}[F])$ satisfying M_2, M_3, M_4, and M_6. The resolvent operator of $\tilde{X}$ is K_α^0.*

The proof is achieved by applying the same method as in [13] more carefully.

Another transformation by φ_t is killing or the formation of subprocess. The next theorem is a special case of a theorem valid under weaker conditions on φ_t (cf. [2], [9] and [12]).

THEOREM 3.2. *There is a subprocess $\dot{X}=(\dot{x}_t, \dot{\zeta}, \dot{\mathfrak{M}}_t, \dot{\mathbf{P}}_a, \dot{\theta}_t)$ of X corresponding to the multiplicative functional $e^{-\varphi_t}$ and satisfying $M_2\sim M_7$. The resolvent operator of $\dot{X}$ is G_α^1.*

More generally, as indicated in [12], $\{U_\alpha^\lambda\colon \alpha>0\}$ is the system of the resolvent operators for the process obtained through time change by φ_t and killing by $\lambda\psi_t$.

The rest of the section is devoted to some general properties of invariant, subinvariant and terminal measures.

If m is an invariant measure for X, then

$$\alpha\int_E G_\alpha^0 f(a)m(da)=\int_E f(a)m(da), \qquad \text{for } f\in B_0^+(E),\ \alpha>0. \tag{3.1}$$

Conversely, we have

LEMMA 3.2. *If, for some $\alpha_0>0$,*

$$\alpha_0\int_E G_{\alpha_0}^0 f(a)m(da)=\int_E f(a)m(da), \qquad f\in B_0^+(E), \tag{3.2}$$

then m is an invariant measure for X.

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10) A non-negative function u is said to be α-excessive (relative to X), if $\mathbf{M}_a[e^{-\alpha t}u(x_t)]\leq u(a)$ holds and the left member increases to the right as $t\downarrow 0$.

Proof. (3. 1) for all $\alpha \geqq \alpha_0$ follows from the resolvent equation:

$$G_{\alpha_0}^0 f - G_\alpha^0 f + (\alpha_0 - \alpha) G_{\alpha_0}^0 G_\alpha^0 f = 0. \tag{3. 3}$$

Noting that

$$\int_E G_\alpha^0 f(a) m(da) = \int_0^\infty e^{-\alpha t}\, dt \int_E T_t f(a) m(da) \text{ and } \frac{1}{\alpha}\int_E f(a) m(da) = \int_0^\infty e^{-\alpha t}\, dt \int_E f(a) m(da),$$

we have, by the one-to-one property of the Laplace transform,

$$\int_E T_t f(a) m(da) = \int_E f(a) m(da) \tag{3. 4}$$

for any $t \notin S$, where S is a set of Lebesgue measure zero. Using the second countability, S can be chosen to be common for all $f \in B_0(E)$. Let $t \in S$, we can find t_0 outside S such that $t_0 + t$ is also not in S. Then

$$\int_E T_t f(a) m(da) = \int_E T_{t_0} T_t f(a) m(da) = \int_E T_{t_0+t} f(a) m(da) = \int_E f(a) m(da),$$

and (3. 4) holds for all t. Hence the proof is complete.

LEMMA 3. 3. *A σ-finite measure m is subinvariant for X, if and only if*

$$\alpha \int_E G_\alpha^0 f(a) m(da) \leqq \int_E f(a) m(da) \tag{3. 5}$$

holds for all $\alpha > 0$ and $f \in C_0^+(E)$.

Proof is essentially found in Hunt [4]. "Only if" part is obvious, so we give the proof of "if" part. First, let us prove

$$\alpha e^{-\beta t} \int_E T_t G_{\alpha+\beta}^0 f(a) m(da) \leqq \alpha \int_E G_{\alpha+\beta}^0 f(a) m(da), \tag{3. 6}$$

for any $\alpha, \beta > 0$ and $f \in C_0^+(E)$. Define (non-negative) measure m_0 by

$$\int_E h(a) m_0(da) = \int_E (h(a) - \alpha G_{\alpha+\beta}^0 h(a)) m(da).$$

Then, since

$$\int_E G_{\alpha+\beta}^0 g(a) m(da) = \int_E G_\beta^0 g(a) m_0(da)$$

for any $g \in B^+(E)$, we have

$$\alpha e^{-\beta t} \int T_t G_{\alpha+\beta}^0 f(a) m(da) = \alpha e^{-\beta t} \int G_{\alpha+\beta}^0 T_t f(a) m(da)$$

$$= \alpha e^{-\beta t} \int G_\beta^0 T_t f(a) m_0(da) \leqq \alpha \int G_\beta^0 f(a) m_0(da) = \alpha \int G_{\alpha+\beta}^0 f(a) m(da),$$

namely (3. 6). Make $\alpha \to \infty$ in (3. 6). The left side estimates as

$$\liminf_{\alpha\to\infty} \alpha e^{-\beta t}\int_E T_t G^0_{\alpha+\beta} f(a)m(da) = \liminf_{\alpha\to\infty} \alpha e^{-\beta t}\mathbf{M}_m[G^0_{\alpha+\beta} f(x_t)] \geqq e^{-\beta t}\mathbf{M}_m[f(x_t)],$$

using Fatou theorem, and the right side is

$$\liminf_{\alpha\to\infty} \alpha\int_E G^0_{\alpha+\beta} f(a)m(da) \leqq \int_E f(a)m(da)$$

by (3. 5). Hence we have

$$\int_E T_t f(a)m(da) \leqq \int_E f(a)m(da), \qquad \text{for all } t>0.$$

The terminal measure of the killed process $\dot{X}$ corresponding to $e^{-\varphi_t}$ is represented by K^0_1, that is,

THEOREM 3. 3. *Suppose that X is conservative and satisfies M_8, then*

$$\dot{\mathbf{M}}_a[f(\dot{x}_{\zeta-0}): \dot{\zeta}<\infty] = K^0_1 f(a) \tag{3. 7}$$

holds for all $a\in E$ and $f\in B(E)$.

Proof. Without loss of generalites, f is assumed to be in $C(E)$. Then we have

$$K^0_1 f(a) = \mathbf{M}_a\left[\int_0^\infty f(x_t)d(-e^{-\varphi_t})\right] = \mathbf{M}_a\left[\int_0^\infty f(x_{t-0})d(-e^{-\varphi_t})\right],$$

since the discontinuity points of $x_t(\omega)$ are at most countable. Hence,

$$\begin{aligned} K^0_1 f(a) &= \mathbf{M}_a[\lim_{h\downarrow 0}\sum_{i=0}^\infty f(x_{ih})(e^{-\varphi_{ih}} - e^{-\varphi_{(i+1)h}})] \\ &= \lim_{h\downarrow 0}\sum_{i=0}^\infty \dot{\mathbf{M}}_a[f(\dot{x}_{ih}): ih<\dot{\zeta}\leqq(i+1)h] \\ &= \dot{\mathbf{M}}_a[f(\dot{x}_{\zeta-0}): 0<\dot{\zeta}<\infty] = \dot{\mathbf{M}}_a[f(\dot{x}_{\zeta-0}): \dot{\zeta}<\infty], \end{aligned}$$

so that (3. 7) is proved.

