LINEAR MAPPINGS AMONG THE FUNCTION CLASSES ON RIEMANN SURFACES

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Introduction

Let W be an open Riemann surface and G a non-compact subregion of W, whose relative boundary ∂G consists of at most a countably infinite number of analytic components which do not cluster at any compact part of W. Let R be a collection of compact or non-compact subregions of G all boundary components of which consists of a finite number of analytic curves and cluster at only the ideal boundary of W. We assume further that any point on the relative boundary is freely accessible from both sides of curves.

Let $\{W_n\}$ be a sequence of compact domains satisfying the following conditions:
(1) $W_n \subset W_{n+1}$,
(2) ∂W_n consists of a finite number of components each of which consists of a finite number of analytic curves, and
(3) for any compact subset K there exists such a number $n_0(K)$ that K is contained in W_n for any $n \geqq n_0(K)$.
We say this sequence $\{W_n\}$ an exhaustion of W, however our notion of exhaustion presented here is different from that usually availed. Indeed there may exist a finite number of islands any member of which does not separate any ideal boundary element and a stronger condition $\overline{W}_n \subset W_{n+1}$ than the one claimed in (1) is not postulated here.

We shall make use of the following notations:

H: a class of harmonic functions, which is equal to zero on the relative boundary if it exists. P: a class of positive functions. B: a class of bounded functions. $A(X)$: some function class A defined in X.

We shall introduce several linear mappings among several families of functions some of which are well known.

For each $u \in PH(G)$ the upper envelope of the set of members of $PH(R)$ which are dominated by u in R is itself a member of $PH(R)$. We define a mapping T of $PH(G)$ into $PH(R)$ by requiring that for each $u \in PH(G)$ Tu shall be the above upper envelope. In another direction, we define for each $U \in PH(R)$, its standard subharmonic extension U^* by $U^* = U$ in R, $=0$ in $G-R$. Let Q_R denote the subset of $PH(R)$ consisting of those U for which U^* admits a harmonic majorant on G. We define the map S from Q_R into $PH(G)$ by requiring that SU shall be the least harmonic majorant of U^* on G. It is immediate that both T and S are positively linear. Then we see that S is univalent and $TSU = U$ for any $U \in Q_R$. If $v \in PH(G)$ is dominated by some member of $S(Q_R)$, then $v \in S(Q_R)$. Now we can define a mapping

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ST from $PH(G)$ into itself.

Refering to the exhaustion $\{W_n\}$ we shall redefine the above mappings S, T and ST. Let G_n and R_n be $G\frown W_n$ and $R\frown W_n$, respectively. For each $u\in PH(G)$ we shall define a positive harmonic function u_n in R_n such that $u_n=0$ on $\partial R\frown W_n$ and $u_n=u$ on $R\frown\partial W_n$. Then u_n is decreasing with increasing n and $0\leqq u_n\leqq u$ in R_n, therefore u_n converges to either a non-zero function or zero function. We can say that this mapping $u\rightarrow\lim u_n$ coincides with the earlier T and $Tu\leqq u$. For each $U\in PH(R)$ we shall define a positive harmonic function U_n in G_n such that $U_n=0$ on $\partial G\frown W_n+\partial W_n\frown(G-\bar{R})$ and $U_n=U$ on $R\frown\partial W_n$. Then U_n is increasing with increasing n and U_n converges to either a function which is not constant ∞ or to a constant ∞. We can say that the mapping $U\rightarrow\lim U_n$ coincides with S, if $\lim U_n\not\equiv\infty$, and $SU\geqq U$ in R and that the image set $T(PH(G))$ excluded $\{0\}$ coincides with $Q_R-\{0\}$ and any non-zero element of $PH(R)$ having an image element in $PH(G)$ coincides with $Q_R-\{0\}$. Therefore the mapping ST: $PH(G)\rightarrow PH(G)$ can be redefined as follows: For any $u\in PH(G)$ we construct a positive harmonic function V_n in G_n such that $V_n=0$ on $\partial G\frown W_n+\partial W_n\frown(G-\bar{R})$ and $V_n=Tu$ on $\partial W_n\frown R$. Then V_n converges to either a non-zero function or constant zero. We can say that $\lim V_n=STu>0$ in G if $\lim V_n\not\equiv 0$ and this is the case if and only if $Tu\not\equiv 0$. Evidently we have $STu\leqq u$. For the above facts we recommend [2], [3] and [5].

In connection with the above facts our problems occured. For any $u\in PH(G)$ or $PHB(G)$ we construct a positive harmonic function X_n in G_n such that $X_n=0$ on $\partial G\frown W_n+\partial W_n\frown(G-R)$ and $X_n=u$ on $\partial W_n\frown R$. Does there exist the limit $\lim X_n$? Does any limit such as $\overline{\lim}\, X_n$ or $\underline{\lim}\, X_n$ coincide with STu? It may be expected that somewhat interesting versions with respect to the relative situation of R in G can be revealed when the problems are negatively solved. We shall discuss the problems in reference to the classes $PHB(G)$ and $PH(G)$ and partly solve them negatively. In Chapter I we shall concern with the linear mappings among the function classes PH. Several notions of niceness of R in G and equivalency of two subsets in G are introduced and their global notions are also introduced. Somewhat troublesome algebraic calculations in order to obtain the relations among various operators are not avoidable for our purposes. In this Chapter the two-sideness condition which will be defined in a later part plays a role. In the last part of this Chapter we shall introduce a notion of tangential path or non-tangential path and we shall prove modification theorems which seem perhaps intuitively evident. However so far as we concern there is no such formulation in any existing bibliography. And these modification theorems and their generalizations play an important role in the next Chapter II. In Chapter II we shall extend our definitions of operators to be more available and we shall offer several geometric or potential-theoretic interpretations of the extended operators. In Chapter III we shall chiefly concern with the class PHB. Here we shall discuss the relations between two classes PH and PHB. In the last part we shall offer several unsolved problems, which seem somewhat important.

Chapter I. Linear mappings in *PH*(*G*)

§1. Preliminary considerations on Martin theory. For $PH(G)$ the major theory due to Martin plays a decisive role. So we shall also refer to the theory [4].

Let W be an arbitrary Riemann surface of hyperbolic type. Let $g(p, q)$ be the Green function of W with pole q. We say $u \in PH(W)$ minimal in Martin's sense if any element $v \in PH(W)$ satisfying an inequality $0 < v \leqq u$ reduces to a proportional element $ku (k > 0)$ of u. Let $K(p, q)$ be the Martin function defined by

$$K(p, q) = \frac{g(p, q)}{g(p_0, q)} \quad (q \neq p_0),$$

where p_0 is a fixed point in W. If $\lim_{n\to\infty} K(p, q_n)$ exists for a non-compact sequence $\{q_n\}$, then we say that $\{q_n\}$ determines an ideal boundary point. If $\{q_n\}$ and $\{q_n'\}$ determine the same limit function, then we say that two sequences determine the same ideal boundary point. The set Δ of these ideal boundary points is called the Martin boundary of W. For $W \smile \Delta$ Martin's metric can be introduced by an integral

$$d(q_1, q_2) = \int_{k_r} \frac{\partial}{\partial n_s} \left| K(p(s), q_1) - K(p(s)\ q_2) \right| ds$$

or any other equivalent forms, where k_r is a circumference with the centre q_0 and with the radius r in terms of a fixed uniformizing parameter. By this metric $W \smile \Delta$ is a compact Hausdorff space, Δ is closed subset and W is an open dense subset of $W \smile \Delta$. And the topology induced by the metric is equivalent to the ordinary surface topology in W. Any minimal function in $PH(W)$ coincides with a function $K(p, q)$, $q \in \Delta$. If this is the case for a $q \in \Delta$, then q is called a minimal point in Martin's sense. The set Δ_1 of all Martin's minimal points is a closed subset in Δ. Then for any $u \in PH(W)$ there is a suitable positive Borel measure σ on Δ for which u can be represented by an integral

$$u(p) = \int_{\Delta} K(p, q)\ d\sigma(q).$$

If $\sigma(\Delta - \Delta_1) = 0$, then σ is called canonical. Any u in $PH(W)$ can be uniquely represented by an integral with a uniquely determined canonical measure σ_u

$$u(p) = \int_{\Delta_1} K(p, q)\ d\sigma_u(q).$$

We are able to make free use of a nice description of the Martin theory by Konstantinescu-Cornea [1]. Let G and R be the ones defined in the introduction. We shall consider only the identity conformal mapping R into G and the class $PH(G)$. Thus all the situation discussed in [1] are fulfiled by their theorem 6 (p. 10 in [1]). For $PH(G)$ any Martin's boundary point q of G defined by some compact point-sequence considering as the one in W, that is, any one lying on some point on the relative boundary ∂G plays no essential role, so we can omit these boundary points from the Martin boundary of G. The remaining set is denoted by Δ or Δ^G, if necessary, and the corresponding Martin's minimal part of Δ is denoted by Δ_1 or Δ_1^G. Then we have the canonical representation theorem:

$$u(p)=\int_{\Delta_1} K(p, q)\ d\sigma_u(q)$$

for any $u\in PH(G)$, where σ_u is a uniquely determined Radon measure on Δ_1. Further Δ_1 and Δ are closed in Δ and in the original Martin boundary of G, respectively.

Two linear mappings T and S between $PH(G)$ and $PH(R)$ are defined as already described in the Introduction. Let $\Delta(R)$, $\Delta_1(R)$ indicate the sets

$$\Delta(R)=\{q\in\Delta,\ T_R^G K(p, q)\neq 0\},\ \Delta_1(R)=\Delta(R)\frown\Delta_1\,.$$

If $u(p)\in PH(G)$ is represented by a canonical integral

$$\int_{\Delta_1} K(p, q)\ d\sigma_u(q)\,,$$

then by theorem 15′ in [1] there holds the representation

$$S_R^G T_R^G u(p)=\int_{\Delta_1(R)} K(p, q)\ d\sigma_u(q)\,.$$

Any point q in Δ_1 is accessible along some non-compact curve in G. This was shown in [1], p. 28. Then any point q in $\Delta_1(R)$ is accessible by a non-compact curve in R which determines the point q. To that end, it is noted that the map T_R^G preserves the minimality if T_R^G has sense. By the definition of $\Delta_1(R)$, $T_R^G K(p, q)\neq 0$, that is, T_R^G has sense for $K(p, q)$. Therefore $T_R^G K(p, q)$ is also a Martin minimal function in $PH(R)$. By this fact some Martin minimal point q' in Δ_1^R is uniquely determined and hence q' is accessible by a curve $\gamma_{q'}$ in R. Then $S_R^G T_R^G K(p, q)\equiv K(p, q)$ by a fact that the minimality is also preserved by the map S_R^G if it has sense. Therefore $\gamma_{q'}$ determines a minimal point in $\Delta_1^G(R)\equiv\Delta_1(R)$ which must coincide with the q. That is, q is accessible by $\gamma_{q'}$. Such a curve is called the defining tail of q.

In order to enter into the discussion of the algebraic properties of λ and μ, we shall give some representation in [1], where λ and μ are defined in the next section.

If u_1, $u_2\in PH(G)$ and

$$u_i=\int_{\Delta_1^G} K(p, q)\ d\sigma_{u_i}(q)\,,\qquad i=1,\ 2$$

and $u_1\leqq u_2$, then there exists a measurable function θ on Δ_1^G, $0\leqq\theta\leqq 1$, for which

$$\sigma_{u_1}(A)=\int_A \theta(q)\ d\sigma_{u_2}(q)$$

for any Borel set A in $\Delta_1{}^G$. If

$$u_i(p)=\int_{\Delta_1^G} K(p, q)\ \theta_i(q)\ d\sigma(q)\,,$$

where θ_i is a non-negative σ-integrable Borel function, then

$$\theta_1 \leqq \theta_2 : [\sigma],$$

that is, $\theta_1 \leqq \theta_2$ holds almost everywhere on Δ_1^G with respect to σ-measure. This is the lemma a in [1].

Through this chapter we assume that any open set or subregion D in W or in G has its relative boundary with the following two-sidedness condition in W or in G: Let p be any proper boundary point and $N(p)$ be some suitable neighborhood of p. Then there are only two components one of which belongs to the set D and another of which does not belong to the set D. This definition of two-sidedness can be extended to a more general set.

§2. Linear mappings and their representations. In this section we shall define several linear mappings and discuss their representations.

For each $u \in PH(G)$ we construct a positive harmonic function u_n in G_n such that $u_n = u$ on $\partial W_n \frown \bar{R}$ and $u_n = 0$ on $\partial G \frown W_n + \partial W_n \frown (G - \bar{R})$. Then $u_n \leqq u$ in G_n. Let μu and λu be

$$\inf \varliminf_n u_n \text{ and } \sup \varlimsup_n u_n,$$

respectively, where inf and sup are taken over all exhaustions $(\{W_n\})$ of W, then we have

$$STu \leqq \mu u \leqq \lambda u \leqq u$$

by $u_n \geqq Tu$ in R_n.

