ON THE ENVELOPE OF HOLOMORPHY OF A GENERALIZED TUBE IN C^n

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In 1937 Stein [10] proved that the envelope of holomorphy of a tube-domain in C^2 coincides with its envelope of convexity. We can find no difficulty in extending the above Stein's proof to the case in C^n. In 1938 Bochner [2] obtained the above Stein's Theorem quite independently in C^n. Later Hitotumatu [7] gave a new and elegant proof and Bremermann [5] extended the above Stein's Theorem in complex Banach spaces.

The main purpose of the present paper is to extend the above Stein's Theorem to a generalized tube in C^n. The main method is based on the Levi's problem and the convergence theorem concerning the domain of holomorphy.

For two n-tuples $x=(x_1, x_2,\cdots, x_n)$ and $y=(y_1, y_2,\cdots, y_n)$ of real numbers, we shall use the notation $z=x+iy$ by putting $z=(z_1, z_2,\cdots, z_n)$ and $z_j=x_j+iy_j$ $(1\leqq j\leqq n)$. The space of n real variables $x_1, x_2,\cdots$ and x_n is denoted by R_x^n and the space of n complex variables $z_1, z_2,\cdots$ and z_n is denoted by C_z^n or simply by C^n.

Let A and B be subsets of R_x^n and R_y^n respectively. Then $A\times B$ is called a *generalized tube* in C_z^n where $z=x+iy$. A is called its *real base* and B is called its *imaginary base*. $A\times R_y^n$ is called simply a *tube* in C_z^n.

Concerning a tube in C^n we have the following theorem [10].

STEIN'S THEOREM. *The envelope of holomorphy of an open connected tube in C^n coincides with its geometrical envelope of convexity.*

LEMMA 1. *If an open connected generalized tube $A\times\{(y_1, y_2,\cdots, y_n); a_j<y_j<b_j$ $(j=1, 2,\cdots, n)\}$ is a domain of holomorphy, then $A\times\{(y_1, y_2,\cdots, y_n); a_j+c_j<y_j<b_j+c_j$ $(j=1, 2,\cdots, n)\}$ is also a domain of holomorphy for any real number c_j.*

Proof. Since the holomorphic mapping Φ defined by $\Phi(z)=(z_1+ic_1, z_2+ic_2,\cdots, z_n+ic_n)$ is a bi-holomorphic mapping of the closure of the former onto that of the latter, we have our Lemma. q. e. d.

LEMMA 2. *If an open connected generalized tube $T=A\times\{(y_1, y_2,\cdots, y_n); a_j<y_j<b_j$ $(j=1, 2,\cdots, n)\}$ is not a domain of holomorphy, then for any positive integer k $(1\leqq k\leqq n)$ and for any real number d_k such that $(a_k+b_k)/2<d_k\leqq b_k$, $T_1=T\cap[A$*

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$\times \{(y_1, y_2, \cdots, y_n) \in R_y^n;\ a_k < y_k < d_k\}]$ *is not a domain of holomorphy.*

Proof. Suppose that T_1 is a domain of holomorphy. Then from Lemma 1, $T_2 = T \cap [A \times \{(y_1, y_2, \cdots, y_n) \in R_y^n;\ a_k + b_k - d_k < y_k < b_k\}]$ is a domain of holomorphy.

Since T_1 and T_2 are domains of holomorphy which unite themselves, that is, $\overline{T_1 - T_1 \cap T_2} \cap \overline{T_2 - T_1 \cap T_2} = \phi$, $T = T_1 \cup T_2$ is a domain of holomorphy from [4], [8] or [9]. Thus we have arrived at a contradiction. q. e. d.

LEMMA 3. *If an open connected generalized tube* $A \times \{(y_1, y_2, \cdots, y_n);\ a_j < y_j < b_j\ (j = 1, 2, \cdots, n)\}$ *is not a domain of holomorphy, then for any positive integer* k $(1 \leqq k \leqq n)$ *and for arbitrary real numbers* a_k' *and* b_k' *such that* $0 < b_k' - a_k' \leqq b_k - a_k$, $A \times \{(y_1, y_2, \cdots, y_n);\ a_1 < y_1 < b_1, \cdots, a_{k-1} < y_{k-1} < b_{k-1}, a_k' < y_k < b_k', a_{k+1} < y_{k+1} < b_{k+1}, \cdots, a_n < y_n < b_n\}$ *is not a domain of holomorphy.*

Proof. From our assumption there exists an integer $p > 0$ such that $(b_k - a_k)2^{-p} < b_k' - a_k' \leqq (b_k - a_k)2^{1-p}$. We can prove our Lemma by induction with respect to p making use of Lemma 1 and 2. q. e. d.