Theorem 3. 3 implies that the terminal measure of $\dot{X}$ is identical with the Green measure (of order one) of $\tilde{X}$. This is the basis for the next

THEOREM 3. 4. *Suppose that X is conservative and satisfies M_8, and let n be a σ-finite measure on E. Then the following two statements are mutually equivalent;*
(i) *n is itself the terminal measure of $(\dot{X}, n)$;*
(ii) *n is concentrated on F and an invariant measure for $\tilde{X}$.*

Proof. Assume that n satisfies (i). Then we have, by Theorem 3. 3,

$$\int_E K_1^0 f(a)n(da) = \int_E f(a)n(da) \tag{3. 8}$$

for all $f \in B_0(E)$. Since $\mathbf{P}_a[x_{\tau_t} \in F]=1$, we have $n(E\backslash F)=0$. For,

$$\int_F f(a)n(da) = \int_E K_1^0(\chi_F f)(a)n(da) = \mathbf{M}_n\left[\int_0^\infty e^{-\varphi_t}(\chi_F f)(x_t)d\varphi_t\right]$$

$$= \mathbf{M}_n\left[\int_0^\xi e^{-t}(\chi_F f)(x_{\tau_t})dt\right] = \mathbf{M}_n\left[\int_0^\xi e^{-t} f(x_{\tau_t})dt\right]$$

$$= \int_E K_1^0 f(a)n(da) = \int_E f(a)n(da).$$

Hence (3. 8) is now read as

$$\int_F K_1^0 f(a)n(da) = \int_F f(a)n(da),$$

implying n is an invariant measure for $\tilde{X}$ by Lemma 3. 2.

§ 4. Representation of G_α^λ and K_α^λ.

In this section we make two assumptions. The first one is the existence of a σ-finite measure m and a non-negative (possibly infinite) $\mathfrak{B}\times\mathfrak{B}$-measurable function $g_{\alpha_0}(a, b)$ such that, for any $f \in B_0(E)$,

$$G_{\alpha_0}^0 f(a) = \int_E g_{\alpha_0}(a, b) f(b)m(db). \tag{4. 1}$$

Here α_0 is a fixed non-negative number and the both sides of (4. 1) are supposed to be finite. The function $g_{\alpha_0}(a, b)$ is assumed to be, as a function of a, α_0-excessive and α_0-harmonic in $E\backslash b$.[11),12)] Given a continuous non-negative additive functional $\varphi_t(\omega)$, our second assumption[13)] is the existence of a σ-finite measure n satisfying

$$\mathbf{M}_a\left[\int_0^\zeta e^{-\alpha_0 t}\, d\varphi_t\right] = \int_E g_{\alpha_0}(a, b)n(db) < \infty, \qquad a \in E. \tag{4. 2}$$

We assume that (4. 2) does not identically reduce to zero.

We shall prove that the measures m and n together with appropriate modifications of $g_{\alpha_0}(a, b)$ permit us to represent the operators $G_{\alpha_0}^\lambda$ and $K_\lambda^{\alpha_0}$.

THEOREM 4. 1. *For all $f \in B(E)$, $K_0^{\alpha_0} f$ is finite and represented as*

$$K_0^{\alpha_0} f(a) = \int_E g_{\alpha_0}(a, b) f(b)n(db), \qquad a \in E. \tag{4. 3}$$

11) A function u is said to be α-harmonic (relative to X) in $E\backslash b$, if $\mathbf{M}_a[e^{-\alpha\sigma}u(x_\sigma)] = u(a)$, $\sigma=\sigma_U$, holds for every open set U containing b.

12) Sufficient conditions for this assumption are treated in Kunita-Watanabe [6].

13) For a sufficient condition, see Meyer [9].

A similar theorem is found in Meyer [9], and a close problem is also treated by Motoo [10]. In the Brownian case, it is also proved in McKean-Tanaka [7]. Our proof is based on the next fundamental lemma which, as well as its proof, is due to Tanaka (cf. [5]).

LEMMA 4. 1. *If n has no mass outside of an open set U, then*

$$\mathbf{P}_a[\varphi_t=0, \text{ for all } t\in[0, \sigma_U)]=1, \qquad a\in E. \tag{4. 4}$$

Proof. Put

$$p(a)=\mathbf{M}_a\left[\int_0^\zeta e^{-\alpha_0 t}\, d\varphi_t\right]. \tag{4. 5}$$

By the strict Markov property and (4. 2) we have

$$\begin{aligned}\mathbf{M}_a\left[\int_0^{\sigma_U} e^{-\alpha_0 t}\, d\varphi_t\right]&=\mathbf{M}_a\left[\int_0^\zeta e^{-\alpha_0 t}\, d\varphi_t\right]-\mathbf{M}_a\left[\int_{\sigma_U}^\zeta e^{-\alpha_0 t}\, d\varphi_t\right]\\ &=p(a)-\mathbf{M}_a[e^{-\alpha_0\sigma_U}p(x_{\sigma_U})]\\ &=\int_E (g_{\alpha_0}(a, b)-\mathbf{M}_a[e^{-\alpha_0\sigma_U}g_{\alpha_0}(x_{\sigma_U}, b)])n(db).\end{aligned}$$

The integration in the last member can be restricted to U and vanishes since $g_{\alpha_0}(a, b)$ is α_0-harmonic in $E\backslash b$. Thus (4. 4) is proved.

According to the work of Meyer [9], a non-negative and finitely valued function $u(a)$ has the representation $u(a)=\mathbf{M}_a[\phi_{\zeta-0}^{(\alpha_0)}]$ by a continuous non-negative additive functional $\phi_t^{(\alpha_0)}$ of order α_0, if and only if it has the following properties: (i) u is α_0-excessive; (ii) If $\{\sigma_n\}$ is a non-decreasing sequence of random variables independent of the future,

$$\mathbf{M}_a[e^{-\alpha_0\sigma_n}u(x_{\sigma_n})]\downarrow \mathbf{M}_a[e^{-\alpha_0\sigma}u(x_\sigma)], \qquad n\uparrow\infty$$

where $\sigma=\lim\sigma_n$. Moreover, $\phi_t^{(\alpha_0)}$ is determined by u uniquely up to $\mathbf{P}_a$-measure zero for all $a\in E$. These remarkable results imply

LEMMA 4. 2. *For any $f\in B(E)$, there exists a continuous non-negative additive functional $\varphi_t^f(\omega)$ such that*

$$\mathbf{M}_a\left[\int_0^\zeta e^{-\alpha_0 t}\, d\varphi_t^f\right]=\int_E g_{\alpha_0}(a, b)f(b)n(db) \tag{4. 6}$$

holds. Such φ_t^f is unique up to $\mathbf{P}_a$-measure zero for all $a\in E$.