For each $u \in PH(G)$ we construct a positive harmonic function $u(n, m)$ in $G_{n,m} = G_m - \overline{(R_m - \bar{R}_n)}$ such that $u(n, m) = u$ on $\partial W_n \frown R + \partial R \frown (W_m - \bar{W}_n) + \partial G \frown W_m, = 0$ on $\partial W_m \frown (G - \bar{R})$. Then $u \geqq u(n, m) \geqq u(n, m-1)$ in $G_{n, m-1}$ and hence $\lim_{m \to \infty} u(n, m) = u(n)$ exists. And further $u(n)$ decreases with increasing n and hence tends to a harmonic function in G. Putting this limit function $\lambda_1 u$, then $\lambda_1 u \in PH(G)$ and $\lambda_1 u \leqq u$ in G.

Let $u^{n,m}$ be a positive harmonic function in $G^{n,m} = G_m - \overline{(G - \bar{R}) \frown (W_m - \bar{W}_n)}$ with boundary value u on $\partial W_m \frown R + \partial G \frown W_n$ and 0 on $\partial W_n \frown (G - \bar{R}) + \partial R \frown (W_m - \bar{W}_n)$. Evidently we have $u^{n,m} \geqq u^{n,m+1}$ and $u^{n+1,m} \geqq u^{n,m}$, which imply the existence of a function

$$\lim_{n \to \infty} \lim_{m \to \infty} u^{n,m},$$

which belongs to the class $PH(G)$. Then we put this limit function $\mu_1 u$. Evidently we have that $\mu_1 u \leqq \lambda_1 u \leqq u$ and λ_1 and μ_1 are positively linear.

Next we shall put $\mu^* u = STu$ and $\lambda^* u = u - \mu^*{}_{G-\bar{R}}^{G} u$, where we indicate the linear mappings λ, μ between $PH(G)$ into itself with reference to R by λ_R^G, μ_R^G, respectively, and so on.

Let u be canonically represented by an integral

$$u(p) = \int_{\Delta_1} K(p, q) \, d\sigma_u(q).$$

Let Δ_{1R}^G be a set of minimal points belonging to Δ_1^G which can be defined by some non-compact point-sequence lying in R. This is an F_σ set. Then we put

$$\bar{\lambda}_R^G u = \int_{\Delta_{1R}^G} K(p, q)\ d\sigma_u(q)\,, \quad \bar{\mu}_R^G u = u - \bar{\lambda}_{G-\bar{R}}^G u\,.$$

Let r be a minimal point in Δ_1^G. Let $\bar{N}_\varepsilon(r)$ be an ε-neighborhood of r in Martin's metric and $N_\varepsilon = \bar{N}_\varepsilon(r) \frown G$. Then $T_{N_\varepsilon}^G K_G(p, r) > 0$ for a minimal function $K_G(p. r)$. See [1], p. 28.

THEOREM 1. *For any $u \in PH(G)$ there holds a system of inequalities*

$$\bar{\lambda}_R^G u \geqq \lambda^{*G}_R u \geqq \lambda_R^G u = \lambda_{1R}^G u$$
$$\geqq \mu_R^G u = \mu_{1R}^G u \geqq \mu^{*G}_R u \geqq \bar{\mu}_R^G u\,.$$

Proof. By the maximum principle we have $u(n, m) \geqq u_n$ in G_n whence follows $u(n) \geqq u_n$ in G_n. This shows $\lambda_1 u \geqq \overline{\lim}\, u_n$ in G for an exhaustion $\{W_n\}$ of W. On the other hand $\lambda_1 u$ is independent of the exhaustion, that is, two different exhaustions determine the same $\lambda_1 u$ for any given $u \in PH(G)$. Therefore we have

$$\lambda_1 u \geqq \sup \overline{\lim}\ u_n = \lambda u\,.$$

For any $\varepsilon > 0$ and for any compact set K, there exist two numbers N and N_1 for which

$$|u(n, m) - u(n)| < \varepsilon/2, \qquad |u(n) - \lambda_1 u| < \varepsilon/2$$

hold in K for any $n > N$ and $m > N_1 (m > n)$ And hence

$$|u(n, m) - \lambda_1 u| < \varepsilon$$

for any $m > n > N_1$ in K. From the original exhaustion $\{W_n\}$ we shall construct a new exhaustion. Let p be the sum $n + m$, then two inequalities $n < n'$ and $m < m'$ imply an inequality $p < p'$. Putting $W'_p = W_m \frown (W - \bar{R}) \smile W_n$, $\{W'_p\}$ is an exhaustion of W. For a sequence $\varepsilon/2^{j-1}$ we can determine a sequence of pairs of integers (n_j, m_j) such that $n_j < n_{j+1}$, $m_j < m_{j+1}$ from the above consideration. And hence we can construct a sequence of integers $\{p_j\}$ by $\{n_j + m_j\}$ and a corresponding exhaustion $\{W'_{p_j}\}$. For this exhaustion $\{W'_{p_j}\}$ we can construct u_{p_j} and $\overline{\lim}\ u_{p_j}$. Then we can say

$$\lambda u \geqq \overline{\lim_{j\to\infty}}\ u_{p_j}$$

in G by the definition of λu and $u_{p_j} = u(n_j, m_j)$. On the other hand

$$|u_{p_j} - \lambda_1 u| < \varepsilon/2^{j-1}$$

in K, whence follows that

$$\lambda_1 u \leqq \overline{\lim_{j\to\infty}}\,(u_{p_j} + \varepsilon/2^{j-1}) \leqq \lambda u$$

in K. Since K is an arbitrary compact set in G, we have

$$\lambda_1 u \leqq \lambda u$$

in G. Therefore we see that $\lambda_1 u = \lambda u$ in G for any $u \in PH(G)$.

Similarly we can prove the following fact:

$$\mu_1 u = \mu u$$

for any $u \in PH(G)$.

Further by the definitions of linear mappings λ, λ_1, and μ, μ_1 we can easily prove that

$$\lambda_R^G u = \lambda_{1R}^G u = u - \mu_{1,G-\bar{R}}^G u = u - \mu_{G-\bar{R}}^G u$$

remain valid for any $u \in PH(G)$.

Further we have $u \geqq T_R^G u$ in R and hence $u_n \geqq (T_R^G u)$ in $G \frown W_n$, which implies $\lim_n u_n \geqq S_R^G T_R^G u$ for any exhaustion $\{W_n\}$. Therefore we have $\mu_R^G u = \inf_{(\{W_n\})} \underline{\lim}_n u_n \geqq S_R^G T_R^G u = \mu^*{}_R^G u$. The following fact is a simple consequence of the above fact:

$$\lambda^*{}_R^G u = u - \mu^*{}_{G-\bar{R}}^G u \geqq u - \mu_{G-\bar{R}}^G u = \lambda_R^G u .$$

Next we shall prove $\varDelta_1^G(R) \supseteqq \varDelta_1^G - \varDelta_{1,G-\bar{R}}^G$. Let r be a point in $\varDelta_1^G - \varDelta_{1,G-\bar{R}}^G$, then $d(r, G-\bar{R}) > 0$ in Martin's metric in $G \smile \varDelta^G$. Indeed, if it is not so, then there is a point-sequence $\{q_n\}$ in $G-\bar{R}$ and hence in $G-R$ which tends to the r. This sequence defines the point r in $\varDelta_1^G$. This implies that $r \in \varDelta_{1,G-\bar{R}}^G$, which is untenable. Thus any point r in $\varDelta_1^G - \varDelta_{1,G-\bar{R}}^G$ has a positive distance from $G-\bar{R}$ and hence $G-R$. Therefore we can construct a neighborhood $\bar{N}(r)$ of r satisfying $d(p, r) < d(r, G-\bar{R}) = d(r, G-R)$ for any $p \in \bar{N}(r)$. $N(r) = \bar{N}(r) \frown G \subset R$ is thus obtained. Then $T_{N(r)}^G K_G(p, r) > 0$ and hence $T_R^G K_G(p, r) \geqq T_{N(r)}^G K_G(p, r) > 0$. This shows that $r \in \varDelta_1^G(R)$. By this result or by a similar method we have $\varDelta_{1R}^G \supseteqq \varDelta_1^G - \varDelta_1^G(G-\bar{R})$. Therefore we have

$$\mu^*{}_R^G u = \int_{\varDelta_1^G(R)} K_G(p, q)\, d\sigma_u(q) \geqq \int_{\varDelta_1^G - \varDelta_{1,G-\bar{R}}^G} K_G(p, q)\, d\sigma_u(q)$$

$$= u(p) - \int_{\varDelta_{1,G-\bar{R}}^G} K_G(p, q)\, d\sigma_u(q)$$

$$= u(p) - \bar{\lambda}_{G-\bar{R}}^G u(p) = \bar{\mu}_R^G u .$$

Further we have $\lambda^*{}_R^G u = u - \mu^*{}_{G-\bar{R}}^G u \leqq u - \bar{\mu}_{G-\bar{R}}^G u = \bar{\lambda}_R^G u$.

REMARK 1. In the inequality $\mu^*{}_R^G u \geqq \bar{\mu}_R^G u$ there is an example excluding the equality. In the following Fig. 1 q has a defining tail in $R \subset G$. Therefore $\mu^*{}_R^G K_G(p, q) = K_G(p, q) > 0$. However q also belongs to $\varDelta_{1,G-\bar{R}}^G$ and hence q does not belong to $\varDelta_1^G - \varDelta_{1,G-\bar{R}}^G$. This implies that $\bar{\mu}_R^G K_G(p, q) = 0$. This also implies $\bar{\lambda}_{G-\bar{R}}^G K_G(p, q) \gneqq \lambda^*{}_{G-\bar{R}}^G K_G(p, q) = 0$.

REMARK 2. In the inequality $\mu^*{}_R^G u \geqq \bar{\mu}_R^G u$ there are some examples for which the equality occurs for any $u \in PH(G)$. See Fig. 2 and 3.

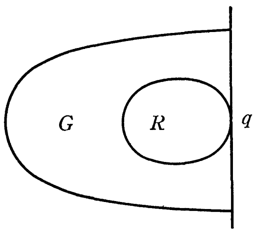
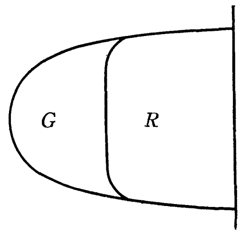
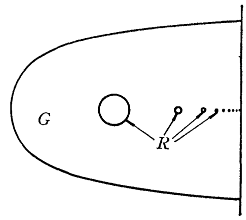


Fig. 1



$\mu^{*G}_R u = \bar{\mu}^G_R u > 0$
Fig. 2



$\mu^{*G}_R u = \bar{\mu}^G_R u = 0$
Fig. 3

REMARK 3. It should be remember that ∂R is freely accessible from both sides, that is, R satisfies the two-sidedness condition. This condition is essential for the inequality $\mu^{*G}_R \geqq \bar{\mu}^G_R$. In fact, if it is not so, then we cannot conclude that

$$N(r) = \bar{N}(r) \frown G \subset R$$

even if $d(p, r) < d(p, G-\bar{R})$ for any $p \in N(r)$, since $d(p, G-\bar{R}) \gneqq d(p, G-R) = 0$ in some example which is easy to construct. If $d(p, G-R) > 0$, then the situation is the same as in Theorem 1 and this implies $N(r) \subseteqq \bar{R}$ and hence $N(r) \subseteqq R$.

REMARK 4. Let $\mathring{\lambda}^G_R u$ be an operator defined by

$$\mathring{\lambda}^G_R u = \int_{\Delta^G_1 - \overline{\Delta^G_1(G-\bar{R})}} K_G(p, q)\, d\sigma_u(q)$$

for

$$u = \int_{\Delta^G_1} K_G(p, q)\, d\sigma_u(q),$$

where $\overline{\Delta^G_1(G-\bar{R})}$ is the closure in Martin's topology. Evidently we have $\lambda^{*G}_R u \geqq \mathring{\lambda}^G_R u$. On the other hand $\Delta^G_1(G-\bar{R}) \subseteqq \Delta^G_{1,G-\bar{R}}$ and the later one is closed in Δ^G_1 and hence $\overline{\Delta^G_1(G-\bar{R})} \subseteqq \Delta^G_{1,G-\bar{R}}$. This implies $\Delta^G_1 - \overline{\Delta^G_1(G-\bar{R})} \supseteqq \Delta^G_1 - \Delta^G_{1,G-\bar{R}}$. Therefore we have $\mathring{\lambda}^G_R u \geqq \bar{\mu}^G_R u$.

