SOME EXPANSION THEOREMS FOR STOCHASTIC PROCESSES, I

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1. Let $\mathcal{E}(t)$ $(-\infty<t<\infty)$ be a continuous stationary stochastic process of the second order (in the wide sense) with mean zero; that is,

$$E\{\mathcal{E}(t+u)\overline{\mathcal{E}(t)}\}=\rho(u) \tag{1. 1}$$

is a continuous function of u only, and

$$E\{\mathcal{E}(t)\}=0,\ -\infty<t<\infty. \tag{1. 2}$$

$\rho(u)$ is called the correlation function of $\mathcal{E}(t)$. We have, then,

$$\mathcal{E}(t)=\int_{-\infty}^{\infty} e^{it\lambda}\,dZ(\lambda) \tag{1. 3}$$

and

$$\rho(u)=\int_{-\infty}^{\infty} e^{iu\lambda}\,dF(\lambda), \tag{1. 4}$$

where $F(\lambda)$ is a bounded non-decreasing function such that $F(\infty)-F(-\infty)=\rho(0)=E\{|\mathcal{E}(t)|^2\}$, and $Z(\lambda)$ is an orthogonal process such that $E\{|Z(\lambda')-Z(\lambda)|^2\}=F(\lambda'-0)-F(\lambda-0)$. $F(\lambda)$ and $Z(\lambda)$ are called the spectral function and the random spectral function of $\mathcal{E}(t)$ respectively.

Let

$$X(t)=f(t)+\mathcal{E}(t),\ -\infty<t<\infty, \tag{1. 5}$$

and consider

$$n\int_{-\infty}^{\infty} X(t-s)\,K(ns)\,ds=\int_{-\infty}^{\infty} X\left(t-\frac{s}{n}\right)K(s)\,ds, \tag{1. 6}$$

where $f(t)$ and $K(s)$ are numerical valued functions. Kawata [5] has shown that if (i) $f(s)/(1+|s|^{3/2})\in L^1(-\infty,\infty)$, (ii) $f(t+u)-f(t)=O(u)$ for small u, (iii) $(1+|s|)\,K(s)$

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$\in L^1(-\infty, \infty)$, (iv) $K(s)$ is bounded and $o(|s|^{-3/2})$ as $|s|\to\infty$ and (v)

$$\int_{-\infty}^{\infty} |\lambda|\, dF(\lambda)<\infty,$$

then it holds that

$$E\left\{\left|n\int_{-\infty}^{\infty} X(t-s)\,K(ns)\,ds-X(t)\int_{-\infty}^{\infty} K(s)\,ds\right|^2\right\}=o\left(\frac{1}{n}\right) \quad \text{as } n\to\infty. \tag{1.7}$$

In the following, we shall make this result more complete.

2. Let $F(\lambda)$ be the spectral function of a continuous stationary process $\mathcal{E}(t)$. If

$$\int_{-\infty}^{\infty} \lambda^{2r}\, dF(\lambda)<\infty, \tag{2.1}$$

r being a positive integer, then

$$\mathcal{E}^{(k)}(t)=\underset{h\to 0}{\mathrm{l.i.m.}}\ \frac{\mathcal{E}^{(k-1)}(t+h)-\mathcal{E}^{(k-1)}(t)}{h} \tag{2.2}$$

exists for $k=1, 2, \cdots, r$, where $\mathcal{E}^{(0)}(t)\equiv\mathcal{E}(t)$. Now, we shall prepare the following lemma which has been proved in [4], section 3.

LEMMA 1. *Under the condition* (2.1), *we have with probability* 1 *that*

$$\mathcal{E}^{(r)}(t)=\underset{h\to 0}{\mathrm{l.i.m.}}\ h^{-r}\sum_{k=0}^{r}\binom{r}{k}(-1)^{r-k}\mathcal{E}(t+kh). \tag{2.3}$$

THEOREM 1. *Let $H(u)$ be of bounded variation in $(-\infty, \infty)$. If*

$$\int_{-\infty}^{\infty} |u|^{r+\alpha}|\, dH(u)\,|<\infty \tag{2.4}$$

and

$$\int_{-\infty}^{\infty} |\lambda|^{2r+2\alpha}\, dF(\lambda)<\infty, \tag{2.5}$$

then we have

$$E\left\{\left|\int_{-\infty}^{\infty} \mathcal{E}\left(t-\frac{s}{n}\right)dH(s)-\sum_{k=0}^{r}\frac{(-1)^k\mathcal{E}^{(k)}(t)}{k!\,n^k}\int_{-\infty}^{\infty} s^k dH(s)\right|^2\right\} \tag{2.6}$$

$$=o\left(\frac{1}{n^{2r+2\alpha}}\right) \qquad as \qquad n\to\infty,$$

where r is a non-negative integer and α *is a constant satisfying* $0\leqq\alpha<1$.