Proof of Theorem 4. 1. Let V be a closed set and φ_t^V denote φ_t^f when $f=\chi_V$. Put $U_n=\{a\colon \rho(a, V)<1/n\}$ where ρ is a metric compatible with the topology of E. Then we have $\mathbf{P}_a[\sigma_{U_n}\uparrow\sigma_V]=1$, on acount of M_5 and closedness of V. Therefore we have, by Lemma 4. 1,

$$\mathbf{P}_a[\varphi_t^V=0, \quad \text{for all} \quad t\in[0, \sigma_V)]=1, \qquad \text{for } a\in E. \tag{4. 7}$$

We shall prove that

(4. 8) $$\mathbf{P}_a\left[\varphi_t^V=\int_0^t \chi_V(x_s)d\varphi_s^V,\ \text{for all } t\in[0,\zeta)\right]=1,\quad \text{for}\quad a\in E.$$

Put $U=\{a: \rho(a, V)>\varepsilon\}$ where ε is a positive constant. Define a sequence of random variables independent of the future;

$$\tau_1=\sigma_U,$$

$$\tau_{2n}=\begin{cases} \tau_{2n-1}+\theta_{\tau_{2n-1}}\sigma_V & \text{if}\quad \tau_{2n-1}<\zeta \\ \zeta & \text{if}\quad \tau_{2n-1}\geqq\zeta, \end{cases}$$

$$\tau_{2n+1}=\begin{cases} \tau_{2n}+\theta_{\tau_{2n}}\sigma_U & \text{if}\quad \tau_{2n}<\zeta \\ \zeta & \text{if}\quad \tau_{2n}\geqq\zeta, \end{cases}$$

for $n=1, 2, \cdots$. By virtue of M_5, we have $\lim_{n\to\infty}\tau_n=\zeta$, $\mathbf{P}_a$-almost certainly. Hence

$$\mathbf{M}_a\left[\int_0^\zeta \chi_U(x_s)d\varphi_s^V\right]=\mathbf{M}_a\left[\sum_{n=1}^\infty\int_{\tau_{2n-1}}^{\tau_{2n}} \chi_U(x_s)d\varphi_s^V\right]$$

$$\leqq\sum_{n=1}^\infty \mathbf{M}_a[\varphi_{\tau_{2n}-0}^V-\varphi_{\tau_{2n-1}}^V: \tau_{2n-1}<\zeta]=\sum_{n=1}^\infty \mathbf{M}_a[\mathbf{M}_{x\tau_{2n-1}}[\varphi_{\sigma_V-0}^V]]=0$$

by (4. 7). Letting ε tend to zero, we have

$$\mathbf{M}_a\left[\int_0^\zeta \chi_{E\setminus V}(x_s)d\varphi_s^V\right]=0,$$

so that

(4. 9) $$\mathbf{M}_a[\varphi_{t'-0}^V]=\mathbf{M}_a\left[\int_0^{t'} \chi_V(x_s)d\varphi_s^V\right].^{14)}$$

The both sides in (4. 9) are finite since they do not exceed

$$e^{\alpha_0 t}\mathbf{M}_a\left[\int_0^{t'} e^{-\alpha_0 s}\, d\varphi_s^V\right]\leqq e^{\alpha_0 t}\int_V g_{\alpha_0}(a, b)n(db)<\infty.$$

From this we can easily see (4. 8).

Next we shall prove

(4. 10) $$\mathbf{M}_a\left[\int_0^\zeta e^{-\alpha_0 t}\chi_V(x_t)d\varphi_t\right]=\int_V g_{\alpha_0}(a, b)n(db),$$

from which we can derive (4. 3) with the standard use of Dynkin's lemma ([2] Lemma 1. 2). Let V_1 be a closed set outside V. Using

$$\int_0^{t'} e^{-\alpha_0 s}\, d\varphi_s^V=\int_0^{t'} e^{-\alpha_0 s}\chi_V(x_s)d\varphi_s^V\leqq\int_0^{t'} e^{-\alpha_0 s}\chi_V(x_s)d\varphi_s$$

---

14) Notation: $t'=t\wedge\zeta(\omega)$.

and the same relation replacing V by V_1, we have

$$0 \leqq \mathbf{M}_a\left[\int_0^\zeta e^{-\alpha_0 s}\chi_V(x_s)d\varphi_s - \int_0^\zeta e^{-\alpha_0 s}\,d\varphi_s^V\right]$$

$$\leqq \mathbf{M}_a\left[\int_0^\zeta e^{-\alpha_0 s}\,d\varphi_s - \int_0^\zeta e^{-\alpha_0 s}\,d\varphi_s^V - \int_0^\zeta e^{-\alpha_0 s}\,d\varphi_s^{V_1}\right]$$

$$= \int_E g_{\alpha_0}(a, b)(1-\chi_V(b)-\chi_{V_1}(b))n(db).$$

Make V_1 swell to $E\setminus V$, then the last member above tends to zero and we get (4. 10). Thus the proof of Theorm 4. 1 is complete.

Put

$$g_{\alpha_0}^\lambda(a, b) = g_{\alpha_0}(a, b) - \lambda\mathbf{M}_a\left[\int_0^\zeta e^{-\alpha_0 t-\lambda\varphi t}g_{\alpha_0}(x_t, b)d\varphi_t\right] \tag{4. 11}$$

when the right-hand side is well-defined. Then the following theorem holds.

THEOREM 4. 2. *Fix $a\in E$ and $\lambda \geqq 0$. Then, for $(m+n)$-almost every b, $g_{\alpha_0}^\lambda(a, b)$ is well-defined, finite, non-negative, and not greater than $g_{\alpha_0}(a, b)$. And we have*

$$G_{\alpha_0}^\lambda f(a) = \int_E g_{\alpha_0}^\lambda(a, b)f(b)m(db), \qquad f\in B_0(E), \tag{4. 12}$$

and

$$K_\lambda^{\alpha_0} f(a) = \int_E g_{\alpha_0}^\lambda(a, b)f(b)n(db), \qquad f\in B(E). \tag{4. 13}$$

Proof. Without loss of generality we suppose $f\in B_0^+(E)$. Changing the order of integrations we get

$$\int_E \mathbf{M}_a\left[\int_0^\zeta e^{-\alpha_0 t-\lambda\varphi t}g_{\alpha_0}(x_t, b)d\varphi_t\right]f(b)m(db)$$

$$= \mathbf{M}_a\left[\int_0^\zeta e^{-\alpha_0 t-\lambda\varphi t}G_{\alpha_0}^0 f(x_t)d\varphi_t\right] = K_\lambda^{\alpha_0}G_{\alpha_0}^0 f(a).$$

On the other hand we have

$$G_{\alpha_0}^\lambda f - G_{\alpha_0}^0 f + \lambda K_\lambda^{\alpha_0}G_{\alpha_0}^0 f = 0. \tag{4. 14}$$

This is a special case of (2. 13) if $\alpha_0>0$, and if $\alpha_0=0$, is obtained by approximation of α_0 from above. (4. 14) implies finiteness of each term, and we can see that (4. 12) holds and that, for m-almost every b, $g_{\alpha_0}^\lambda(a, b)$ is well-defined, finite, non-negative and not greater than $g_{\alpha_0}(a, b)$. The rest of the proof is similar if we use Theorem 4. 1.

§ 5. The properties of measure n——sufficient conditions.

We continue to make the same assumptions as in the preceding section. Further we assume that α_0 is positive and that for any $b \in E$, m satisfies

$$\alpha_0 \int_E m(da) g_{\alpha_0}(a, b) = 1, \tag{5.1}$$

which implies, by Lemma 3.2, that m is an invariant measure for X and

$$\alpha \int_E G_\alpha^0 f(a) m(da) = \int_E f(a) m(da), \tag{5.2}$$

for any $\alpha > 0$ and $f \in B_0(E)$.

We shall study what properties n has for the processes $\tilde{X}$ and $\dot{X}$ obtained through time change and killing by φ_t, respectively. First we prepare a lemma due to M. Motoo.[15] Our proof is different from his.

LEMMA 5.1. *n is concentrated on F, i.e., $n(E \setminus F) = 0$.*

Proof. By virtue of Theorem 4.1,

$$\int_{E \setminus F} g_{\alpha_0}(a, b) n(db) = \mathbf{M}_a\left[\int_0^\zeta e^{-\alpha_0 t} \chi_{E \setminus F}(x_t) d\varphi_t\right] = \mathbf{M}_a\left[\int_0^{\zeta - 0} e^{-\alpha_0 \tau_t} \chi_{E \setminus F}(x_{\tau_t}) dt\right] = 0,$$

since $\mathbf{P}_a[x_{\tau_t} \in F] = 1$. Integrate with m, then use (5.1), we have $n(E \setminus F) = 0$ immediately.