We have easily the following lemma from Lemma 3.

LEMMA 4. *If an open connected generalized tube* $A \times \{(y_1, y_2, \cdots, y_n);\ a_j < y_j < b_j\ (j = 1, 2, \cdots, n)\}$ *is not a domain of holomorphy, then for arbitrary real numbers* a_j' *and* b_j' *such that* $0 < b_j' - a_j' \leqq b_j - a_j\ (j = 1, 2, \cdots, n)$, $A \times \{(y_1, y_2, \cdots, y_n);\ a_j' < y_j < b_j'\ (j = 1, 2, \cdots, n)\}$ *is not a domain of holomorphy.*

LEMMA 5. *If a domain* A *in* R_x^n *is not geometrically convex, then there exists a positive number* a_0 *such that for every* $a \geqq a_0$, $A \times \{(y_1, y_2, \cdots, y_n) \in R_y^n;\ -a < y_1 < a\}$ *is not a domain of holomorphy in* C_z^n.

Proof. Suppose that our Lemma does not hold. Then there exists a sequence $\{a_p;\ p = 1, 2, 3, \cdots\}$ of positive numbers satisfying the following conditions: $a_1 < a_2 < \cdots < a_p < a_{p+1} < \cdots$; $a_p \to \infty$ as $p \to \infty$; and $T_p = A \times \{(y_1, y_2, \cdots, y_n) \in R_y^n;\ -a_p < y_1 < a_p\}$ is a domain of holomorphy for every p. Then from the convergence theorem of Behnke and Stein [1], $\lim_{p \to \infty} T_p = A \times R_y^n$ is a domain of holomorphy. Thus from Stein's Theorem we have arrived at a contradiction. q. e. d.

LEMMA 6. *If a domain* A *in* R_x^n *is not geometrically convex, then there exists a positive number* a_0 *such that for every* $a \geqq a_0$ $T = A \times \{(y_1, y_2, \cdots, y_n);\ -a < y_j < a\ (j = 1, 2, \cdots, n)\}$ *is not a domain of holomorphy in* C_z^n.

Proof. For any integer p $(I \leqq p \leqq n)$ we shall put $T_p = A \times \{(y_1, y_2, \cdots, y_n) \in R_y^n;\ -a < y_1 < a, -a < y_2 < a, \cdots, -a < y_p < a\}$. We can prove our Lemma by induction with respect to p quite similarly to Lemma 5. q. e. d.

PROPOSITION 1. *Let* A *be a domain in* R_x^n *and* a_j *and* b_j *be arbitrary real numbers such that* $a_j < b_j$ *for* $j = 1, 2, \cdots, n$. *Then the generalized tube* $T = A \times \{(y_1,$

$y_2, \cdots, y_n); a_j<y_j<b_j$ $(j=1, 2, \cdots, n)\}$ is a domain of holomorphy in C_z^n, if and only if A is geometrically convex.

Proof. The necessity of our Proposition follows from Lemmas 4 and 6.

If A is geometrically convex, then from Stein's Theorem $A\times R_y^n$ and $R_x^n\times\{(y_1, y_2, \cdots, y_n); a_j<y_j<b_j$ $(j=1, 2, \cdots, n)\}$ are domains of holomorphy. Therefore their intersection T is a domain of holomorphy from [6]. q. e. d.

LEMMA 7. *Let A be a domain in R_x^n and B be a convex domain in R_y^n. Then the envelope of holomorphy H of $T=A\times B$ in C_z^n is also a generalized tube with the imaginary base B in C_z^n.*

Proof. Since B is a convex domain in R_y^n, $R_x^n\times B$ is a domain of holomorphy from Stein's Theorem. Then $H\cap(R_x^n\times B)$ is a domain of holomorphy from [6] as intersection of two domains of holomorphy. Since $T\subset H\cap(R_x^n\times B)\subset H$ and H is the envelope of holomorphy of T, we have $H=H\cap(R_x^n\times B)$.