Let u be canonically represented

$$u(p) = \int_{\Delta^G_1} K(p, q)\, d\sigma_u(q),$$

then

$$\bar{\lambda}^G_R u = \int_{\Delta^G_{1,R}} K(p, q)\, d\sigma_u(q), \qquad \lambda^{*G}_R u = \int_{\Delta^G_1 - \Delta^G_1(G-\bar{R})} K(p, q)\, d\sigma_u(q),$$

$$\mu^{*G}_R u = \int_{\Delta^G_1(R)} K(p, q)\, d\sigma_u(q), \qquad \bar{\mu}^G_R u = \int_{\Delta^G_1 - \Delta^G_{1,G-\bar{R}}} K(p, q)\, d\sigma_u(q).$$

However these are also represented by the canonical integrals

$$\bar{\lambda}_R^G u=\int_{\varDelta_1^G} K(p,q)\,d\sigma_{\bar{\lambda}_R^G u}(q),\quad \lambda^{*G}_R u=\int_{\varDelta_1^G} K(p,q)\,d\sigma_{\lambda^{*G}_R u}(q)$$

$$\mu^{*G}_R u=\int_{\varDelta_1^G} K(p,q)\,d\sigma_{\mu^{*G}_R u}(q),\quad \bar{\mu}_R^G u=\int_{\varDelta_1^G} K(p,q)\,d\sigma_{\bar{\mu}_R^G u}(q).$$

Therefore by the above fact

$$\sigma_{\bar{\lambda}_R^G u}(A)=\int_A \theta_1(q)\,d\sigma_u(q),\quad \sigma_{\lambda^{*G}_R u}(A)=\int_A \theta_2(q)\,d\sigma_u(q),$$

$$\sigma_{\mu^{*G}_R u}(A)=\int_A \theta_3(q)\,d\sigma_u(q),\quad \sigma_{\bar{\mu}_R^G u}(A)=\int_A \theta_4(q)\,d\sigma_u(q)$$

for any Borel set A, where $\theta_1, \theta_2, \theta_3$ and θ_4 are measurable on $\varDelta_1^G$ and satisfy the following conditions

$$\theta_1=\begin{cases}1 & \text{on } \varDelta_{1R}^G\\ 0 & \text{on } \varDelta_1^G-\varDelta_{1R}^G,\end{cases}\qquad \theta_2=\begin{cases}1 & \text{on } \varDelta_1^G-\varDelta_1^G(G-\bar{R})\\ 0 & \text{on } \varDelta_1^G(G-\bar{R}).\end{cases}$$

$$\theta_3=\begin{cases}1 & \text{on } \varDelta_1^G(R)\\ 0 & \text{on } \varDelta_1^G-\varDelta_1^G(R),\end{cases}\qquad \theta_4=\begin{cases}1 & \text{on } \varDelta_1^G-\varDelta_{1,G-\bar{R}}^G\\ 0 & \text{on } \varDelta_{1,G-\bar{R}}^G.\end{cases}$$

If $u\leqq v$ and $u, v\in PH(G)$, then two canonical measures σ_u, σ_v satisfy an equation

$$\sigma_u(A)=\int_A \theta(q)\,d\sigma_v(q),\qquad 0\leqq\theta\leqq 1,$$

and hence $\sigma_u(A)\leqq\sigma_v(A)$ for any Borel set A.

In the sequel we shall use the words "for any λ" and "for any μ" in the sense of "for $\bar{\lambda}$, λ^*" and "for $\bar{\mu}$, μ^*".

If $u\leqq v$ and $u, v\in PH(G)$, then $\lambda u\leqq\lambda v$ and $\mu u\leqq\mu v$ for any λ and for any μ. Further we have for any u and v in $PH(G)$ $\lambda(u\wedge v)=\lambda u\wedge\lambda v$ and $\lambda(u\vee v)=\lambda u\vee\lambda v$, $\mu(u\wedge v)=\mu u\wedge\mu v$ and $\mu(u\vee v)=\mu u\vee\mu v$ for any λ and for any μ. For these it should be noted the lemma b in [1]. Further, if $R_1\supset R_2$ in G, then there hold $\bar{\lambda}_{R_1}^G u\geqq\bar{\lambda}_{R_2}^G u$, $\bar{\mu}_{R_1}^G u\geqq\bar{\mu}_{R_2}^G u$ and $\lambda^{*G}_{R_1} u\geqq\lambda^{*G}_{R_2} u$, $\mu^{*G}_{R_1} u\geqq\mu^{*G}_{R_2} u$ for any $u\in PH(G)$. To that end, by the representation of these operators it is sufficient to prove the following relations:

$$\varDelta_{1R_1}^G\supseteqq\varDelta_{1R_2}^G \quad\text{and}\quad \varDelta_1^G(R_1)\supseteqq\varDelta_1^G(R_2)$$

for any $R_1\supset R_2$. By the definition of $\varDelta_{1R_2}^G$, $q\in\varDelta_{1R_2}^G$ implies the existence of the defining point-sequence $\{q_n\}$ in R_2 by which q in $\varDelta_1^G$ is defined. This sequence also belongs to R_1 and defines the same q. If $q\in\varDelta_1^G(R_2)$, then $T_{R_2}^G K_G(p,q)>0$. However $T_{R_2}^G$ is equal to $T_{R_2}^{R_1} T_{R_1}^G$, which implies $T_{R_1}^G K_G(p,q)>0$. This shows that $q\in\varDelta_1^G(R_1)$. $G-\bar{R}_1\subset G-\bar{R}_2$ implies $\varDelta_1^G-\varDelta_{1,G-\bar{R}_1}^G\supseteqq\varDelta_1^G-\varDelta_{1,G-\bar{R}_2}^G$ and $\varDelta_1^G-\varDelta_1^G(G-\bar{R}_1)\supseteqq\varDelta_1^G-\varDelta_1^G(G-\bar{R}_2)$ which are equivalent to $\bar{\mu}_{R_1}^G u\geqq\bar{\mu}_{R_2}^G u$ and $\lambda^{*G}_{R_1} u\geqq\lambda^{*G}_{R_2} u$ for any $u\in PH(G)$, respectively.

If u_n converges to u monotonically in $PH(G)$, then $\bar{\lambda}_R^G u_n$, $\lambda^{*G}_R u_n$, $\mu^{*G}_R u_n$ and $\bar{\mu}_R^G u_n$

converge to $\bar{\lambda}_R^G u$, $\lambda^{*G}_R u$, $\mu^{*G}_R u$ and $\bar{\mu}_R^G u$ monotonically, respectively. Indeed this can be proved in the following manner: Let u_n and u be represented by the canonical integrals

$$\int_{\Delta_1^G} K(p, q)\ d\sigma_{u_n}(q) \quad \text{and} \quad \int_{\Delta_1^G} K(p, q)\ d\sigma_u(q),$$

respectively, and $u_n \leqq u$, then $\sigma_{u_n}(A) \leqq \sigma_u(A)$ for any Borel set A or more precisely

$$\sigma_{u_n}(A) = \int_A \theta_n(q)\ d\sigma_u(q)$$

for some measurable function θ_n, $0 \leqq \theta_n \leqq 1$. Thus we have

$$u_n(p) = \int_{\Delta_1^G} K(p, q)\ \theta_n(q)\ d\sigma_u(q).$$

Further, if u_n tends to u increasingly, then $\theta_n(q)$ is monotone increasing with n and tends to 1 almost everywhere in Δ_1^G. Then

$$\begin{aligned}\bar{\lambda}_R^G u_n(p) &= \int_{\Delta_{1R}^G} K(p, q)\ d\sigma_{u_n}(q) \\ &= \int_{\Delta_{1R}^G} K(p, q)\ \theta_n(q)\ d\sigma_u(q)\end{aligned}$$

tends to

$$\int_{\Delta_{1R}^G} K(p, q)\ d\sigma_u(q) = \bar{\lambda}_R^G u(p)$$

For any other cases similar proof can be carried out.

Let u be a minimal in $PH(G)$ and if $\bar{\mu}_R^G u > 0$, then $u = \lambda_R^G u = \lambda^{*G}_R u = \mu^{*G}_R u = \bar{\mu}_R^G u$. In fact u is equal to $k\ K_G(p, q)$, $q \in \Delta_1^G$. $\bar{\mu}_R^G u = cu = ck\ K_G(p, q)$, since $\bar{\mu}_R^G u > 0$ implies the equality. On the other hand

$$u(p) = \int_{\Delta_1^G} K_G(p, r)\ d\sigma_u(r), \quad \sigma_u(A) = \begin{cases} k & \text{if} \quad q \in A \\ 0 & \text{if} \quad q \notin A. \end{cases}$$

This implies

$$cu = \bar{\mu}_R^G u = \int_{\Delta_1^G - \Delta_{1, G-\bar{R}}^G} K_G(p, r)\ d\sigma_u(r)$$

and hence $q \in \Delta_1^G - \Delta_{1,G-\bar{R}}^G$. Thus $\bar{\mu}_R^G u = k\ K_G(p, q) = u$.

If $\mu^{*G}_R u > 0$ for a minimal u in $PH(G)$, then $u = \bar{\lambda}_R^G u = \lambda^{*G}_R u = \mu^{*G}_R u$. If $\lambda^{*G}_R u > 0$ for a minimal u, then $u = \bar{\lambda}_R^G u = \lambda^{*G}_R u$. If $\bar{\lambda}_R^G u > 0$ for a minimal u, then $u = \bar{\lambda}_R^G u$. Further these facts imply that the minimality is preserved by any operators if the resulting function is positive in G.

Let u be singular in $PH(G)$, that is, $0 \leqq v \leqq u$ for any $v \in THB(G)$ implies $v = 0$. Then $\bar{\lambda}_R^G u$, $\lambda^{*G}_R u$, $\mu^{*G}_R u$ and $\bar{\mu}_R^G u$ are also singular if they are positive. This is evident by the singularity of u and by theorem 1.

§3. The *P*-niceness and their relations. In this section we shall define several sorts of P-niceness of R in G by making use of the operators and discuss their relations.

If $\lambda^{*G}_{R} u = \mu^{*G}_{R} u$ for any $u \in PH(G)$ remains true, then we say R a *P-nice subset* in G. If this holds for any G with some relative boundary ∂G, then R is called *extremally P-nice*. If $\bar{\lambda}^{G}_{R} u = \bar{\mu}^{G}_{R} u$ remains true for any $u \in PH(G)$, then R is called *most P-nice* in G. If this is the case for any G with some relative boundary ∂G, then we say that R is *extremally most P-nice*.

The following figure shows that our original plan of investigation should be carefully done by its pathological nature.

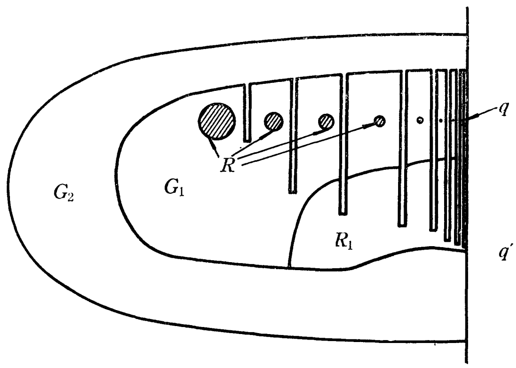


Fig. 4

In any simply-connected domain G any accessible boundary element in the sense of Carathéodory corresponds to a Martin minimal point and inversely any Martin minimal point corresponds to some accessible boundary point. Thus any defining sequence of boundary element in the sense of Carathéodory defines a Martin minimal point. The proof of this fact is easy to perform by the Riemann mapping theorem and the invariance of Martin's whole theory under any univalent onto conformal mapping. In our above example, in G_2 q is a point of $\Delta_1^{G_2}$ which is different from q' which is also a point of $\Delta_1^{G_2}$. However q coincides with q' as a point of $\Delta_1^{G_1}$ which consists of only one point q'. Therefore the set R defines a Martin minimal point $q' = q$ in G_1. These considerations show that $\mu^{*G_2}_{R} K_{G_2}(p, q) = 0$ and $\lambda^{*G_2}_{R} K_{G_2}(p, q) = K_{G_2}(p, q)$. However $\mu^{*G_1}_{R} K_{G_1}(p, q) = k\ \mu^{*G_1}_{R} K_{G_1}(p, q') = 0$ and $\lambda^{*G_1}_{R} K_{G_1}(p, q) = k\ K_{G_1}(p, q')$. Further we have $\Delta_{1,R_1}^{G_1} = q'$ and $\Delta_1^{G_1}(R_1) = q' = \Delta_{1,G_1-\bar{R}_1}^{G_1}$, which shows that $\bar{\lambda}^{G_1}_{R_1} u = \lambda^{*G_1}_{R_1} u = \mu^{*G_1}_{R_1} u > 0$ for $u = K_{G_1}(p, q')$ and $\bar{\mu}^{G_1}_{R_1} u = 0$. Thus R_1 is not most P-nice but P-nice in G_1. Evidently R_1 is not P-nice in G_2.