THEOREM 5.1. *For any $\alpha > 0$ and $f \in B_0(E)$, we have*

$$\alpha \int_E K_0^\alpha f(a) m(da) = \int_F f(a) n(da). \tag{5.3}$$

Proof. We prove this for $f \in B_0^+(E)$. First note that

$$K_\beta^\alpha f - K_\beta^{\alpha_0} f + (\alpha - \alpha_0) G_\alpha^\beta K_\beta^{\alpha_0} f = 0, \tag{5.4}$$

which is a special case of (2.12). Integrate the formula with m and let β tend to zero. Then by the monotone convergence theorem, the second term tends to $-\int_E K_0^{\alpha_0} f(a) m(da)$, which is equal to $-(1/\alpha_0)\int_F f(a) n(da)$ by (5.1), Theorem 4.1 and Lemma 5.1. Similarly, the third term tends to $(1/\alpha_0 - 1/\alpha)\int_F f(a) n(da)$, while the first tends to $\int_E K_0^\alpha f(a) m(da)$. This proves (5.3).

THEOREM 5.2. *For any $\alpha, \beta > 0$ and $f \in B_0^+(E)$, n and m satisfy*

$$\alpha \int_F K_\alpha^0 f(a) n(da) \leq \int_F f(a) n(da), \tag{5.5}$$

$$\alpha \int_F G_0^\alpha f(a) n(da) \leq \int_E f(a) m(da), \tag{5.6}$$

15) Private communication.

(5. 7) $$\beta\int_E G_\beta^\alpha f(a)m(da)\leqq\int_E f(a)m(da).$$

(5. 5) *and* (5. 7) *mean that* n *and* m *are subinvariant measures for* X *and* $\dot{X}$, *respectively.*

Proof. Integrating the formula $K_0^\beta f-K_\alpha^\beta f-\alpha K_0^\beta K_\alpha^\beta=0$ by the measure βm, and using the preceding theorem, we obtain

(5. 8) $$\int_F f(a)n(da)-\beta\int_E K_\alpha^\beta f(a)m(da)-\alpha\int_F K_\alpha^\beta f(a)n(da)=0.$$

Letting β decrease to zero, (5. 5) follows. To prove (5. 6), we start from the formula

(5. 9) $$G_\alpha^0 f-G_\beta^\alpha f+(\alpha-\beta)G_\alpha^0 G_\beta^\alpha f-\alpha K_0^\alpha G_\beta^\alpha f=0,$$

a special case of (2. 13). Integrating with m, then using the preceding theorem and (5. 2), we have

(5. 10) $$\int_E f(a)m(da)-\beta\int_E G_\beta^\alpha f(a)m(da)-\alpha\int_F G_\beta^\alpha f(a)n(da)=0,$$

from which (5. 6) follows as β tends to zero. (5. 7) is a direct consequence of (5. 10). Lemma 3. 3 completes the proof.

In many cases, we can replace inequalities in (5. 5) and (5. 6) by equalities. We shall give some sufficient conditions. The necessary and sufficient conditions are treated in §7 in more restricted situations.

THEOREM 5. 3. *Suppose that at least one of the following conditions is satisfied:*

(i) *m is a finite measure;*

(ii) *n is a finite measure and it holds that for any $a\in F$,*

(5. 11) $$\mathbf{P}_a[\varphi_{\zeta-0}=\infty]=1;$$

(iii) *X is conservative and recurrent and G_α^0 maps $B(E)$ into $C(E)$;*

(iv) *There is a constant $k>0$ such that $km(A)\leqq n(A)$ for every $A\in\mathfrak{B}$;*

(v) *For some $\alpha>0$, $\int_E K_\alpha^0 f(a)m(da)$ is finite for every $f\in C_0^+(E)$;*

(vi) *For some $\alpha>0$, it holds that*

(5. 12) $$\lim_{\beta\downarrow 0}\beta\int_E K_\alpha^\beta f(a)m(da)=0,\qquad \textit{for}\quad f\in C_0^+(E).$$

Then, for all $\alpha>0$, we have

(5. 13) $$\alpha\int_F K_\alpha^0 f(a)n(da)=\int_F f(a)n(da),\qquad \textit{for}\quad f\in B_0(E),$$

and

(5. 14) *n is an invariant measure for $\tilde{X}$.*

Furthermore, if X is conservative and satisfies M_8, then

(5. 15) *the terminal measure of $(\dot{X}, n)$ is n itself.*

Conversely, (5. 14) *implies* (5. 12) *for all* $\alpha>0$.

Proof. Keeping in mind the formula (5. 8) and Lemma 3. 2, (5. 12) for some α, (5. 12) for all α, (5. 13) for some α, (5. 13) for all α, and (5. 14) are mutually equivalent. As to (5. 15) Theorem 3. 4 is applied. Since $||K_\alpha^0 f|| \leqq (1/\alpha)\,||f||$, (i) is sufficient for (v), which obviously implies (vi). Noting that $\int_F K_\alpha^0 f(a) n(da) < \infty$ for $f \in B_0^+(E)$ by (5. 5), (iv) is also sufficient for (v). Hence it remains to prove that (ii) and (iii) are sufficient conditions. Now assume (ii) and let $0 \leqq f \leqq 1$. It follows that

$$\alpha \int_F K_\alpha^0(1-f)(a) n(da) \leqq \int_F (1-f(a)) n(da),$$

which, combined with (5. 5), implies (5. 13), since $\alpha K_\alpha^0 1(a)=1$ on F. In order to prove the sufficiency of (iii), we need a

LEMMA 5. 2.[16] *If* (iii) *is assumed, any continuous additive functional φ_t, not identically zero, satisfies*

(5. 16) $$\mathbf{P}_a[\varphi_\infty=\infty]=1, \qquad \textit{for } a \in E.$$

Proof. Put $u(a)=1-\mathbf{M}_a[e^{-\varphi_\infty}]$. Then u is a bounded excessive function, for $\mathbf{M}_a[u(x_t)]=1-\mathbf{M}_a[e^{-(\varphi_\infty-\varphi_t)}]$. Lower semi-continuity of u follows since G_α^0 maps $B(E)$ into $C(E)$. Take a and b arbitrarily. Then

$$u(a) \geqq \mathbf{M}_a[u(x_{\sigma_U})], \qquad \text{for any open } U \text{ containing } b.$$

Letting $U \downarrow b$, we find

$$u(a) \geqq \liminf_{c \to b} u(c) \geqq u(b)$$

by the recurrence and lower semi-continuity. Thus u is a constant function. Hence

$$\text{const.}=\mathbf{M}_a[e^{-\varphi_\infty}]=\mathbf{M}_a[\mathbf{M}_{x_t}[e^{-\varphi_\infty}]]=\mathbf{M}_a[e^{-(\varphi_\infty-\varphi_t)}] \to \mathbf{P}_a[\varphi_\infty<\infty] \text{ as } t \to \infty,$$

so that $\mathbf{P}_a[\varphi_\infty=0 \text{ or } \infty]=1$ and $\mathbf{P}_a[\varphi_\infty=0]=\text{const.}$ On the other hand $\mathbf{P}_a[\varphi_\infty=0]=0$ for $a \in F$ and the proof of the lemma is complete.