If we put $\tilde{A}_y=\{x; (x, y)\in H\}$ for each $y\in B$, then we have $H=\{(x, y); x\in\tilde{A}_y, y\in B\}$. Since A is the real base of T, we have $A\subset\tilde{A}_y$ for each $y\in B$. Let K be any compact subset of B and $\tilde{A}_K$ be the open kernel of the intersection of all $\tilde{A}_y$ for $y\in K$. Obviously it holds that $A\subset\tilde{A}_K\subset\tilde{A}_y$ for each $y\in K$.

Now we shall show that $\tilde{A}_K\times R_y^n$ is a domain of holomorphy. Since K is compact in B, there exists $b>0$ such that $\{y'; |y_j'-y_j|<b$ $(j=1, 2, \cdots, n)\}\subset B$ for each $y\in K$. At first we shall prove that $\tilde{A}_K\times\{y; |y_j|<b/2$ $(j=1, 2, \cdots, n)\}$ is a domain of holomorphy.

Let x^0 be any boundary point of $\tilde{A}_K$. For any neighbourhood U of x^0 in R_x^n, there exists a point x' in U such that x' is a boundary point of $\tilde{A}_{y^0}$ for some $y^0\in K$. Let y^1 be any point such that $|y_j^1|<b/2$ $(j=1, 2, \cdots, n)$.

Since H is a domain of holomorphy, $H\cap[R_x^n\times\{y; |y_j-y_j^0+y_j^1|<b/2\}]=\{z=x+iy; x\in\tilde{A}_y, |y_j-y_j^0+y_j^1|<b/2$ $(j=1, 2, \cdots, n)\}$ is a domain of holomorphy as intersection of two domains of holomorphy from [6]. Therefore $H_{b,y^1}=\{z=x+iy; x\in\tilde{A}_{y'}, y_j'=y_j-y_j^1+y_j^0, |y_j|<b/2$ $(j=1, 2, \cdots, n)\}$ is a domain of holomorphy, because there exists a bi-holomorphic mapping $\Phi(z)=(z_1+i(y_1^0-y_1^1), z_2+i(y_2^0-y_2^1), \cdots, z_n+i(y_n^0-y_n^1))$ of the closure of H_{b,y^1} onto that of the domain of holomorphy as cited above.

Therefore there exists a holomorphic function in H_{b,y^1} which is unbounded in any neighbourhood of the point (x', y^1). This function is holomorphic in $\tilde{A}_K\times\{y; |y_j|<b/2$ $(j=1, 2, \cdots, n)\}$. Since U is any neighbourhood of x^0 and x' is a point in U, there exists a holomorphic function in $\tilde{A}_K\times\{y; |y_j|<b/2$ $(j=1, 2, \cdots, n)\}$ which is unbounded in any neighbourhood of the point (x^0, y^1), that is, (x^0, y^1) has the frontier property in Bochner-Martin's sense [3]. Therefore any boundary point of $\tilde{A}_K\times\{y; |y_j|<b/2\}$ has the frontier property. Hence $\tilde{A}_K\times\{y; |y_j|<b/2$ $(j=1, 2, \cdots, n)\}$ is a domain of holomorphy from [3]. Consequently, $\tilde{A}_K\times R_y^n$ is a domain of holomorphy from Proposition 1 and Stein's Theorem.

Then $H\cap(\tilde{A}_K\times R_y^n)$ is a domain of holomorphy satisfying $T\subset H\cap(\tilde{A}_K\times R_y^n)\subset H$. Since H is the envelope of holomorphy of T, we have $H=H\subset(\tilde{A}_K\times R_y^n)$. This implies $\tilde{A}_y=\tilde{A}_K$ for any $y\in K$. Since K is any compact subset of B, H is a generalized tube in C_z^n. q. e. d.