Next figure 5 shows that the notion of most P-nice in G is also a relative notion. Indeed in this figure $\Delta_{1,R}^{G_2}$ is a closed segment $\overline{qr}$ and $\Delta_{1,G_2-\bar{R}}^{G_2}$ coincides with two closed segments $\overline{sq'}$ and $\overline{rt}$. Thus $\Delta_1^{G_2} - \Delta_{1,G_2-\bar{R}}^{G_2}$ is an open segment $\overline{q'r}$. Therefore R is not most P-nice in G_2. However R is most P-nice in G_1.

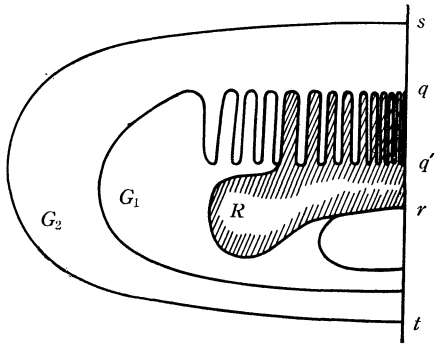


Fig 5

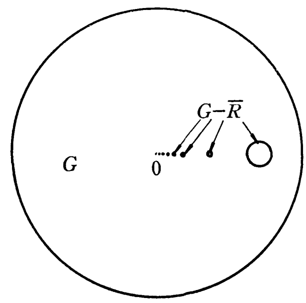


Fig. 6

Let G be the punctured unit disc $\{|z|<1\}-0$ and $G-\bar{R}$ is an infinite number of disc $\{|z-1/n|<1/10^{n!}\}$, $n=2, 3, \ldots$, then Δ_1^G consists of only one point $\{0\}$ and $\Delta_1^G(R)$ also consists of only one point $\{0\}$ by the irregularity of the origin in R. $\Delta_1^G(G-\bar{R})$ is empty and hence $\Delta_1^G-\Delta_1^G(G-\bar{R})$ consists of only one point $\{0\}$. Thus R is P-nice in G. However both $\Delta_{1,R}^{\;G}$ and $\Delta_{1,G-\bar{R}}^{\;G}$ consists of only one point $\{0\}$ and hence $\Delta_1^G-\Delta_{1,G-\bar{R}}^G$ is empty. This shows that R is not most P-nice in G. This example shows further that the extremal P-niceness of R does not imply the extremal most P-niceness of R. In Fig. 6 the irregularity condition for 0 in R is essential. If it is not the case, then $\Delta_1^G(R)$ reduces to an empty set. Further in this case R satisfies the conditions $\bar{\lambda}_R^G u=\lambda^{*}{}_R^G u=u=k\log 1/|z|$ and $\bar{\mu}_R^G u=\mu^{*}{}_R^G u=0$. For these phenomena we recommend our earlier paper [6].

We shall here introduce several other P-niceness. We make use of the following terminologies:

R is *exterior P-nice* in G if $\bar{\lambda}_R^G u=\lambda^{*}{}_R^G u$ for any $u\in PH(G)$,

R is *interior P-nice* in G if $\mu^{*}{}_R^G u=\bar{\mu}_R^G u$ for any $u\in PH(G)$ and

R is *ordinary P-nice* in G if it is exterior and interior P-nice in G.

Evidently if R is most P-nice in G, then it is ordinary P-nice in G. If any element $v\in PH(R)$ satisfies $S_R^G v\in PH(G)$, then T_R^G: $PH(G)\rightarrow PH(R)$ is an onto positively linear map and vice versa. If this is the case, then R is called *onto P-nice* in G. All of these notions are not absolute.

Let R be most P-nice in G, then we have

$$\Delta_{1,R}^{\;G}=\Delta_1^G-\Delta_1^G(G-\bar{R})=\Delta_1^G(R)=\Delta_1^G-\Delta_{1,G-\bar{R}}^{\;G},$$

$$\Delta_{1,G-\bar{R}}^{\;G}=\Delta_1^G(G-\bar{R}).$$

Let r be a Martin minimal point in Δ_1^G whose one defining point-sequence $\{q_n\}$ belongs to R, that is, $r\in\Delta_{1,R}^{\;G}$, then $d_G(r, q_n)$ the Martin distance from q_n to r tends to zero when $n\rightarrow\infty$. $\Delta_{1,R}^{\;G}=\Delta_1^G(R)$ implies $T_R^G K_G(p, r)>0$. Thus $K_R(p, r')$ is determines as a minimal function, that is, a minimal point $r' \in \Delta_1^R$ is determined. If the r is also defined by a sequence $\{q_n\} \in G-\bar{R}$, then $r \in \Delta_{1,G-\bar{R}}^{G} = \Delta_1^G(G-\bar{R})$. This implies that $T_{G-\bar{R}}^{G} K_G(p, r) > 0$. Since $\Delta_1^G(R) \frown \Delta_1^G(G-\bar{R}) = \phi$, $T_R^G K_G(p, r) > 0$ and $T_{G-\bar{R}}^{G} K_G(p, r) > 0$ lead to a contradiction. Thus there is no point sequence in $G-\bar{R}$ by which r is defined. Therefore there is a small neighborhood $\overline{N(r)}$ of r for which $N(r) = \overline{N(r)} \frown G$ entirely belongs to R. Let r and s be two different points in $\Delta_{1,R}^{G}$, then r' and s' are different in Δ_1^G. Indeed $T_R^G K_G(p, r) = K_R(p, r')$ and $T_R^G K_G(p, s) = K_R(p, s')$ are minimal and $r' = s'$ implies $K_G(p, r) = S_R^G T_R^G K_G(p, r) = S_R^G K_R(p, r') = S_R^G K_R(p, s') = S_R^G T_R^G K_G(p, s) = K_G(p, s)$, whence follows $r = s$ in Δ_1^G. This is a contradiction. Let q be any point in $\overline{N(r)} \frown \Delta_1^G$, then $d_G(q, r) < d_G(\partial N(r), r)$ and $d_G(q, \partial N(r)) > \delta > 0$. Let $N_\varepsilon(q)$ be a point set in G satisfying an inequality $d_G(q, p) < \varepsilon < \delta$. Then $N_\varepsilon(q) \subset N(r) \subset R$. Thus

$$0 < T_{N_\varepsilon(q)}^{G} K_G(p, q) = T_{N_\varepsilon(q)}^{R} T_R^G K_G(p, q)$$

implies

$$T_R^G K_G(p, q) > 0\,,$$

whence follows $q \in \Delta_1^G(R)$. This fact shows that any connected component of Δ_1^G belongs to either $\Delta_1^G(R)$ or $\Delta_1^G(G-\bar{R})$.

THEOREM 2. *If R is most P-nice in G, then any connected component of Δ_1^G belongs to either $\Delta_1^G(R) \equiv \Delta_{1,R}^{G}$ or $\Delta_1^G(G-\bar{R}) \equiv \Delta_{1,G-\bar{R}}^{G}$. $\Delta_{1,R}^{G}$ is imbedded in Δ_1^R by T_R^G, that is, there is a univalent correspondence induced by T_R^G which carries $\Delta_{1,R}^{G}$ into Δ_1^R. If R is onto P-nice in G, then Δ_1^R is imbedded in Δ_1^G by S_R^G and its image is equal to $\Delta_1^G(R)$ and vice versa.*

Proof. It is sufficient to prove the last part. Let r' be any point in Δ_1^R, then by the onto P-niceness assumption of R in G there exists a minimal function $K_G(p, r)$ such that $T_R^G K_G(p, r) = K_R(p, r')$ and hence $K_G(p, r) = S_R^G K_R(p, r')$, where $K_R(p, r')$ is a minimal function in R corresponding to the r'. Thus $r \in \Delta_1^G(R)$. By the minimality preserving property of T_R^G, this correspondence $r' \to r$ is univalent. Any point r in $\Delta_1^G(R)$ corresponds to a point r' in Δ_1^R. Therefore the map induced by S_R^G carries Δ_1^R univalently onto $\Delta_1^G(R)$. Inverse statement is evident.

By T_R^G, $\Delta_1^G(R)$ can be imbedded into Δ_1^R. Thus there is a univalent correspondence T_R^G: $\Delta_1^G(R) \to \Delta_1^R$. If r corresponds to r', $r \in \Delta_1^G(R)$, $r' \in \Delta_1^R$ by T_R^G. Let $\overline{N_\varepsilon(r)}$ and $\overline{N_\delta(r')}$ be two neighborhoods of r in $G \smile \Delta^G$ and of r' in $R \smile \Delta^R$ defined by $d_G(p, r) < \varepsilon$ and $d_R(p, r') < \delta$ in the respective Martin metrices. Let $N_\varepsilon(r)$ and $N_\delta(r')$ be $\overline{N_\varepsilon(r)} \frown G$ and $\overline{N_\delta(r')} \frown R$, respectively. We prove the following relation: $N_\varepsilon(r) \frown N_\delta(r') \neq \phi$ for any ε and δ. If it is not so, then there are some ε and some δ for which $N_\varepsilon(r) \frown N_\delta(r') = \phi$. On the other hand

$$T_R^G K_G(p, r) = K_R(p, r')$$

for two minimals $K_G(p, r)$ and $K_R(p, r')$. Thus we have

$$0 < T^{G}_{N_\varepsilon(r)} K_G(p\ q)$$

and

$$0 < T^{R}_{N_\delta(r')} K_R(p, r') = T^{R}_{N_\delta(r')} T^{G}_{R} K_G(p, r) = T^{G}_{N_\delta(r')} K_G(p, r).$$

Since $N_\varepsilon(r) \frown N_\delta(r') = \phi$ and $K_G(p, r)$ is minimal in $PH(G)$, the above two inequalities are untenable. Thus $N_\varepsilon(r) \frown N_\delta(r') \neq \phi$ for any ε and for any δ. This implies that we can select the same defining tail of r and r' in R.

In the next place we shall show that the following two implications

$$\text{onto } P\text{-niceness} \rightleftarrows P\text{-niceness}$$

are false in general.

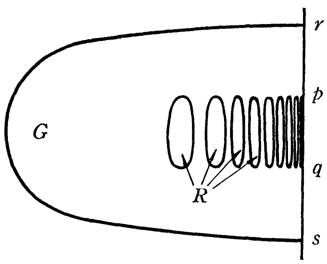


Fig. 7

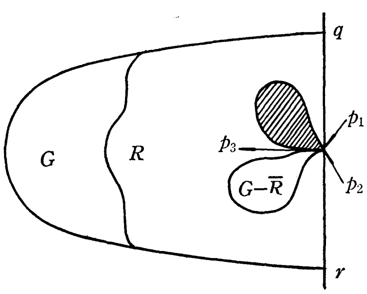


Fig. 8

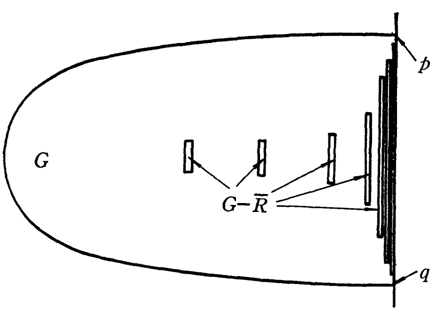


Fig. 9

In Fig. 7 we have $\Delta_1^G = [r, s]$, $\Delta_1^G(R) = \phi$, $\Delta_1^G(G - \bar{R}) = [r, p) \smile (q, s]$ and $\Delta_1^G - \Delta_1^G(G - \bar{R}) = [p, q]$. This shows that R is not P-nice in G. However $\Delta_1^R = \Delta^R = \phi$ implies the onto P-niceness of R in G.

In Fig. 8 $\Delta_1^G = [q, p_1] \smile [p_2, r]$, $\Delta_1^G(R) = [q, p_1] \smile [p_2, r]$. This implies the P-niceness of R in G and further the exterior P-niceness of R in G. However $\Delta_1^R = [q, p_1] \smile (p_3) \smile [p_2, r]$ implies $\Delta_1^R \supsetneqq \Delta_1^G(R)$, that is, R is not onto P-nice in G. (The shaded part is a component of the exterior of G.)