Before proving this theorem, we shall give an explanation on the definition of the integral

$$\int_{-\infty}^{\infty} \mathcal{E}\left(t-\frac{s}{n}\right)dH(s).$$

In this paper, we are often concerned the integral of the type

$$\int_{-\infty}^{\infty} Y(s)\, dL(s),$$

where $Y(s)$ is a stochastic process with $E\{|Y(s)|^2\}<\infty$ and $L(s)$ is of bounded variation in any finite interval. This integral is taken here as

$$\underset{\substack{A\to-\infty\\ B\to\infty}}{\text{l.i.m.}}\int_{A}^{B} Y(s)\, dL(s),$$

where l.i.m. means the limit in the mean of order 2 and the finite integral in this definition is also as a Riemann-Stieltjes integral, the limit process being taken as l.i.m..

Proof of Theorem 1. The existence of the integral

$$\int_{-\infty}^{\infty} \mathcal{E}\left(t-\frac{s}{n}\right)dH(s)$$

in the above sense can be seen easily. (See [3].) By Lemma 1, we have with probability 1 that

$$\mathcal{E}^{(k)}(t)=\underset{m\to\infty}{\text{l.i.m.}}\,(-m)^k\sum_{\nu=0}^{k}(-1)^{k-\nu}\binom{k}{\nu}\,\mathcal{E}\left(t-\frac{\nu}{m}\right)$$

(2. 7)

$$=\underset{m\to\infty}{\text{l.i.m.}}\int_{-\infty}^{\infty}\mathcal{E}\left(t-\frac{s}{n}\right)du_{m,k}(s)$$

for $k=1, 2, \cdots, r$, where

(2.8)
$$u_{m,k}(s)=\begin{cases}0 & \text{for } s<0,\\ (-m)^k\sum_{\nu=0}^{[ms/n]}(-1)^{k-\nu}\binom{k}{\nu} & \text{for } s\geqq 0,\end{cases}$$

so that we have

$$I \equiv \int_{-\infty}^{\infty} \mathcal{E}\left(t-\frac{s}{n}\right) dH(s) - \sum_{k=0}^{r} \frac{(-1)^k \mathcal{E}^{(k)}(t)}{k!\,n^k} \int_{-\infty}^{\infty} s^k \, dH(s)$$

(2. 9)

$$= \underset{m\to\infty}{\text{l.i.m.}} \int_{-\infty}^{\infty} \mathcal{E}\left(t-\frac{s}{n}\right) dL_m(s) \qquad \text{(a.s.)},$$

where

$$L_m(s) = H(s) - \sum_{k=0}^{r} \frac{(-1)^k u_{m,k}(s)}{k!\,n^k} \int_{-\infty}^{\infty} \sigma^k \, dH(\sigma).$$

We thus get

$$E\{|I|^2\} = \lim_{m\to\infty} E\left\{\left|\int_{-\infty}^{\infty} \mathcal{E}\left(t-\frac{s}{n}\right) dL_m(s)\right|^2\right\}$$

(2. 10)

$$= \lim_{m\to\infty} E\left\{\int_{-\infty}^{\infty} \mathcal{E}\left(t-\frac{s}{n}\right) dL_m(s) \overline{\int_{-\infty}^{\infty} \mathcal{E}\left(t-\frac{\sigma}{n}\right) dL_m(\sigma)}\right\}.$$

Since

$$\int_{-\infty}^{\infty} \mathcal{E}\left(t-\frac{s}{n}\right) dL_m(s) = \underset{\substack{A\to-\infty \\ B\to\infty}}{\text{l.i.m.}} \int_{A}^{B} \mathcal{E}\left(t-\frac{s}{n}\right) dL_m(s)$$

and

$$\int_{A}^{B} \mathcal{E}\left(t-\frac{s}{n}\right) dL_m(s)$$

is also the limit of the Riemann sum

$$\sum_{\nu} \mathcal{E}\left(t-\frac{s_\nu}{n}\right) \{L_m(s_\nu) - L_m(s_{\nu-1})\}$$

in the mean of l.i.m., it follows from (2. 10) that

$$E\{|I|^2\} = \lim_{m\to\infty} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\left\{ \mathcal{E}\left(t-\frac{s}{n}\right) \overline{\mathcal{E}\left(t-\frac{\sigma}{n}\right)}\right\} dL_m(s)\, \overline{dL_m(\sigma)}$$

$$= \lim_{m\to\infty} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho\left(\frac{\sigma-s}{n}\right) dL_m(s)\, \overline{dL_m(\sigma)}$$