Making use of the above lemma, let us finish the proof of Theorem 5. 3. Take a compact set V so large that

$$\mathbf{M}_a\left[\int_0^\infty e^{-\alpha_0 t} \chi_V(x_t) d\varphi_t\right]=\int_V g_{\alpha_0}(a, b) n(db)$$

is not identically zero. Put $\bar{\varphi}_t=\int_0^t \chi_V(x_s) d\varphi_s$, then the previous lemma implies that $\mathbf{P}_a[\bar{\varphi}_\infty=\infty]=1$. Hence, by the suffciency of (ii), we have

16) This lemma as well as its proof is due to H. Tanaka (private communication).

(5. 17) $$\lim_{\beta\downarrow 0}\beta\int_E \bar{K}_\alpha^\beta f(a)m(da)=0, \qquad \text{for} \quad f\in B_0(E),$$

where

$$\bar{K}_\alpha^\beta f(a)=\mathbf{M}_a\left[\int_0^\infty {}^{-\alpha\bar{\varphi}_t-\beta t} f(x_t)d\bar{\varphi}_t\right].$$

For any $f\in B_0^+(E)$, take V containing the support of f. Then,

$$\bar{K}_\alpha^\beta f(a)=\mathbf{M}_a\left[\int_0^\infty e^{-\alpha\bar{\varphi}_t-\beta t} f(x_t)d\varphi_t\right]\geqq K_\alpha^\beta f(a),$$

whence,

$$\lim_{\beta\downarrow 0}\beta\int_E K_\alpha^\beta f(a)m(da)=0,$$

which completes the proof.

We cannot prove the analogous sufficient conditions for the validity of

(5. 18) $$\alpha\int_F G_0^\alpha f(a)n(da)=\int_E f(a)m(da), \qquad \text{for} \quad \alpha>0,\ f\in B_0(E),$$

until some additional assumptions are imposed in the next section. Here we mention only the following

THEOREM 5. 4. *Suppose that* (iv) *holds true or that one of the followings is satisfied*:

(vii) $\int_E G_0^\alpha f(a)m(da)$ *is finite for every* $\alpha>0$ *and* $f\in C_0^+(E)$;

(viii) *For every* $\alpha>0$,

(5. 19) $$\lim_{\beta\downarrow 0}\beta\int_E G_\beta^\alpha f(a)m(da)=0, \qquad \textit{for} \quad f\in C_0^+(E).$$

Then, (5. 18) *holds true and*

(5. 20) *the Green measure of* $(\dot{X},\ n)$ *is the measure* m.

(viii) *is also a necessary condition.*

These are proved similarly to the corresponding parts of the proof of Theorem 5. 3 by making use of (5. 10).

§6. The adjoints of $\tilde{X}$ and $\dot{X}$.

We shall study the adjoint processes of $\tilde{X}$ and $\dot{X}$ with respect to appropriate measures. For the purpose we need to assume the existence of the adjoint process $\hat{X}$ of X and continuous additive functional $\hat{\varphi}_t$ of $\hat{X}$.[17] To be precise, let X and φ_t

17) We denote the quantities of $\hat{X}$ by putting $\wedge$ on.

be originally given, we assume the existence of $\hat{X}$, $\hat{\varphi}_t$, $g_\alpha(a, b)$, m, and n with the following properties: 1) $\hat{X}$ is a Markov process with state space $(E, \mathfrak{B})$ satisfying $M_1 \sim M_7$. 2) $\hat{\varphi}_t$ is a continuous non-negative additive functional of $\hat{X}$. 3) m and n are σ-finite measures on E. 4) $\{g_\alpha(a, b): \alpha>0\}$ is a family of non-negative (possibly infinite) $\mathfrak{B}\times\mathfrak{B}$-measurable functions such that, as a function of a, $g_\alpha(a, b)$ is α-harmonic in $E\backslash b$ and α-excessive relative to X, while, as a function of b, α-harmonic in $E\backslash a$ and α-excessive relative to $\hat{X}$. 5) G_α^0 and $\hat{G}_\alpha^0$ is expressed as

$$G_\alpha^0 f(a)=\int_E g_\alpha(a, b)f(b)m(db), \tag{6.1}$$

$$\hat{G}_\alpha^0 f(b)=\int_E f(a)m(da)g_\alpha(a, b).^{18)} \tag{6.2}$$

6) For some $\alpha_0>0$

$$K_0^{\alpha_0}1(a)\equiv \mathbf{M}_a\left[\int_0^\zeta e^{-\alpha_0 t}\, d\varphi_t\right]=\int_E g_{\alpha_0}(a, b)n(db) \tag{6.3}$$

holds and is bounded in a, and similarly, for some $\hat{\alpha}_0>0$,

$$\hat{K}_0^{\hat{\alpha}_0}1(b)\equiv \hat{\mathbf{M}}_b\left[\int_0^{\hat{\zeta}} e^{-\hat{\alpha}_0 t}\, d\hat{\varphi}_t\right]=\int_E n(da)g_{\hat{\alpha}_0}(a, b) \tag{6.4}$$

holds and is bounded in b. We assume that (6. 3) and (6. 4) are not identically zero.

(6. 1) and (6. 2) imply that X and $\hat{X}$ are mutually adjoint with respect to m. We shall prove that $\tilde{X}$ and $\hat{\tilde{X}}$, which is obtained from $\hat{X}$ through time change by $\hat{\varphi}_t$, are mutually adjoint with respect to n, and that $\dot{X}$ and $\hat{\dot{X}}$, killed process of $\hat{X}$ by $\hat{\varphi}_t$, are mutually adjoint with respect to m. These results have intimate connections with a part of Hunt (§17, §18 in [4]) and Meyer [8].

By Theorem 2. 2, it follows from the assumption above that $K_0^\alpha f$ and $\hat{K}_0^\alpha f$ are bounded for any $\alpha>0$ and $f\in B(E)$.

LEMMA 6. 1. *If U is open in the intrinsic topology[19] induced by X or $\hat{X}$, then* $m(U)>0$.

Proof is immediate, because $G_\alpha^0\chi_U(a)>0$ or $\hat{G}_\alpha^0\chi_U(a)>0$ if $a\in U$.

Let us denote $F_0=F\cap\hat{F}$ where $\hat{F}=\{a: \hat{\mathbf{P}}_a[\hat{\tau}_0>0]=0\}$. Then,

LEMMA 6. 2. *n is concentrated on F_0, i.e., $n(E\backslash F_0)=0$.*

Proof. To prove $n(E\backslash F)=0$ under the present assumption, we may carry over the proof of Lemma 5. 1, replacing the use of (5. 1) by

$$\int_E m(da)g_{\alpha_0}(a, b)=\hat{G}_{\alpha_0}^0 1(b)>0.$$

18) As to the sufficient conditions for this, see Hunt [4] and Meyer [8].

19) Defined in Dynkin [3].

Similarly, we have $n(E\backslash\hat{F})=0$.