Even if R is a domain, the onto-P-niceness of R in G does not imply the P-niceness of R in G. In Fig. 9 $\Delta_1^G = [p, q]$, $\Delta_1^G(R) = (p) \smile (q)$, $\Delta_1^G - \Delta_1^G(G - \bar{R}) = [p, q]$ and hence R is not P-nice in G. However $\Delta_1^R = (p) \smile (q)$ $(\subset \Delta^R = [p, q])$ implies the onto P-niceness of R in G.

THEOREM 3. *Let R be most P-nice in G and $\Delta_1^G \equiv \Delta^G$, then R is onto P-nice in G.*

In order to prove this theorem we shall prepare the following two facts:

(i). If R is onto P-nice in G_1 and G_1 is onto P-nice in G_2, then R is onto P-nice in G_2. If R is onto P-nice in G_2 and $R \subset G_1 \subset G_2$, then R is onto P-nice in G_1.

(ii). If the proper part of ∂R in G lies in a compact part of W, then R is onto P-nice and simultaneously most P-nice in G.

Proof of (i). The first part is evident. Let u be any element of $PH(R)$, then there exists an element v such that $T_R^{G_2}v = u$ and $S_R^{G_2}u = v$. Let w be an element of $PH(G_1)$ satisfying $w = T_{G_1}^{G_2}v$, then $T_R^{G_1}w = T_R^{G_1}T_{G_1}^{G_2}v = T_R^{G_2}v = u$. This implies the desired result.

Proof of (ii). By the assumption the proper part of ∂R divides G into two parts. We can assume with no loss of generality that the proper part of ∂R consists of analytic curves. Then we can construct a continuous superharmonic function S in G satisfying the condition $u \leqq S$ in R. In this process the boundedness of $(\partial u/\partial n)\, ds$ on the proper part of ∂R in G is essential and the non-vanishing property of the harmonic measure $\omega(p, \partial R, G-\bar{R})$ is also essential. Then $S_R^G u \leqq S$ in G. Therefore we have $S_R^G u \in PH(G)$. This implies that $T_R^G S_R^G u = u$ and hence R is onto P-nice in G. The proper part of ∂R lies in a compact part of W, therefore $\Delta_1^G(R) = \Delta_{1,R}^{\ G}$, $\Delta_1^G(G-\bar{R}) = \Delta_{1,G-\bar{R}}^{\ G}$ and hence

$$\begin{aligned}\Delta_1^G(G-\bar{R}) + \Delta_1^G(R) \subseteqq \Delta_1^G \subseteqq \Delta_{1,R}^{\ G} + \Delta_{1,G-\bar{R}}^{\ G} \\ = \Delta_1^G(R) + \Delta_1^G(G-\bar{R}).\end{aligned}$$

This implies that R is most P-nice in G.

Proof of theorem 3. By theorem 2, there is no point in $\Delta_1^G \equiv \Delta^G$ being accessible by two point-sequences one of which belongs to R and another of which belongs to $G-R$. Therefore any point of Δ^G is not accessible by a point-sequence belonging to a proper part of ∂R. Thus a proper part of ∂R in G lies in a compact part in W. By (ii) R is onto P-nice in G.

In the first part of (i) we can neither exclude the onto P-niceness of G_1 in G_2 nor replace it by the P-niceness of G_1 in G_2.

THEOREM 4. *Let W belong to the class O_G and G be a subregion of W. If R is a subregion of G whose boundary ∂R clusters irregularly at the ideal boundary of W, then R is onto P-nice and P-nice and exterior P-nice in G.*

Proof. Let r be any point of Δ_1^R, then there exists a defining point-sequence $\{r_n\}$ of r in R. Then

$$g_R(p, r_n) \leqq g_G(p, r_n), \qquad \lim_{n\to\infty} g_R(p, r_n) > 0$$

imply the irregularity of a minimal point $\bar{r}$ on Δ_1^G which is defined by $\{r_n\}$. Thus r corresponds to $\bar{r}$ by S_R^G. This is equivalent to a fact that R is onto P-nice in G. Let $\bar{r}$ be any point of Δ_1, then there is a defining sequence $\{r_n\}$ of $\bar{r}$ in R. Then any convergent subsequence $\{r_{\nu_n}\}$ of $\{r_n\}$ defines a point r in Δ^R and further $g_R(p, r) \equiv \lim_{n\to\infty} g_R(p, r_{\nu_n}) > 0$ and $g_R(p, r_{\nu_n}) \leqq g_G(p. r_{\nu_n})$. This implies that r is an irregular minimal point of $\Delta_{1R}^{\ G}$ and $T_R^G g_G(p, r) \geqq k g_R(p, r) > 0$. Thus r is also a minimal point of Δ^R and $\bar{r} \in \Delta_1^G(R)$. This implies the desired two P-niceness of R in G.

§4. Algebraic calculations. In this section we shall discuss the algebraic properties of $\bar{\lambda}$, λ^*, μ^* and $\bar{\mu}$.

(1) $\bar{\lambda}_R^G u + \bar{\mu}_{G-\bar{R}}^G u = u$ and $\lambda^{*G}_R u + \mu^{*G}_{G-\bar{R}} u = u$.

(2) $\mu^{*G}_R \mu^{*G}_R u = \mu^{*G}_R \lambda^{*G}_R u = \mu^{*G}_R \bar{\lambda}_R^G u = \lambda^{*G}_R \mu^{*G}_R u = \bar{\lambda}_R^G \mu^{*G}_R u = \mu^{*G}_R u$, $\lambda^{*G}_R \lambda^{*G}_R u = \bar{\lambda}_R^G \lambda^{*G}_R u$ $= \lambda^{*G}_R \bar{\lambda}_R^G u = \lambda^{*G}_R u$, $\bar{\lambda}_R^G \bar{\lambda}_R^G u = \bar{\lambda}_R^G u$, $\bar{\mu}_R^G \bar{\mu}_R^G u = \bar{\mu}_R^G \bar{\lambda}_R^G u = \bar{\mu}_R^G \mu^{*G}_R u = \bar{\mu}_R^G \lambda^{*G}_R u = \mu^{*G}_R \bar{\mu}_R^G u$ $= \lambda^{*G}_R \bar{\mu}_R^G u = \bar{\lambda}_R^G \bar{\mu}_R^G u = \bar{\mu}_R^G u$.

Proof. $\mu^{*G}_R \mu^{*G}_R u = S_R^G T_R^G S_R^G T_R^G u = S_R^G T_R^G u = \mu^{*G}_R u$. The following inequalities $\mu^{*G}_R u \leqq \lambda^{*G}_R u \leqq \bar{\lambda}_R^G u \leqq u$ imply two systems of inequalities

$$\mu^{*G}_R u = \mu^{*G}_R \mu^{*G}_R u \leqq \mu^{*G}_R \lambda^{*G}_R u \leqq \mu^{*G}_R \bar{\lambda}_R^G u \leqq \mu^{*G}_R u$$

and

$$\mu^{*G}_R u = \mu^{*G}_R \mu^{*G}_R u \leqq \lambda^{*G}_R \mu^{*G}_R u \leqq \bar{\lambda}_R^G \mu^{*G}_R u \leqq \mu^{*G}_R u .$$

These imply the desired results. Further we have

$$\lambda^{*G}_R \lambda^{*G}_R u = \lambda^{*G}_R(u - \mu^{*G}_{G-\bar{R}} u) = \lambda^{*G}_R u - \lambda^{*G}_R \mu^{*G}_{G-\bar{R}} u$$
$$= \lambda^{*G}_R u - \mu^{*G}_{G-\bar{R}} u + \mu^{*G}_{G-\bar{R}} \mu^{*G}_{G-\bar{R}} u = \lambda^{*G}_R u .$$

We have the following two inequalities

$$\lambda^{*G}_R u = \lambda^{*G}_R \lambda^{*G}_R u \leqq \lambda^{*G}_R \bar{\lambda}_R^G u \leqq \lambda^{*G}_R u$$

and

$$\lambda^{*G}_R u = \lambda^{*G}_R \lambda^{*G}_R u \leqq \bar{\lambda}_R^G \lambda^{*G}_R u \leqq \lambda^{*G}_R u$$

by the inequalities $\lambda^{*G}_R u \leqq \bar{\lambda}_R^G u \leqq u$. We have the following representation

$$\bar{\lambda}_R^G u = \int_{\Delta_{1R}^G} K(p, q)\ d\sigma_u(q) = \int_{\Delta_1^G} K(p, q)\ d\sigma_{\bar{\lambda}_R^G u}(q)$$
$$= \int_{\Delta_1^G} K(p, q)\ \theta_1(q)\ d\sigma_u(q), \quad \theta_1(q) = \begin{cases} 1 & \text{on } \Delta_{1,R}^G, \\ 0 & \text{on } \Delta_1^G - \Delta_{1,R}^G. \end{cases}$$

Therefore we have

$$\bar{\lambda}_R^G \bar{\lambda}_R^G u = \int_{\Delta_1^G} K(p, q)\ \theta_1(q)\ d\sigma_{\bar{\lambda}_R^G u}(q)$$
$$= \int_{\Delta_1^G} K(p, q)\ \theta_1(q)\ \theta_1(q)\ d\sigma_u(q)$$
$$= \int_{\Delta_1^G} K(p, q)\ \theta_1(q)\ d\sigma_u(q) = \bar{\lambda}_R^G u .$$

By the above identity we can say that

$$\bar{\mu}_R^G \bar{\mu}_R^G u = \bar{\mu}_R^G u - \bar{\mu}_R^G \bar{\lambda}_{G-\bar{R}}^G u = \bar{\mu}_R^G u - \bar{\lambda}_{G-\bar{R}}^G u + \bar{\lambda}_{G-\bar{R}}^G \bar{\lambda}_{G-\bar{R}}^G u = \bar{\mu}_R^G u .$$

The inequalities $\bar{\mu}^G_R u \leqq \mu^{*G}_R u \leqq \lambda^{*G}_R u \leqq \bar{\lambda}^G_R u \leqq u$ imply the desired fact by operating $\bar{\mu}^G_R$ and by substituting u by $\bar{\mu}^G_R u$.

(3) If $R_1 \frown R_2 = \phi$, then $\mu^{*G}_{R_1} \mu^{*G}_{R_2} u = \bar{\mu}^G_{R_1} \bar{\mu}^G_{R_2} u = 0$. Further we have $\mu^{*G}_{G-\bar{R}} \mu^{*G}_R u = \mu^{*G}_{G-\bar{R}} \lambda^{*G}_R u = \lambda^{*G}_R \mu^{*G}_{G-\bar{R}} u = \lambda^{*G}_{G-\bar{R}} \mu^{*G}_R u = \mu^{*G}_R \lambda^{*G}_{G-\bar{R}} u = 0$ and $\bar{\mu}^G_{G-\bar{R}} \bar{\mu}^G_R u = \bar{\mu}^G_{G-\bar{R}} \mu^{*G}_R u = \mu^{*G}_R \bar{\mu}^G_{G-\bar{R}} u = \bar{\mu}^G_{G-\bar{R}} \bar{\lambda}^G_R u = \bar{\lambda}^G_R \bar{\mu}^G_{G-\bar{R}} u = 0$.

(4) $\lambda^{*G}_{G-\bar{R}} \lambda^{*G}_R u = \lambda^{*G}_R \lambda^{*G}_{G-\bar{R}} u = \lambda^{*G}_R u - \mu^{*G}_R u = \lambda^{*G}_{G-\bar{R}} u - \mu^{*G}_{G-\bar{R}} u$ and $\bar{\lambda}^G_{G-\bar{R}} \bar{\lambda}^G_R u = \bar{\lambda}^G_R \bar{\lambda}^G_{G-\bar{R}} u = \bar{\lambda}^G_R u - \bar{\mu}^G_R u = \bar{\lambda}^G_{G-\bar{R}} u - \bar{\mu}^G_{G-\bar{R}} u$ for any $u \in PH(G)$.