(2. 11)

$$= \lim_{m\to\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{i(\sigma-s)\lambda/n}\, dF(\lambda)\right) dL_m(s)\, \overline{dL_m(\sigma)}$$

$$= \lim_{m\to\infty}\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty} e^{-is\lambda/n}\, dL_m(s)\right|^2 dF(\lambda).$$

Now, we have

$$\int_{-\infty}^{\infty} e^{-is\lambda/n}\, dL_m(s)$$

$$= \int_{-\infty}^{\infty} e^{-is\lambda/n}\, dH(s) - \sum_{k=0}^{r}\frac{m^k}{k!\,n^k}\left[\sum_{\nu=0}^{k}(-1)^{k-\nu}\binom{k}{\nu} e^{-i\nu\lambda/m}\right]\int_{-\infty}^{\infty} s^k\, dH(s)$$

(2. 12)

$$= \int_{-\infty}^{\infty} e^{-is\lambda/n}\, dH(s) - \sum_{k=0}^{r}\frac{m^k}{k!\,n^k}(e^{-i\lambda/m}-1)^k\int_{-\infty}^{\infty} s^k\, dH(s)$$

$$= \int_{-\infty}^{\infty}\left[e^{-is\lambda/n} - \sum_{k=0}^{r}\frac{(sm)^k}{k!\,n^k}(e^{-i\lambda/m}-1)^k\right] dH(s)$$

and

$$\left|e^{-is\lambda/n} - \sum_{k=0}^{r}\frac{(sm)^k}{k!\,n^k}(e^{-i\lambda/m}-1)^k\right|$$

(2. 13)

$$\leqq 1 + \sum_{k=0}^{r}\frac{|sm|^k}{k!\,n^k}\cdot\left|\frac{\lambda}{m}\right|^k = 1 + \sum_{k=0}^{r}\frac{|s\lambda|^k}{k!\,n^k} \equiv A(\lambda, s).$$

Since $A(\lambda, s)$ is independent of m and

$$\lim_{m\to\infty}\left\{e^{-is\lambda/n} - \sum_{k=0}^{r}\frac{(sm)^k}{k!\,n^k}(e^{-i\lambda/m}-1)^k\right\}$$

$$= e^{-is\lambda/n} - \sum_{k=0}^{r}\frac{(-is\lambda)^k}{k!\,n^k},$$

it holds by Lebesgue's convergence theorem that

(2. 14) $$\lim_{m\to\infty}\int_{-\infty}^{\infty} e^{-is\lambda/n}\, dL_m(s) = \int_{-\infty}^{\infty}\left[e^{-is\lambda/n} - \sum_{k=0}^{r}\frac{(-is\lambda)^k}{k!\,n^k}\right] dH(s).$$

On the other hand, we see by (2. 4) that

$$\left|\int_{-\infty}^{\infty} e^{-is\lambda/n}\, dL_m(s)\right|^2 \leqq \left(\int_{-\infty}^{\infty} A(\lambda, s)\,|\, dH(s)\,|\right)^2 \equiv B(\lambda) < \infty$$

and the polynomial $B(\lambda)$ of λ, whose degree is $2r$, is independent of m so that (2. 11) implies with (2. 5) and (2. 14) that

$$E\{|\,I\,|^2\} = \int_{-\infty}^{\infty} \lim_{m\to\infty} \left|\int_{-\infty}^{\infty} e^{-is\lambda/n}\, dL_m(s)\right|^2 dF(\lambda)$$

$$= \int_{-\infty}^{\infty} \left|\int_{-\infty}^{\infty} \left[e^{-is\lambda/n} - \sum_{k=0}^{r} \frac{(-is\lambda)^k}{k\,!\, n^k}\right] dH(s)\right|^2 dF(\lambda)$$

$$(2.\ 15) \qquad = \int_{-\infty}^{\infty} dF(\lambda) \left|\int_{-\infty}^{\infty} dH(s) \int_0^s \frac{(-i\lambda)^r}{(r-1)\,!\, n^r} (e^{-ix\lambda/n} - 1)(s-x)^{r-1}\, dx\right|^2$$