PROPOSITION 2. *Let A be a domain in R_x^n, $\tilde{A}$ be its geometrical envelope of convexity and a_j and b_j be real numbers such that $a_j<b_j$ for $j=1, 2,\cdots, n$. Then the envelope of holomorphy of $A\times\{y; a_j<y_j<b_j\ (j=1, 2,\cdots, n)\}$ in C_z^n is $\tilde{A}\times\{y; a_j<y_j<b_j\ (j=1, 2,\cdots, n)\}$.*

Proof. From Lemma 7 the envelope of holomorphy of $A\times\{y; a_j<y_j<b_j\ (j=1, 2,\cdots, n)\}$ is a generalized tube with the imaginary base $\{y; a_j<y_j<b_j\ (j=1, 2,\cdots, n)\}$ and is denoted by $E\times\{y; a_j<y_j<b_j\ (j=1, 2,\cdots, n)\}$. From Proposition 1, E must be a geometrically convex domain containing A. Conversely if E' is a geometrically convex domain containing A, then from Proposition 1 $E'\times\{y; a_j<y_j<b_j\}$ is a domain of holomorphy containing $A\times\{y; a_j<y_j<b_j\ (j=1, 2,\cdots, n)\}$. Therefore E is the geometrical envelope of convexity $\tilde{A}$ of A. q. e. d.

LEMMA 8. *Let $A\times B$ be an open connected generalized tube in C_z^n, and $\tilde{A}$ and $\tilde{B}$ be, respectively, the geometrical envelopes of convexity of A in R_x^n and of B in R_y^n. Then any holomorphic function in $A\times B$ is analytically continued in $\tilde{A}\times\tilde{B}$.*

Proof. It suffices to prove that any holomorphic function in $A\times B$ is analytically continued in $\tilde{A}\times B$.

Let y^0 be any point of B. Since B is open, there exists a positive number a such that $B_0=\{y; |y_j-y_j^0|<a\ (j=1, 2,\cdots, n)\}\subset B$. From Proposition 2, the envelope of holomorphy of $A\times B^0$ is $\tilde{A}\times B^0$. Therefore any holomorphic function in $A\times B^0$ is analytically continued in $\tilde{A}\times B^0$. Any holomorphic function in $A\times B$ is holomorphic in $A\times B^0$ and hence is analytically continued in $\tilde{A}\times B$. q. e. d.

PROPOSITION 3. *Let $A\times B$ be an open connected generalized tube in C_z^n, and $\tilde{A}$ and $\tilde{B}$ be, respectively, the geometrical envelopes of convexity of A in R_x^n and of B in R_y^n. Then the envelope of holomorphy of $A\times B$ is $\tilde{A}\times\tilde{B}$.*

Proof. Any holomorphic function in $A\times B$ is analytically continued in $\tilde{A}\times\tilde{B}$ from Lemma 8. From Stein's Theorem $\tilde{A}\times R_y^n$ and $R_x^n\times\tilde{B}$ are domains of holomorphy. Therefore their intersection $\tilde{A}\times\tilde{B}=(\tilde{A}\times R_y^x)\cap(R_x^n\times\tilde{B})$ is a domain of holomorphy from [6]. Hence $\tilde{A}\times\tilde{B}$ is the envelope of holomorphy of $A\times B$. q. e. d.

Let S be any subset of C^n which is not necessarily open. A holomorphic function in some neighbourhood of S is called a *holomorphic function in* S.

If S and T are subsets of C^n such that $S\subset T$ and any holomorphic function in S is analytically continued in T, then T is called an *analytic completion* of S. We say that S has the *maximal analytic completion* $\tilde{S}$, if there exists a subset $\tilde{S}$ of C^n satisfying the following conditions:

(1) $\tilde{S}$ is an analytic completion of S;

(2) If T is an analytic completion of S, then $T\subset\tilde{S}$.

If the intersection of the envelopes of holomorphy of all domains containing S is univalent in C^n, then it is the maximal analytic completion of S. Conversely, if there exists the maximal analytic completion of S, then it coincides with the intersection of the envelopes of holomorphy of all domains containing S.

Therefore we have the following theorem from Proposition 3.

THEOREM. *The geometrical envelope of convexity of a connected generalized tube in C^n is its maximal analytic completion.*

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