Proof.
$$\lambda^{*G}_{G-\bar{R}} \lambda^{*G}_R u = \lambda^{*G}_R u - \mu^{*G}_R \lambda^{*G}_R u = \lambda^{*G}_R u - \mu^{*G}_R u$$
$$= \int_{\Delta^G_1 - \Delta^G_1(G-\bar{R})} K(p, q)\, d\sigma_u(q) - \int_{\Delta^G_1(R)} K(p, q)\, d\sigma_u(q)$$
$$= \int_{\Delta^G_1 - \Delta^G_1(G-\bar{R}) - \Delta^G_1(R)} K(p, q)\, d\sigma_u(q),$$
$$\bar{\lambda}^G_{G-\bar{R}} \bar{\lambda}^G_R u = \bar{\lambda}^G_R u - \bar{\mu}^G_R \bar{\lambda}^G_R u = \bar{\lambda}^G_R u - \bar{\mu}^G_R u$$
$$= \int_{\Delta^G_{1,R}} K(p, q)\, d\sigma_u(q) - \int_{\Delta^G_1 - \Delta^G_{1,G-\bar{R}}} K(p, q)\, d\sigma_u(q).$$

(5)
$$\lambda^{*G}_{G-\bar{R}} \bar{\lambda}^G_R u = \bar{\lambda}^G_R \lambda^{*G}_{G-\bar{R}} u = \bar{\lambda}^G_R u - \mu^{*G}_R u$$
$$= \lambda^{*G}_{G-\bar{R}} u - \bar{\mu}^G_{G-\bar{R}} u = u - (\bar{\mu}^G_{G-\bar{R}} + \mu^{*G}_R) u,$$
$$\lambda^{*G}_R \bar{\lambda}^G_{G-\bar{R}} u = \bar{\lambda}^G_{G-\bar{R}} \lambda^{*G}_R u = \lambda^{*G}_R u - \bar{\mu}^G_R u$$
$$= \bar{\lambda}^G_{G-\bar{R}} u - \mu^{*G}_{G-\bar{R}} u = u - (\mu^{*G}_{G-\bar{R}} + \bar{\mu}^G_R) u$$
for any $u \in PH(G)$.

Proof.
$$\lambda^{*G}_{G-\bar{R}} \bar{\lambda}^G_R u = \bar{\lambda}^G_R u - \mu^{*G}_R \bar{\lambda}^G_R u = \bar{\lambda}^G_R u - \mu^{*G}_R u$$
$$= \int_{\Delta^G_{1R} - \Delta^G_1(R)} K(p, q)\, d\sigma_u(q).$$
$$\lambda^{*G}_R \bar{\lambda}^G_{G-\bar{R}} u = \lambda^{*G}_R u - \lambda^{*G}_R \bar{\mu}^G_R u = \lambda^{*G}_R u - \bar{\mu}^G_R u$$
$$= \int_{\Delta^G_{1,G-\bar{R}} - \Delta^G_1(G-\bar{R})} K(p, q)\, d\sigma_u(q).$$

(6)
$$\lambda^{*G}_{G-\bar{R}} (\bar{\lambda}^G_R - \mu^{*G}_R) u = (\bar{\lambda}^G_R - \mu^{*G}_R) u, \quad \bar{\lambda}^G_{G-\bar{R}} (\lambda^{*G}_R - \bar{\mu}^G_R) u = (\lambda^{*G}_R - \bar{\mu}^G_R) u$$
for any $u \in PH(G)$.

(7) Let G_1 and G_2 $(\supset G_1)$ be two subregions of W containing R, then for each $u \in PH(G_1)$ satisfying $S^{G_2}_{G_1} u \in PH(G_2)$, we have
$$T^{G_2}_{G_1} \mu^{*G_2}_{G_1} S^{G_2}_{G_1} u = \mu^{*G_1}_R u \text{ and } T^{G_2}_{G_1} \lambda^{*G_2}_R S^{G_2}_{G_1} u \geqq \lambda^{*G_1}_R u.$$
And for each $v \in PH(G_2)$ satisfying $T^{G_2}_{G_1} v > 0$ in G_1, we have

$$\mu^{*G_2}_{R}v = S^{G_2}_{G_1}\mu^{*G_1}_{R}T^{G_2}_{G_1}v \text{ and } \lambda^{*G_2}_{R}v \geqq S^{G_2}_{G_1}\lambda^{*G_1}_{R}T^{G_2}_{G_1}v.$$

Proof.
$$T^{G_2}_{G_1}\mu^{*G_2}_{R}S^{G_2}_{G_1} = T^{G_2}_{G_1}S^{G_2}_{R}T^{G_1}_{R}T^{G_2}_{G_1}S^{G_2}_{G_1}$$
$$= T^{G_2}_{G_1}S^{G_2}_{G_1}S^{G_1}_{R}T^{G_1}_{R} = S^{G_1}_{R}T^{G_1}_{R} = \mu^{*G_1}_{R}.$$
$$S^{G_2}_{G_1}\mu^{*G_1}_{R}T^{G_2}_{G_1} = S^{G_2}_{G_1}S^{G_1}_{R}T^{G_1}_{R}T^{G_2}_{G_1} = S^{G_2}_{R}T^{G_2}_{R} = \mu^{*G_2}_{R}.$$

By the definition of $S^{G_2}_{G_1}$ we have $S^{G_2}_{G_1}u \geqq u$ in G_1 and hence $\lambda^{*G_2}_{R}S^{G_2}_{G_1}u \geqq \lambda^{*G_1}_{R}u$ in G_1. This implies that $T^{G_2}_{G_1}\lambda^{*G_2}_{R}S^{G_2}_{G_1}u \geqq \lambda^{*G_1}_{R}u$ in G_1. For any $v \in PH(G_2)$ we have $v \geqq T^{G_2}_{G_1}v$ in G_1 and hence $\lambda^{*G_2}_{R}v \geqq \lambda^{*G_1}_{R}T^{G_2}_{G_1}v$ in G_1. Therefore we have $\lambda^{*G_2}_{R}v \geqq S^{G_2}_{G_1}\lambda^{*G_1}_{R}T^{G_2}_{G_1}v$ in G_2.

Here it should be noted that the map $T^{G_2}_{G_1}$: $PH(G_2) \rightarrow PH(G_1)$ is not onto in general and the map $S^{G_2}_{G_1}$: $PH(G_1) \rightarrow PH(G_2)$ is not defined for any element of $PH(G_1)$. This is an important different point between the case PH and the case PHB, which will be treated in Chapter III.

(8) $\lambda^{*G_2}_{R}\lambda^{*G_2}_{G_1}u = \lambda^{*G_2}_{R}u$, $\mu^{*G_2}_{R}\lambda^{*G_2}_{G_1}u = \mu^{*G_2}_{R}\mu^{*G_2}_{G_1}u = \mu^{*G_2}_{R}u$ and $\mu^{*G_2}_{R}u \leqq \lambda^{*G_2}_{R}\mu^{*G_2}_{G_1}u \leqq \lambda^{*G_2}_{R}u$ for any $u \in PH(G_2)$. If G_1 is P-nice in G_2, then $\lambda^{*G_2}_{R}\mu^{*G_2}_{G_1}u = \lambda^{*G_2}_{R}u$.

Proof. $u \geqq \lambda^{*G_2}_{G_1}u \geqq \lambda^{*G_2}_{R}u$ imply $\lambda^{*G_2}_{R}u \geqq \lambda^{*G_2}_{R}\lambda^{*G_2}_{G_1}u \geqq \lambda^{*G_2}_{R}\lambda^{*G_2}_{R}u = \lambda^{*G_2}_{R}u$. $\mu^{*G_2}_{R}u \leqq \mu^{*G_2}_{G_1}u \leqq \lambda^{*G_2}_{G_1}u \leqq u$ imply $\mu^{*G_2}_{R}u = \mu^{*G_2}_{R}\mu^{*G_2}_{R}u \leqq \mu^{*G_2}_{R}\mu^{*G_2}_{G_1}u \leqq \mu^{*G_2}_{R}\lambda^{*G_2}_{G_1}u \leqq \mu^{*G_2}_{R}u$ and $\mu^{*G_2}_{R}u = \mu^{*G_2}_{R}\mu^{*G_2}_{R}u \leqq \lambda^{*G_2}_{R}\mu^{*G_2}_{G_1}u \leqq \lambda^{*G_2}_{R}u$. If G_1 is P-nice in G_2, then $\mu^{*G_2}_{G_1}u = \lambda^{*G_2}_{G_1}u \geqq \lambda^{*G_2}_{R}u$. Hence $\lambda^{*G_2}_{R}u = \lambda^{*G_2}_{R}\lambda^{*G_2}_{R}u \leqq \lambda^{*G_2}_{R}\mu^{*G_2}_{G_1}u \leqq \lambda^{*G_2}_{R}u$.

(9) $\bar{\lambda}^{G_2}_{R}\bar{\lambda}^{G_2}_{G_1}u = \bar{\lambda}^{G_2}_{R}u$, $\bar{\mu}^{G_2}_{R}\bar{\lambda}^{G_2}_{G_1}u = \bar{\mu}^{G_2}_{R}\bar{\mu}^{G_2}_{G_1}u = \bar{\mu}^{G_2}_{R}u$ and $\bar{\mu}^{G_2}_{R}u \leqq \bar{\lambda}^{G_2}_{R}\bar{\mu}^{G_2}_{G_1}u \leqq \bar{\lambda}^{G_2}_{R}u$ for any $u \in PH(G_2)$. If G_1 is most P-nice in G_2, then $\bar{\lambda}^{G_2}_{R}\bar{\mu}^{G_2}_{G_1}u = \bar{\lambda}^{G_2}_{R}u$.

Proof. We can similarly prove this as in (8).

(10) $\lambda^{*G_2}_{R}\bar{\lambda}^{G_2}_{G_1}u = \lambda^{*G_2}_{R}u$, $u^{*G_2}_{R}\bar{\lambda}^{G_2}_{G_1}u = \mu^{*G_2}_{R}u$, $\bar{\mu}^{G_2}_{R}\lambda^{*G_2}_{G_1}u = \bar{\mu}^{G_2}_{R}\mu^{*G_2}_{G_1}u = \bar{\mu}^{G_2}_{R}u$, $\bar{\mu}^{G_2}_{R}u \leqq \mu^{*G_2}_{R}\bar{\mu}^{G_2}_{G_1}u \leqq \mu^{*G_2}_{R}u$, $\lambda^{*G_2}_{R}u \leqq \bar{\lambda}^{G_2}_{R}\lambda^{*G_2}_{G_1}u \leqq \bar{\lambda}^{G_2}_{R}u$, $\bar{\mu}^{G_2}_{R}u \leqq \lambda^{*G_2}_{R}\bar{\mu}^{G_2}_{G_1}u \leqq \lambda^{*G_2}_{R}u$ and $\lambda^{*G_2}_{R}u \leqq \bar{\lambda}^{G_2}_{R}\mu^{*G_2}_{G_1}u \leqq \bar{\lambda}^{G_2}_{R}u$ for any $u \in PH(G_2)$. If G_1 is most P-nice in G_2, then $\mu^{*G_2}_{R}\bar{\mu}^{G_2}_{G_1}u = \mu^{*G_2}_{R}u$, $\bar{\lambda}^{G_2}_{R}\lambda^{*G_2}_{G_1}u = \bar{\lambda}^{G_2}_{R}\mu^{*G_2}_{G_1}u = \bar{\lambda}^{G_2}_{R}u$ and $\lambda^{*G_2}_{R}\bar{\mu}^{G_2}_{G_1}u = \lambda^{*G_2}_{R}u$.

Proof. $\lambda^{*G_2}_{R}u = \lambda^{*G_2}_{R}\lambda^{*G_2}_{G_1}u \leqq \lambda^{*G_2}_{R}\bar{\lambda}^{G_2}_{G_1}u \leqq \lambda^{*G_2}_{R}u$. $\mu^{*G_2}_{R}u = \mu^{*G_2}_{R}\lambda^{*G_2}_{G_1}u \leqq \mu^{*G_2}_{R}\bar{\lambda}^{G_2}_{G_1}u \leqq \mu^{*G_2}_{R}u$. $\bar{\mu}^{G_2}_{R}u = \bar{\mu}^{G_2}_{R}\bar{\mu}^{G_2}_{G_1}u \leqq \bar{\mu}^{G_2}_{R}\mu^{*G_2}_{G_1}u \leqq \bar{\mu}^{G_2}_{R}\lambda^{*G_2}_{G_1}u \leqq \bar{\mu}^{G_2}_{R}\bar{\lambda}^{G_2}_{G_1}u \leqq \bar{\mu}^{G_2}_{R}u$. $\bar{\mu}^{G_2}_{R}u = \bar{\mu}^{G_2}_{R}\bar{\mu}^{G_2}_{G_1}u \leqq \mu^{*G_2}_{R}\bar{\mu}^{G_2}_{G_1}u \leqq \mu^{*G_2}_{R}u$. $\lambda^{*G_2}_{R}u = \lambda^{*G_2}_{R}\lambda^{*G_2}_{G_1}u \leqq \bar{\lambda}^{G_2}_{R}\lambda^{*G_2}_{G_1}u \leqq \bar{\lambda}^{G_2}_{R}u$. $\bar{\mu}^{G_2}_{R}u \leqq \mu^{*G_2}_{R}\bar{\mu}^{G_2}_{G_1}u \leqq \lambda^{*G_2}_{R}\bar{\mu}^{G_2}_{G_1}u \leqq \lambda^{*G_2}_{R}\mu^{*G_2}_{G_1}u \leqq \lambda^{*G_2}_{R}u$. $\lambda^{*G_2}_{R}u \leqq \bar{\lambda}^{G_2}_{R}\lambda^{*G_2}_{G_1}u \leqq \bar{\lambda}^{G_2}_{R}\mu^{*G_2}_{G_1}u \leqq \bar{\lambda}^{G_2}_{R}\lambda^{*G_2}_{G_1}u \leqq \bar{\lambda}^{G_2}_{R}u$.
If G_1 is most P-nice in G_2, then $\bar{\mu}^{G_2}_{G_1} = \mu^{*G_2}_{G_1} = \lambda^{*G_2}_{G_1} = \bar{\lambda}^{G_2}_{G_1}$ and hence $\mu^{*G_2}_{R}\bar{\mu}^{G_2}_{G_1}u = \mu^{*G_2}_{R}\bar{\lambda}^{G_2}_{G_1}u = \mu^{*G_2}_{R}u$, $\bar{\lambda}^{G_2}_{R}\mu^{*G_2}_{G_1}u = \bar{\lambda}^{G_2}_{R}\bar{\mu}^{G_2}_{G_1}u = \bar{\lambda}^{G_2}_{R}\lambda^{*G_2}_{G_1}u = \bar{\lambda}^{G_2}_{R}\bar{\lambda}^{G_2}_{G_1}u = \bar{\lambda}^{G_2}_{R}u$ and $\lambda^{*G_2}_{G_1}\bar{\mu}^{G_2}_{G_1}u = \lambda^{*G_2}_{R}\bar{\lambda}^{G_2}_{G_1}u = \lambda^{*G_2}_{R}u$.