LEMMA 6. 3. *For any* $\alpha>0$ *and* $f\in B(E)$,

$$K_0^\alpha f(a)=\int_{F_0} g_\alpha(a, b)f(b)n(db), \tag{6. 5}$$

$$\hat{K}_0^\alpha f(b)=\int_{F_0} f(a)n(da)g_\alpha(a, b). \tag{6. 6}$$

Proof. Note that

$$\begin{aligned} g_\beta(a, b)&=g_\alpha(a, b)+(\alpha-\beta)\int_E g_\alpha(a, c)m(dc)g_\beta(c, b) \\ &=g_\alpha(a, b)+(\alpha-\beta)\int_E g_\beta(a, c)m(dc)g_\alpha(c, b) \end{aligned} \tag{6. 7}$$

holds for all a and b if $\alpha>\beta$. For, (6. 7) is evident from the resolvent equation for any a and m-almost every b but, by excessivity, the both sides of (6. 7) are continuous in b in the intrinsic topology induced by $\hat{X}$. Hence, by Lemma 6. 1, (6. 7) is valid for all a and b.

Let us prove (6. 5), while (6. 6) is proved in the same way. If $\alpha=\alpha_0$, (6. 5) is found in Theorem 4. 1. If $\alpha<\alpha_0$, then, using (6, 7),

$$\begin{aligned} &\int_{F_0} g_\alpha(a, b)f(b)n(db) \\ &=\int g_{\alpha_0}(a, b)f(b)n(db)+(\alpha_0-\alpha)\int\int g_\alpha(a, c)m(dc)g_{\alpha_0}(c, b)f(b)n(db) \\ &=K_0^{\alpha_0}f(a)+(\alpha_0-\alpha)G_\alpha^0K_0^{\alpha_0}f(a)=K_0^\alpha f(a). \end{aligned}$$

If $\alpha>\alpha_0$, then

$$\begin{aligned} &K_0^\alpha f(a)+(\alpha-\alpha_0)G_\alpha^0K_0^{\alpha_0}f(a)=K_0^{\alpha_0}f(a)=\int g_{\alpha_0}(a, b)f(b)n(db) \\ &=\int g_\alpha(a, b)f(b)n(db)+(\alpha-\alpha_0)\int\int g_\alpha(a, c)m(dc)g_{\alpha_0}(c, b)f(b)n(db) \\ &=\int g_\alpha(a, b)f(b)n(db)+(\alpha-\alpha_0)G_\alpha^0K_0^{\alpha_0}f(a), \end{aligned}$$

which completes the proof.

Define $g_\alpha^\lambda(a, b)$ as in §4 and $\hat{g}_\alpha^\lambda(a, b)$ similarly, i.e.,

$$g_\alpha^\lambda(a, b)=g_\alpha(a, b)-\lambda\mathbf{M}_a\left[\int_0^\zeta e^{-\alpha t-\lambda\varphi t}g_\alpha(x_t, b)d\varphi_t\right], \tag{6. 8}$$

$$\hat{g}_\alpha^\lambda(a, b)=g_\alpha(a, b)-\lambda\hat{\mathbf{M}}_b\left[\int_0^{\hat{\zeta}} e^{-\alpha t-\lambda\hat{\varphi}t}g_\alpha(a, \hat{x}_t)d\hat{\varphi}_t\right]. \tag{6. 9}$$

Then, as shown in Theorem 4. 2,

$$K_\lambda^\alpha f(a)=\int_{F_0} g_\alpha^\lambda(a, b) f(b) n(db), \tag{6. 10}$$

$$\hat{K}_\lambda^\alpha f(b)=\int_{F_0} f(a) n(da) \hat{g}_\alpha^\lambda(a, b), \tag{6. 11}$$

$$G_\alpha^\lambda f(a)=\int_E g_\alpha^\lambda(a, b) f(b) m(db), \tag{6. 12}$$

$$\hat{G}_\alpha^\lambda f(b)=\int_E f(a) m(da) \hat{g}_\alpha^\lambda(a, b), \tag{6. 13}$$

for all $\alpha>0$, $\lambda\geqq 0$ and $f\in B(E)$.

THEOREM 6. 1. *For any* $\alpha\geqq 0$ *and* $\lambda\geqq 0$ *such that* $\alpha+\lambda>0$ *and for any* $f, g \in B_0(E)$, *we have*

$$\int_{F_0} K_\lambda^\alpha f(a) g(a) n(da)=\int_{F_0} f(b) \hat{K}_\lambda^\alpha g(b) n(db). \tag{6. 15}$$

For any $\alpha>0$ *and* $\lambda\geqq 0$,

$$\int_E G_\alpha^\lambda f(a) g(a) m(da)=\int_E f(b) \hat{G}_\alpha^\lambda g(b) m(db) \tag{6. 16}$$

holds. In other words, the processes with $\{K_\lambda^\alpha\colon \lambda>0\}$ *and* $\{\hat{K}_\lambda^\alpha\colon \lambda>0\}$ *as their resolvents are mutually adjoint with respect to* n, *and the processes with resolvents* $\{G_\alpha^\lambda\colon \alpha>0\}$ *and* $\{\hat{G}_\alpha^\lambda\colon \alpha>0\}$ *are mutually adjoint with respect to* m.

Proof. Fix $\alpha>0$. In order to obtain (6. 15), it is sufficient to verify that

$$\iint g(a) n(da) g_\alpha^\lambda(a, b) f(b) n(db)=\iint g(a) n(da) \hat{g}_\alpha^\lambda(a, b) f(b) n(db). \tag{6. 17}$$

First we note that $g_\alpha(a, b)\geqq g_\alpha^\lambda(a, b)\geqq 0$ and $g_\alpha(a, b)\geqq \hat{g}_\alpha^\lambda(a, b)\geqq 0$ for $n\times n$-almost every (a, b). Rewriting $K_\lambda^\alpha f=K_0^\alpha f-\lambda K_0^\alpha K_\lambda^\alpha f$, we have

$$\int g_\alpha^\lambda(a, b) f(b) n(db)=\int g_\alpha(a, b) f(b) n(db)-\lambda\iint g_\alpha(a, c) n(dc) g_\alpha^\lambda(c, b) f(b) n(db).$$

On the other hand, integrating $\hat{K}_\lambda^\alpha g=\hat{K}_0^\alpha g-\lambda \hat{K}_\lambda^\alpha \hat{K}_0^\alpha g$ with fn, we have, since g is arbitrary,

$$\int \hat{g}_\alpha^\lambda(a, b) f(b) n(db)=\int g_\alpha(a, b) f(b) n(db)-\lambda\iint g_\alpha(a, c) n(dc) \hat{g}_\alpha^\lambda(c, b) f(b) n(db)$$

for n-almost every a. Put

$$u(a)=\int (g_\alpha^\lambda(a, b)-\hat{g}_\alpha^\lambda(a, b)) f(b) n(db).$$

Then $u\in L_\infty(dn)$ and we have, for n-almost every a,

$$u(a)=-\lambda\int\int g_\alpha(a, c)n(dc)(g_\alpha^\lambda(c, b)-\hat{g}_\alpha^\lambda(c, b))f(b)n(db)$$

$$=-\lambda\int g_\alpha(a, c)u(c)n(dc)=-\lambda K_0^\alpha u(a).$$

Hence, $\|u\|_\infty\leqq\lambda\|K_0^\alpha\|\,\|u\|_\infty$.[20] Therefore, for $\lambda<1/\|K_0^\alpha\|$, $\|u\|_\infty=0$, i.e., $u=0$ n-almost everywhere. Thus (6. 17) is verified for $\lambda<1/\|K_0^\alpha\|$.