$$\leqq \frac{1}{[(r-1)\,!\,]^2} \int_{-\infty}^{\infty} dF(\lambda)$$

$$\cdot \left(\int_{-\infty}^{\infty} |\, dH(s)\,| \int_0^{|s|} \left|\frac{\lambda}{n}\right|^r \left|\frac{x\lambda}{n}\right|^{\alpha} |e^{-ix\lambda/n} - 1\,|^{1-\alpha} (|\,s\,| - x)^{r-1}\, dx\right)^2$$

$$\leqq \frac{1}{n^{2r+2\alpha}[(r-1)\,!\,]^2} \int_{-\infty}^{\infty} |\,\lambda\,|^{2r+2\alpha}\, dF(\lambda)$$

$$\cdot \left(\int_{-\infty}^{\infty} |\, dH(s)\,| \int_0^{|s|} |\, e^{-ix\lambda/n} - 1\,|^{1-\alpha}\, x^{\alpha}(\,|\,s\,| - x)^{r-1}\, dx\right)^2,$$

because

$$e^{-is\lambda/n} - \sum_{k=0}^{r-1} \frac{(-is\lambda)^k}{k\,!\, n^k} = \frac{(-i\lambda)^r}{(r-1)\,!\, n^r} \int_0^s e^{-ix\lambda/n} (s-x)^{r-1}\, dx.$$

Since it holds

$$|\, e^{-ix\lambda/n} - 1\,|^{1-\alpha} \leqq 2^{1-\alpha},$$

$$\int_0^{|s|} |\, e^{-ix\lambda/n} - 1\,|^{1-\alpha}\, x^{\alpha}(|\,s\,| - x)^{r-1}\, dx \leqq 2^{1-\alpha}\,|\,s\,|^{r+\alpha}$$

and

$$\left(\int_{-\infty}^{\infty} |\, dH(s)\,| \int_0^{|s|} |\, e^{-ix\lambda/n} - 1\,|^{1-\alpha}\, x^{\alpha}(|\,s\,| - x)^{r-1} dx\right)^2$$

$$\leqq 2^{2-2\alpha}\left(\int_{-\infty}^{\infty} | s |^{r+\alpha} | dH(s) |\right)^2,$$

repeated application of Lebesgue's theorem gives with (2. 4) and (2.5) that

$$\lim_{n\to\infty}\int_0^{|s|} x^\alpha | e^{-ix\lambda/n}-1 |^{1-\alpha} (| s |-x)^{r-1}\, dx=0,$$

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} | dH(s) | \int_0^{|s|} | e^{-ix\lambda/n}-1 |^{1-\alpha} x^\alpha(| s |-x)^{r-1}\, dx=0$$

and

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} | \lambda |^{2r+2\alpha}\, dF(\lambda)\left(\int_{-\infty}^{\infty} | dH(s) | \int_0^{|s|} | e^{-ix\lambda/n}-1 |^{1-\alpha} x^\alpha(| s |-x)^{r-1}\, dx\right)^2=0.$$

Therefore we have

(2. 16) $$\lim_{n\to\infty} n^{2r+2\alpha} E\{| I |^2\} =0,$$

which proves the theorem. The proof of the case $r=0$ will similarly be done with slight modifications.

LEMMA 2. *Let $H(u)$ be a function satisfying the conditions in Theorem* 1, *and $f(u)$ be Lebesgue-Stieltjes integrable with respect to the measure $| dH(u) |$. If we assume that*

(2. 17) $$f(t+u)=\sum_{k=0}^{r} \frac{f^{(k)}(t)}{k!} u^k+o(| u |^{r+\alpha}) \qquad \textit{for small } u$$

and

(2. 18) $$| f(u) | \leqq C(1+| u |^{r+\alpha}) \qquad \textit{for all } u,$$

where C is a positive constant, then we have

(2. 19) $$\int_{-\infty}^{\infty} f\left(t-\frac{s}{n}\right) dH(s)= \sum_{k=0}^{r} \frac{(-1)^k f^{(k)}(t)}{k!\, n^k}\int_{-\infty}^{\infty} s^k\, dH(s)+o\left(\frac{1}{n^{r+\alpha}}\right) \quad \textit{as } n\to\infty.$$

This lemma has been stated in [1]. Let $X(s)=f(s)+\mathcal{E}(s)$, $-\infty<s<\infty$. And, to ensure the existence of the integral

$$\int_{-\infty}^{\infty} X\left(t-\frac{s}{n}\right) dH(s),$$

we assume that $f(u)$ is Riemann-Stieltjes integrable with respect to $dH(u)$. Then, we have immediatly the following

THEOREM 2. *Under the conditions in Theorem* 1 *and Lemma* 2, *we have*

$$E\left\{\left|\int_{-\infty}^{\infty} X\left(t-\frac{s}{n}\right)dH(s)-\sum_{k=0}^{r}\frac{(-1)^k X^{(k)}(t)}{k!\,n^k}\int_{-\infty}^{\infty} s^k\,dH(s)\right|^2\right\}$$

(2. 20)

$$=o\left(\frac{1}{n^{2r+2\alpha}}\right) \qquad \text{as } n\to\infty,$$

where $X^{(k)}(t)=f^{(k)}(t)+\mathcal{E}^{(k)}(t)$ *for* $k=1, 2, \cdots, r$.