(11) For any $u \in PH(G_2)$ $\lambda^{*G_2}_{G_1}\lambda^{*G_2}_{R}u = \lambda^{*G_2}_{R}u$, $\lambda^{*G_2}_{R_1}\mu^{*G_2}_{R}u = \mu^{*G_2}_{G_1}\mu^{*G_2}_{R}u = \mu^{*G_2}_{R}u$ and $\lambda^{*G_2}_{R}u \geqq \mu^{*G_2}_{G_1}\lambda^{*G_2}_{R}u \geqq \mu^{*G_2}_{R}u$. If G_1 is P-nice in G_2, then $\mu^{*G_2}_{G_1}\lambda^{*G_2}_{R}u = \lambda^{*G_2}_{R}u$ and $\lambda^{*G_2}_{G_2-\bar{G}_1}\lambda^{*G_2}_{R}u = 0$.

Proof. $\lambda^{*G_2}_{R}u=\lambda^{*G_2}_{R}\ \lambda^{*G_2}_{R}u \leqq \lambda^{*G_2}_{G_1}\ \lambda^{*G_2}_{R}u \leqq \lambda^{*G_2}_{R}u$. $\mu^{*G_2}_{R}u=\mu^{*G_2}_{R}\ \mu^{*G_2}_{R}u \leqq \mu^{*G_2}_{G_1}\ \mu^{*G_2}_{R}u \leqq \lambda^{*G_2}_{G_1}\ \mu^{*G_2}_{R}u \leqq \mu^{*G_2}_{R}u$. $\mu^{*G_2}_{R}u \leqq \mu^{*G_2}_{R}\ \lambda^{*G_2}_{R}u \leqq \mu^{*G_2}_{G_1}\ \lambda^{*G_2}_{R}u \leqq \lambda^{*G_2}_{R}u$. If G_1 is P-nice in G_2, then $\lambda^{*G_2}_{G_1}=\mu^{*G_2}_{G_1}$ and hence $\mu^{*G_2}_{G_1}\ \lambda^{*G_2}_{R}u = \lambda^{*G_2}_{G_1}\ \lambda^{*G_2}_{R}u = \lambda^{*G_2}_{R}u$. $0 = \lambda^{*G_2}_{R}u-\mu^{*G_2}_{G_1}\ \lambda^{*G_2}_{R}u = \lambda^{*G_2}_{G_2-\bar{G}_1}\ \lambda^{*G_2}_{R}u$.

(12) $\bar{\lambda}^{G_2}_{G_1}\ \bar{\lambda}^{G_2}_{R}u=\bar{\lambda}^{G_2}_{R}u$, $\bar{\lambda}^{G_2}_{G_1}\ \bar{\mu}^{G_2}_{R}u=\bar{\mu}^{G_2}_{G_1}\ \bar{\mu}^{G_2}_{R}u=\bar{\mu}^{G_2}_{R}u$ and $\bar{\lambda}^{G_2}_{R}u \geqq \bar{\mu}^{G_2}_{G_1}\ \bar{\lambda}^{G_2}_{R}u \geqq \bar{\mu}^{G_2}_{R}u$ for any $u \in PH(G_2)$. If G_1 is most P-nice in G_2, then $\bar{\mu}^{G_2}_{G_1}\ \bar{\lambda}^{G_2}_{R}u=\bar{\lambda}^{G_2}_{R}u$ and $\bar{\lambda}^{G_2}_{G_2-\bar{G}_1}\ \bar{\lambda}^{G_2}_{R}u=0$.

Proof. It is quite similarly proved as in (11).

(13) $\bar{\lambda}^{G_2}_{G_1}\ u^{*G_2}_{R}u = \mu^{*G_2}_{R}u$, $\bar{\lambda}^{G_2}_{G_1}\ \lambda^{*G_2}_{R}u = \lambda^{*G_2}_{R}u$, $\lambda^{*G_2}_{G_1}\ \bar{\mu}^{G_2}_{R}u = \mu^{*G_2}_{G_1}\ \bar{\mu}^{G_2}_{R}u = \bar{\mu}^{G_2}_{R}u$, $\lambda^{*G_2}_{R}u \leqq \lambda^{*G_2}_{G_1}\ \bar{\lambda}^{G_2}_{R}u \leqq \bar{\lambda}^{G_2}_{R}u$, $\mu^{*G_2}_{R}u \leqq \mu^{*G_2}_{G_1}\ \bar{\lambda}^{G_2}_{R}u \leqq \bar{\lambda}^{G_2}_{R}u$ and $\bar{\mu}^{G_2}_{R}u \leqq \bar{\mu}^{G_2}_{G_1}\ \mu^{*G_2}_{R}u \leqq \bar{\mu}^{G_2}_{G_1}\ \lambda^{*G_2}_{R}u \leqq \lambda^{*G_2}_{R}u$ for any $u \in PH(G_2)$. If G_1 is most P-nice in G_2, then $\bar{\mu}^{G_2}_{G_1}\ \lambda^{*G_2}_{R}u = \bar{\lambda}^{G_2}_{G_1}\ \lambda^{*G_2}_{R}u = \lambda^{*G_2}_{R}u$, $\bar{\mu}^{G_2}_{G_1}\ \mu^{*G_2}_{R}u = \bar{\lambda}^{G_2}_{G_1}\ \mu^{*G_2}_{R}u = \mu^{*G_2}_{R}u$ and $\bar{\mu}^{G_2}_{G_1}\ \bar{\lambda}^{G_2}_{R}u = \mu^{*G_2}_{G_1}\ \bar{\lambda}^{G_2}_{R}u = \lambda^{*G_2}_{G_1}\ \bar{\lambda}^{G_2}_{R}u = \bar{\lambda}^{G_2}_{G_1}\ \bar{\lambda}^{G_2}_{R}u = \bar{\lambda}^{G_2}_{R}u$.

Proof. $\lambda^{*G_2}_{R}u=\lambda^{*G_2}_{G_1}\ \lambda^{*G_2}_{R}u \leqq \bar{\lambda}^{G_2}_{G_1}\ \lambda^{*G_2}_{R}u \leqq \lambda^{*G_2}_{R}u$. $\bar{\mu}^{G_2}_{R}u=\bar{\mu}^{G_2}_{G_1}\ \bar{\mu}^{G_2}_{R}u \leqq \mu^{*G_2}_{G_1}\ \bar{\mu}^{G_2}_{R}u \leqq \lambda^{*G_2}_{G_1}\ \bar{\mu}^{G_2}_{R}u \leqq \bar{\mu}^{G_2}_{R}u$. $\mu^{*G_2}_{R}u=\lambda^{*G_2}_{G_1}\ \mu^{*G_2}_{R}u \leqq \bar{\lambda}^{G_2}_{G_1}\ \mu^{*G_2}_{R}u \leqq \mu^{*G_2}_{R}u$. $\lambda^{*G_2}_{R}u=\lambda^{*G_2}_{G_1}\ \lambda^{*G_2}_{R}u \leqq \lambda^{*G_2}_{G_1}\ \bar{\lambda}^{G_2}_{R}u \leqq \bar{\lambda}^{G_2}_{R}u$. $\mu^{*G_2}_{R}u \leqq \mu^{*G_2}_{G_1}\ \lambda^{*G_2}_{R}u \leqq \mu^{*G_2}_{G_1}\ \bar{\lambda}^{G_2}_{R}u \leqq \lambda^{*G_2}_{G_1}\ \bar{\lambda}^{G_2}_{R}u=\bar{\lambda}^{G_2}_{R}u$, $\bar{\mu}^{G_2}_{R}u=\bar{\mu}^{G_2}_{G_1}\ \bar{\mu}^{G_2}_{R}u \leqq \bar{\mu}^{G_2}_{G_1}\ \mu^{*G_2}_{R}u \leqq \bar{\mu}^{G_2}_{G_1}\ \lambda^{*G_2}_{R}u \leqq \lambda^{*G_2}_{R}u$.

(14) If G_1 is P-nice in G_2, then

$$\lambda^{*G_2}_{R}u = S^{G_2}_{G_1}\ \lambda^{*G_1}_{R}\ T^{G_2}_{G_1}u \text{ and } \lambda^{*G_1}_{R}v = T^{G_2}_{G_1}\ \lambda^{*G_2}_{R}\ S^{G_2}_{G_1}v$$

for any $u \in PH(G_2)$ and for any $v \in PH(G_1)$ satisfying $S^{G_2}_{G_1}v < \infty$.

Proof. By (7) we have for any $u \in PH(G_2)$

$$\lambda^{*G_2}_{R}u \geqq S^{G_2}_{G_1}\ \lambda^{*G_1}_{R}\ T^{G_2}_{G_1}u\,.$$

On the other hand by (8) and (11) we have

$$\begin{aligned}\lambda^{*G_2}_{R}u &= \lambda^{*G_2}_{G_1}\ \lambda^{*G_2}_{R}\ \lambda^{*G_2}_{G_1}u = \mu^{*G_2}_{G_1}\ \lambda^{*G_2}_{R}\ \mu^{*G_2}_{G_1}u = S^{G_2}_{G_1}\ T^{G_2}_{G_1}\ \lambda^{*G_2}_{R}\ S^{G_2}_{G_1}\ T^{G_2}_{G_1}u \\ &= S^{G_2}_{G_1}\ T^{G_2}_{G_1}\ S^{G_2}_{G_1}\ T^{G_2}_{G_1}u - S^{G_2}_{G_1}\ T^{G_2}_{G_1}\ \mu^{*G_2}_{G_2-\bar{R}}\ S^{G_2}_{G_1}\ T^{G_2}_{G_1}u \\ &= S^{G_2}_{G_1}\ T^{G_2}_{G_1}u - S^{G_2}_{G_1}\ T^{G_2}_{G_1}\ u^{*G_2}_{G_2-\bar{R}}\ S^{G_2}_{G_1}\ T^{G_2}_{G_1}u\,,\end{aligned}$$

since $T^{G_2}_{G_1}S^{G_2}_{G_1}T^{G_2}_{G_1}u = T^{G_2}_{G_1}u$ if $T^{G_2}_{G_1}u > 0$. Further since $\mu^{*G_2}_{G_2-\bar{R}}x \geqq \mu^{*G_2}_{G_1-\bar{R}}x$ for any $x \in PH(G_2)$, we have

$$\begin{aligned}S^{G_2}_{G_1}\ T^{G_2}_{G_1}\ \mu^{*G_2}_{G_2-\bar{R}}\ S^{G_2}_{G_1}\ T^{G_2}_{G_1}u &\geqq S^{G_2}_{G_1}\ T^{G_2}_{G_1}\ \mu^{*G_2}_{G_1-\bar{R}}\ S^{G_2}_{G_1}\ T^{G_2}_{G_1}u \\ &= S^{G_2}_{G_1}\ \mu^{*G_1}_{G_1-\bar{R}}\ T^{G_2}_{G_1}u\end{aligned}$$

by (7). Thus we have an inequality

$$\lambda^{*G_2}_{R}u \leqq S^{G_2}_{G_1}\ T^{G_2}_{G_1}u - S^{G_2}_{G_1}\ \mu^{*G_1}_{G_1-\bar{R}}\ T^{G_2}_{G_1}u = S^{G_2}_{G_1}\ \lambda^{*G_1}_{R}\ T^{G_2}_{G_1}u\,,$$

which leads to the desired result if $T^{G_2}_{G_1}u > 0$, $u \in PH(G_2)$. If $T^{G_2}_{G_1}u = 0$, then

$$\lambda^{*G_2}_{R}u = S^{G_2}_{G_1}\ T^{G_2}_{G_1}\ \lambda^{*G_2}_{R}\ S^{G_2}_{G_1}\ T^{G_2}_{G_1}u = 0 = S^{G_2}_{G_1}\ \lambda^{*G_1}_{R}\ T^{G_2}_{G_1}u\,.$$

If $S^{G_2}_{G_1}v < \infty$, $v \in PH(G_1)$, then we put $u = S^{G_2}_{G_1}v$. Then we have $T^{G_2}_{G_1}u = T^{G_2}_{G_1}S^{G_2}_{G_1}v = v > 0$. By the first part we have

$$\lambda^{*G_2}_{R} S^{G_2}_{G_1} v = S^{G_2}_{G_1} \lambda^{*G_1}_{R} v .$$

This is the desired result.