Suppose that (6. 17) is valid for $\lambda=\lambda_0>0$. Rewriting $K_\lambda^\alpha f=K_{\lambda_0}^\alpha f+(\lambda_0-\lambda)K_{\lambda_0}^\alpha K_\lambda^\alpha f$ we have on the one hand

$$\int g_\alpha^\lambda(a, b)f(b)n(db)$$

$$=\int g_\alpha^{\lambda_0}(a, b)f(b)n(db)+(\lambda_0-\lambda)\int\int g_\alpha^{\lambda_0}(a, c)n(dc)g_\alpha^\lambda(c, b)f(b)n(db),$$

and, on the other, integrating $\hat{K}_\lambda^\alpha g=\hat{K}_{\lambda_0}^\alpha g+(\lambda_0-\lambda)\hat{K}_\lambda^\alpha\hat{K}_{\lambda_0}^\alpha g$ with fn, it follows that, since g is arbitrary,

$$\int\hat{g}_\alpha^\lambda(a, b)f(b)n(db)$$

$$=\int\hat{g}_\alpha^{\lambda_0}(a, b)f(b)n(db)+(\lambda_0-\lambda)\int\int\hat{g}_\alpha^{\lambda_0}(a, c)n(dc)\hat{g}_\alpha^\lambda(c, b)f(b)n(db)$$

$$=\int g_\alpha^{\lambda_0}(a, b)f(b)n(db)+(\lambda_0-\lambda)\int\int g_\alpha^{\lambda_0}(a, c)n(dc)\hat{g}_\alpha^\lambda(c, b)f(b)n(db)$$

for n-almost every a. Put

$$v(a)=\int(g_\alpha^\lambda(a, b)-\hat{g}_\alpha^\lambda(a, b))f(b)n(db).$$

Then $v=(\lambda_0-\lambda)K_{\lambda_0}^\alpha v$, implying $\|v\|_\infty\leqq(|\lambda_0-\lambda|/\lambda_0)\|v\|_\infty$. Hence, for $\lambda<2\lambda_0$, $v=0$ n almost everywhere. Accordingly, (6. 17) is true for all $\lambda\geqq0$ and $\alpha>0$. If $\alpha=0$ and $\lambda>0$, (6. 15) is verified through approximation of α from above.

Making use of $G_\alpha^\lambda f=G_\alpha^0 f-\lambda K_0^\alpha G_\alpha^\lambda f$ and $\hat{G}_\alpha^\lambda g=\hat{G}_\alpha^0 g-\lambda\hat{K}_\lambda^\alpha\hat{G}_\alpha^0 g$, the proof of (6. 16) is carried as above except obvious changes. Hence Theroem 6. 1 is proved.

Quite similarly we are able to prove the following formulae.

THEOREM 6. 2. *For any $\alpha\geqq0$ and $\lambda\geqq0$ such as $\alpha+\lambda>0$, and for $f, g\in B_0(E)$, we have*

(6. 18) $$\int_{F_0} G_\alpha^\lambda f(a)g(a)n(da)=\int_E f(b)\hat{K}_\lambda^\alpha g(b)m(db)$$

and

(6. 19) $$\int_E K_\lambda^\alpha f(a)g(a)m(da)=\int_{F_0} f(b)\hat{G}_\alpha^\lambda g(b)n(db).$$

20) The norm in $L_\infty(dn)$ is denoted by $\|\cdot\|_\infty$.

§7. The properties of measure n——necessary and sufficient conditions.

Under the assumptions stated in the first paragraph of §6, we study once more the properties of n treated in §5. Although the assumptions are stronger than those in §5 in several respects, they are weaker in that the invariance of the measure m for X is not assumed.

Concerning the validity of (5.13) we have

THEOREM 7.1. *The following statements are equivalent*:

(7.1) n *is an invariant measure for* $\tilde{X}$;

(7.2) $\tilde{\hat{X}}$ *is conservative*;

(7.3) $\hat{\mathbf{P}}_a[\hat{\zeta}=\infty,\ \hat{\phi}_\infty=\infty]=1$, *for every* $a\in\hat{F}$;

(7.4) $\hat{\mathbf{P}}_a[\hat{\phi}_{\hat{\zeta}-0}=\infty]=1$, *for* n*-almost every* $a\in F_0$.

First, we prepare

LEMMA 7.1. *If* $a\in\hat{F}$, *then* $n(U)>0$ *for any* U *containing* a *and being open in the intrinsic topology induced by* $\hat{X}$.

Proof. Suppose $n(U)=0$. Then, we can prove $\hat{\mathbf{P}}_a[\hat{\phi}_t=0 \text{ for } t<\hat{\sigma}_{E\setminus U}]=1$ in the same way as in the proof of Theorem 4.1. This contradicts to $a\in\hat{F}$, since $\hat{\mathbf{P}}_a[\hat{\sigma}_{E\setminus U}>0]=1$.

Proof of Theorem 7.1. $\hat{\mathbf{P}}_a[\hat{\phi}_{\hat{\zeta}-0}=\infty]=1$ is equivalent to $\alpha\hat{K}_\alpha^0 1(a)=1$. Because

(7.5) $$\int_{F_0} K_\alpha^0 f(a)n(da)=\int_{F_0} f(a)\hat{K}_\alpha^0 1(a)n(da) \qquad \text{for } f\in B_0^+(E),$$

(7.1) and (7.4) are equivalent. Obviously (7.3) implies (7.2), and (7.2) implies (7.4), so that it remains to prove that (7.3) follows from (7.4). Suppose that (7.4) holds. Then, by Lemma 7.1, the point a at which $\alpha\hat{K}_\alpha^0 1(a)=1$ are dense on $\hat{F}$ in the intrinsic topology induced by $\hat{X}$. On the other hand $\alpha\hat{K}_\alpha^0 1(a)$ is continuous in the topology, since it is α-excessive relative to $\hat{X}$. Hence $\alpha\hat{K}_\alpha^0 1(a)=1$ for all $a\in\hat{F}$. Next, let us prove $\hat{\mathbf{P}}_a[\hat{\zeta}=\infty]=1$ on $\hat{F}$. If it be not true, then $\hat{\mathbf{P}}_a[\hat{\phi}_{\hat{t}-0}=\infty]>0$[21] for some t, which is absurd, since $\hat{\mathbf{M}}_a[\hat{\phi}_{\hat{t}}]\leqq e^{\alpha t}\hat{\mathbf{M}}_a[\int_0^{\hat{t}} e^{-\alpha s}d\hat{\phi}_s]\leqq e^{\alpha t}\hat{K}_0^\alpha 1(a)<\infty$. Thus (7.3) holds, completing the proof.

Next we give a theorem concerning (5.18).

THEOREM 7.2. *Each of the following conditions is equivalent to* (5.18):

(7.6) $\hat{\mathbf{P}}_a[\hat{\zeta}=\infty,\ \hat{\phi}_\infty=\infty]=1$ *for every* $a\in E$;

(7.7) $\hat{\mathbf{P}}_a[\hat{\phi}_{\hat{\zeta}-0}=\infty]=1$ *for* m*-almost every* $a\in E$.

Proof is omitted, because it is similar to that of Theorem 7.1 using Lemma

21). $\hat{t}=t\wedge\hat{\zeta}$.

6. 1 and Theorem 6. 2 instead of Lemma 7. 1 and Theorem 6. 1.

COROLLARY. (5. 18) *implies* (5. 13).