3. In this section, we shall note that, by using the random spectral function $Z(\lambda)$ of $\mathcal{E}(t)$, it can be made easy to lead the first half of (2. 15):

$$E\{|I|^2\}=\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty}\left[e^{-is\lambda/n}-\sum_{k=0}^{r}\frac{(-is\lambda)^k}{k!\,n^k}\right]dH(s)\right|^2 dF(\lambda),$$

which was a foundation of the proof of Theorem 1.

If

$$\int_{-\infty}^{\infty}\lambda^2\,dF(\lambda)<\infty,$$

then

(3. 1)
$$\mathcal{E}'(t)=\int_{-\infty}^{\infty} i\lambda e^{i\lambda t}\,dZ(\lambda) \qquad \text{(a. s.).}$$

This is well known [2]. Repeated applications of the method to prove this fact show immediately that if

$$\int_{-\infty}^{\infty}\lambda^{2r}\,dF(\lambda)<\infty,$$

then

(3. 2)
$$\mathcal{E}^{(k)}(t)=\int_{-\infty}^{\infty}(i\lambda)^k e^{i\lambda t}\,dZ(\lambda) \qquad \text{(a. s.)}$$

for $k=0, 1, 2, \cdots, r$. Therefore, under the condition (2. 5), we have

$$P(s)\equiv\mathcal{E}\left(t-\frac{s}{n}\right)-\sum_{k=0}^{r}\frac{(-1)^k\mathcal{E}^{(k)}(t)}{k!\,n^k}s^k$$

(3. 3)

$$=\int_{-\infty}^{\infty}\left[e^{i(t-s/n)\lambda}-e^{it\lambda}\sum_{k=0}^{r}\frac{(-is\lambda)^k}{k!\,n^k}\right]dZ(\lambda) \qquad \text{(a. s.).}$$

Since

$$E\left\{\int_{-\infty}^{\infty} f(\lambda)\, dZ(\lambda) \overline{\int_{-\infty}^{\infty} g(\lambda)\, dZ(\lambda)}\right\} = \int_{-\infty}^{\infty} f(\lambda)\overline{g(\lambda)}\, dF(\lambda) \tag{3. 4}$$

for $f,\ g \in L^2(dF)$,

we have

$$E\{P(s)\overline{P(\sigma)}\} = \int_{-\infty}^{\infty} R(s,\ \lambda)\, \overline{R(\sigma,\ \lambda)}\, dF(\lambda), \tag{3. 5}$$

where

$$R(s,\ \lambda) = e^{i(t-s/n)\lambda} - e^{it\lambda} \sum_{k=0}^{r} \frac{(-is\lambda)^k}{k\,!\, n^k}$$

$$= e^{it\lambda}\left[e^{-is\lambda/n} - \sum_{k=0}^{r} \frac{(-is\lambda)^k}{k\,!\, n^k}\right],$$

and so

$$E\{|I|^2\} = E\left\{\left|\int_{-\infty}^{\infty} P(s)\, dH(s)\right|^2\right\}$$

$$= E\left\{\int_{-\infty}^{\infty} P(s)\, dH(s) \overline{\int_{-\infty}^{\infty} P(\sigma)\, dH(\sigma)}\right\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\{P(s)\overline{P(\sigma)}\}\, dH(s)\, \overline{dH(\sigma)}$$

(3. 6)

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} R(s,\ \lambda)\, \overline{R(\sigma,\ \lambda)}\, dF(\lambda)\right) dH(s)\, \overline{dH(\sigma)}$$

$$= \int_{-\infty}^{\infty} dF(\lambda) \left|\int_{-\infty}^{\infty} R(s,\ \lambda)\, dH(s)\right|^2$$

$$= \int_{-\infty}^{\infty} \left|\int_{-\infty}^{\infty}\left[e^{-is\lambda/n} - \sum_{k=0}^{r} \frac{(-is\lambda)^k}{k\,!\, n^k}\right] dH(s)\right|^2 dF(\lambda).$$

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