The conditions in the above theorems can be stated in terms of $\hat{G}_\alpha^\lambda$. Namely, if $\hat{X}$ is conservative, $\hat{\mathbf{P}}_a[\hat{\varphi}_{\hat{\zeta}-0}=\infty]=1$ if and only if $\lim_{\alpha\downarrow 0}\alpha\hat{G}_\alpha^\lambda 1(a)=0$. For, we have

$$\alpha\hat{G}_\alpha^\lambda 1+\lambda\hat{K}_\lambda^\alpha 1-1=0, \tag{7. 8}$$

which follows from $\hat{G}_\alpha^\lambda 1-\hat{G}_\mu^0 1+(\alpha-\mu)\hat{G}_\alpha^\lambda\hat{G}_\mu^0 1+\lambda\hat{K}_\lambda^\alpha\hat{G}_\mu^0 1=0$ and $\hat{G}_\mu^0 1=1/\mu$.

We give some sufficient conditions for (5. 18), but (ii), (iii) and (iv) are unsatisfactory, since they explicitely include conditions on $\hat{X}$.

THEOREM 7. 3. *Suppose that* (5. 1) *holds. Then, each of the followings is sufficient for* (5. 18):

(i) *m is finite and* $\mathbf{P}_a[\varphi_{\zeta-0}=\infty]=1$ *for any* $a\in E$;

(ii) *m is finite and* $\hat{\mathbf{P}}_a[\hat{\varphi}_{\hat{\zeta}-0}>0]=1$ *for any* $a\in E$;

(iii) *n is finite,* $\mathbf{P}_a[\varphi_{\zeta-0}=\infty]=1$ *for any* $a\in F$, *and* $\hat{\mathbf{P}}_a[\hat{\varphi}_{\hat{\zeta}-0}>0]=1$ *for any* $a\in E$;

(iv) *$\hat{X}$ is conservative and recurrent, and $\hat{G}_\alpha^0$ maps $B(E)$ into $C(E)$.*

Proof. Let (i) be met. Then, by Theorem 5. 4, Theorem 7. 2 and the symmetricity of X and $\hat{X}$,

$$\lim_{\beta\downarrow 0}\beta\int_E \hat{G}_\beta^\alpha f(a)m(da)=0, \qquad \text{for } f\in B_0^+(E).$$

Let a compact set V be sufficiently large, then make $\beta\downarrow 0$ in

$$\beta\int_E G_\beta^\alpha 1(a)m(da)\leqq\beta\int_V G_\beta^\alpha 1(a)m(da)+m(E\setminus V)=\beta\int_E \hat{G}_\beta^\alpha \chi_V(a)m(da)+m(E\setminus V).$$

Thus we have (5. 19), hence (5. 18) is valid. The sufficiency of (ii) or (iii) is proved, combining Theorems 5. 3, 7. 1 and 7. 2 by the lemma below. Lemma 5. 2 and Theorem 7. 2 imply the sufficiency of (iv).

LEMMA 7. 2. *Suppose* $\hat{\mathbf{P}}_a[\hat{\varphi}_{\hat{\zeta}-0}=\infty]=1$ *for* $a\in\hat{F}$. *Then* $\hat{\mathbf{P}}_b[\hat{\varphi}_{\hat{\zeta}-0}=\infty]=1$ *for all* $b\in E$ *at which* $\hat{\mathbf{P}}_b[\hat{\varphi}_{\hat{\zeta}-0}>0]=1$.

Proof. We need only note

$$\begin{aligned}\alpha\hat{K}_\alpha^0 1(b)&=\alpha\hat{\mathbf{M}}_b\left[\int_0^{\hat{\zeta}} e^{-\alpha\hat{\varphi}_t}\,d\hat{\varphi}_t\right]=\alpha\hat{\mathbf{M}}_b\left[\hat{\mathbf{M}}_{\hat{x}_{\hat{\tau}_0}}\left[\int_0^{\hat{\zeta}} e^{-\alpha\hat{\varphi}_t}\,d\hat{\varphi}_t\right]\right]\\&=\hat{\mathbf{P}}_b[\hat{\tau}_0<\hat{\zeta}]=\hat{\mathbf{P}}_b[\varphi_{\hat{\zeta}-0}>0].\end{aligned}$$

In self-adjoint case (i.e. $X=\hat{X}$), the above theorems are reduced to much simpler form. In the case of the Brownian motion, McKean-Tanaka [7] has treated the condition for $\mathbf{P}_a[\varphi_\infty=\infty]=1$.

It may be of use to give a remark that $\mathbf{P}_a[\varphi_{\zeta-0}=\infty]=1$ does not necessarily imply $\hat{\mathbf{P}}_a[\hat{\varphi}_{\zeta-0}=\infty]=1$, as an example shows. Let X and $\hat{X}$ be the deterministic motions on the real line with velocity one to the right and to the left, respectively. Let m be the Lebesgue measure and n be its restriction to $[0, \infty)$. Then they satisfy all the assumptions in the beginning of §6. The additive functionals of X and $\hat{X}$ corresponding to n are

$$\varphi_t=\int_0^t \chi_{[0,\infty)}(x_s)ds \quad \text{and} \quad \hat{\varphi}_t=\int_0^t \chi_{[0,\infty)}(\hat{x}_s)ds.$$

It is easy to see that $\mathbf{P}_a[\varphi_\infty=\infty]=1$ for all a, while $\hat{\mathbf{P}}_a[\hat{\varphi}_\infty<\infty]=1$ for all a, and

$$g_\alpha^\lambda(a, b)=\hat{g}_\alpha^\lambda(a, b)$$
$$=\begin{cases} 0, & \text{for} \quad b>a, \\ e^{-\alpha(b-a)-\lambda(b-a)}, & \text{for} \quad b>a\geqq 0, \\ e^{-\alpha(b-a)-\lambda b}, & \text{for} \quad b>a,\ a<0,\ b\geqq 0, \\ e^{-\alpha(b-a)}, & \text{for} \quad b>a,\ a<0,\ b<0. \end{cases}$$

Therefore,

$$K_\lambda^0 f(a)=e^{\lambda a}\int_a^\infty e^{-\lambda b}f(b)db \qquad \text{for} \quad a\geqq 0,$$

$$\hat{K}_\lambda^0 f(a)=e^{-\lambda a}\int_0^a e^{\lambda b}f(b)db, \qquad \text{for} \quad a\geqq 0,$$

thus,

$$\lambda K_\lambda^0 1(a)=1, \qquad \lambda\int K_\lambda^0 f(a)n(da)=\int(1-e^{-\lambda a})f(a)n(da),$$

$$\lambda \hat{K}_\lambda^0 1(a)=1-e^{-\lambda a}, \quad \lambda\int \hat{K}_\lambda^0 f(a)n(da)=\int f(a)n(da).$$

That is, $\tilde{X}$ is conservative and n is not invariant measure for $\tilde{X}$, though subinvariant. $\tilde{\hat{X}}$ is not conservative but has an invariant measure n. In this example, the state spaces F (of $\tilde{X}$) and $\hat{F}$ (of $\tilde{\hat{X}}$) are not identical, since 0 belongs to F but not to $\hat{F}$.

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Added in proof. 1°. Lemma 3. 3 is found with a different proof in H. Kunita and H. Nomoto, Methods of compactifications relative to Markov processes and their applications, Seminar on Prob. **14** (1962) (Japanese).

2°. Remark to Theorem 3. 1. By a modification of $\tilde{\mathfrak{M}}_t$, $\tilde{X}$ satisfies M_7 also.

3°. Remark to Theorem 3. 2. If φ_t is $\mathfrak{N}_t$-measurable for each $t\geqq 0$, $\dot{X}$ satisfies M_1 also.

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