(15) Let G_1 and G_2 $(\supset G_1)$ be two subregions of W containing R, then for each $u \in PH(G_1)$ satisfying $S^{G_2}_{G_1} u \in PH(G_2)$, there hold

$$T^{G_2}_{G_1} \bar{\mu}^{G_2}_{R} S^{G_2}_{G_1} u = \bar{\mu}^{G_1}_{R} u \text{ and } T^{G_2}_{G_1} \bar{\lambda}^{G_2}_{R} S^{G_2}_{G_1} u \geqq \bar{\lambda}^{G_1}_{R} u .$$

For each $v \in PH(G_2)$, there hold

$$\bar{\mu}^{G_2}_{R} v = S^{G_2}_{G_1} \bar{\mu}^{G_1}_{R} T^{G_2}_{G_1} v \text{ and } \bar{\lambda}^{G_2}_{R} v \geqq S^{G_2}_{G_1} \bar{\lambda}^{G_1}_{R} T^{G_2}_{G_1} v .$$

If G_1 is P-nice in G_2 and $\bar{\lambda}^{G_2}_{G_1} = \lambda^{*G_2}_{G_1}$, then the above two inequalities reduce to the equalities respectively under the same assumption.

Proof. It is sufficient to prove the parts claiming the equalities.

$$\begin{aligned}
\bar{\lambda}^{G_2}_{R} v &= \bar{\lambda}^{G_2}_{G_1} \bar{\lambda}^{G_2}_{R} \bar{\lambda}^{G_2}_{G_1} v = \lambda^{*G_2}_{G_1} \bar{\lambda}^{G_2}_{R} \lambda^{*G_2}_{G_1} v = \mu^{*G_2}_{G_1} \bar{\lambda}^{G_2}_{R} \mu^{*G_2}_{G_1} v \\
&= S^{G_2}_{G_1} T^{G_2}_{G_1} \bar{\lambda}^{G_2}_{R} S^{G_2}_{G_1} T^{G_2}_{G_1} v = S^{G_2}_{G_1} T^{G_2}_{G_1} S^{G_2}_{G_1} T^{G_2}_{G_1} v - S^{G_2}_{G_1} T^{G_2}_{G_1} \bar{\mu}^{G_2}_{G_2-\bar{R}} S^{G_2}_{G_1} T^{G_2}_{G_1} v \\
&= S^{G_2}_{G_1} T^{G_2}_{G_1} v - S^{G_2}_{G_1} T^{G_2}_{G_1} \bar{\mu}^{G_2}_{G_2-\bar{R}} S^{G_2}_{G_1} T^{G_2}_{G_1} v \\
&\leqq S^{G_2}_{G_1} T^{G_2}_{G_1} v - S^{G_2}_{G_1} T^{G_2}_{G_1} \bar{\mu}^{G_2}_{G_1-\bar{R}} S^{G_2}_{G_1} T^{G_2}_{G_1} v \leqq S^{G_2}_{G_1} T^{G_2}_{G_1} v - S^{G_2}_{G_1} \bar{\mu}^{G_1}_{G_1-\bar{R}} T^{G_2}_{G_1} v \\
&= S^{G_2}_{G_1} \bar{\lambda}^{G_1}_{R} T^{G_2}_{G_1} v \leqq \bar{\lambda}^{G_2}_{R} v .
\end{aligned}$$

That is, $\bar{\lambda}^{G_2}_{R} v = S^{G_2}_{G_1} \bar{\lambda}^{G_1}_{R} T^{G_2}_{G_1} v$ holds for any $v \in PH(G_2)$. By the assumption $S^{G_2}_{G_1} u < \infty$, we can transform this identity into the identity

$$T^{G_2}_{G_1} \bar{\lambda}^{G_2}_{R} S^{G_2}_{G_1} u = \bar{\lambda}^{G_1}_{R} u$$

for any $u \in PH(G_1)$ with $S^{G_2}_{G_1} u < \infty$, putting $u = T^{G_2}_{G_1} v$. Because for this we have $S^{G_2}_{G_1} u = S^{G_2}_{G_1} T^{G_2}_{G_1} v = v < \infty$.

Next we shall prove the first equality.

$$\begin{aligned}
T^{G_2}_{G_1} \bar{\mu}^{G_2}_{R} S^{G_2}_{G_1} u &= T^{G_2}_{G_1} S^{G_2}_{G_1} u - T^{G_2}_{G_1} \bar{\lambda}^{G_2}_{G_2-\bar{R}} S^{G_2}_{G_1} u \\
&\leqq T^{G_2}_{G_1} S^{G_2}_{G_1} u - T^{G_2}_{G_1} \bar{\lambda}^{G_2}_{G_2-\bar{R}} S^{G_2}_{G_1} u .
\end{aligned}$$

by $\bar{\lambda}^{G_2}_{G_2-\bar{R}} x \geqq \bar{\lambda}^{G_2}_{G_1-\bar{R}} x$ for any $x \in PH(G_2)$. Since $S^{G_2}_{G_1} u < \infty$ implies $T^{G_2}_{G_1} S^{G_2}_{G_1} u = u$, we can say that

$$T^{G_2}_{G_1} S^{G_2}_{G_1} u - T^{G_2}_{G_1} \bar{\lambda}^{G_2}_{G_2-\bar{R}} S^{G_2}_{G_1} u \leqq u - \bar{\lambda}^{G_1}_{G_1-\bar{R}} u = \bar{\mu}^{G_1}_{R} u ,$$

which shows that $T^{G_2}_{G_1} \bar{\mu}^{G_2}_{R} S^{G_2}_{G_1} u \leqq \bar{\mu}^{G_1}_{R} u$ for any $u \in PH(G_1)$ with $S^{G_2}_{G_1} u < \infty$. Since the inverse inequality is evidently true, we have the desired equality. The remaining identity is a simple transform of this identity.

§5. **Applications.** We shall now apply the above algebraic calculations to investigate the relative situation of R in G.

THEOREM 5. *If R is interior P-nice in G_1 and if $G_1 \subset G_2$, then R is interior P-nice in G_2. If R is interior P-nice in G_2 and G_1 is onto P-nice in G_2, then R is interior P-nice in G_1.*

Proof. By the assumption $\bar{\mu}_R^{G_1} u = \mu^{*G_1}_R u$ holds for any $u \in PH(G_1)$. By (7) and (15) we have

$$\bar{\mu}_R^{G_2} v = S_{G_1}^{G_2} \bar{\mu}_R^{G_1} T_{G_1}^{G_2} v, \qquad \mu^{*G_2}_R v = S^{G_2}_{G_1} \mu^{*G_1}_R T^{G_2}_{G_1} v$$

for any $v \in PH(G_2)$ satisfying $T^{G_2}_{G_1} v > 0$. Then for such a v we have

$$\mu^{G_2}_G v = \mu^{*G_2}_R v .$$

If $T^{G_2}_{G_1} v = 0$, then $u^{*G_2}_R v = S^{G_2}_R T^{G_2}_R v = S^{G_2}_R T^{G_1}_R T^{G_2}_{G_1} v = 0$, which shows that $\mu^{*G_2}_R v = \bar{\mu}^{G_2}_R v = 0$. Therefore in any cases we have the desired result: $\bar{\mu}^{G_2}_R v = \mu^{*G_2}_R v$ for any $v \in PH(G_2)$. By the assumption of the later half $\bar{\mu}^{G_2}_R u = \mu^{*G_2}_R u$ and $S^{G_2}_{G_1} u < \infty$ for any $u \in PH(G_1)$. By (7) and (15) we have

$$\bar{\mu}^{G_1}_R u = T^{G_2}_{G_1} \bar{\mu}^{G_2}_R S^{G_2}_{G_1} u = T^{G_2}_{G_1} \mu^{*G_2}_R S^{G_2}_{G_1} u = \mu^{*G_1}_R u$$

for any $u \in PH(G_1)$. This is the desired result.

As is shown by the following figure, the inverse of the first part does not remain true if no assumption is imposed on G_1. In a neighborhood of p a component of ∂R coincides with ∂G_1 and another component enters into p touching very strongly to Δ_1^G and lying in G_1. Then we have

$$\Delta_1^{G_1} = [p, q], \quad \Delta_{1, G_1-\bar{R}}^{G_1} = [p, q], \quad \Delta_1^{G_1}(R) = (p)$$

and $\Delta_1^{G_1} - \Delta_{1, G_1-\bar{R}}^{G_1} = \phi$. This shows that $\bar{\mu}^{G_1}_R K(r, p) = 0$ and $\mu^{*G_1}_R K(r, p) = K(r, p)$. Thus R is not interior P-nice in G_1, however R is interior P-nice in G_2.

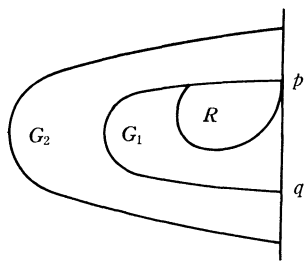


Fig. 10

THEOREM 6. *Let R be exterior P-nice in G_1 and G_1 be most P-nice in G_2, then R is exterior P-nice in G_2. Let R be exterior P-nice in G_2 and G_1 be P-nice and onto P-nice in G_2, then R is exterior P-nice in G_1.*

Proof. Let u be any element of $PH(G_2)$, then

$$\bar{\lambda}^{G_2}_R u = S^{G_2}_{G_1} \bar{\lambda}^{G_1}_R T^{G_2}_{G_1} u = S^{G_2}_{G_1} \lambda^{*G_1}_R T^{G_2}_{G_1} u = \lambda^{*G_2}_R u$$

if $T^{G_2}_{G_1} u > 0$. If $T^{G_2}_{G_1} u = 0$, then

$$\bar{\lambda}^{G_2}_R u = \bar{\lambda}^{G_1}_R \bar{\lambda}^{G_2}_{G_1} u = \bar{\lambda}^{G_1}_R \mu^{*G_2}_{G_1} u = \bar{\lambda}^{G_1}_R S^{G_2}_{G_1} T^{G_2}_{G_1} u = 0,$$

which implies that $\bar{\lambda}^{G_2}_R u = \lambda^{*G_2}_R u = 0$.

By the onto P-niceness of G_1 in G_2 there holds $S^{G_2}_{G_1} u < \infty$ for any $u \in PH(G_1)$, thus we see

$$\bar{\lambda}^{G_1}_R u \leqq T^{G_2}_{G_1} \bar{\lambda}^{G_2}_R S^{G_2}_{G_1} u = T^{G_2}_{G_1} \lambda^{*G_2}_R S^{G_2}_{G_1} u = \lambda^{*G_1}_R u$$

for any $u \in PH(G_1)$. Therefore we have $\bar{\lambda}^{G_1}_R u = \lambda^{*G_1}_R u$ for any $u \in PH(G_1)$.

THEOREM 7. *If R is P-nice in G_1 and G_1 be P-nice in G_2, then R is P-nice in*

G_2. *If R is P-nice in G_2 and G_1 is onto P-nice in G_2 and $G_1 \supset R$, then R is P-nice in G_1.*

Proof. Let u be any element in $PH(G_2)$ satisfying $T_{G_1}^{G_2} u > 0$. Then we have

$$\lambda^{*G_2}_{R} u = S_{G_1}^{G_2} \lambda^{*G_1}_{R} T_{G_1}^{G_2} u = S_{G_1}^{G_2} \mu^{*G_1}_{R} T_{G_1}^{G_2} u = u^{*G_2}_{R} u.$$

If $T_{G_1}^{G_2} u = 0$, then there holds

$$\lambda^{*G_2}_{R} u = \lambda_{R}^{G_2} \lambda^{*G_2}_{G_1} u = \lambda^{*G_2}_{R} \mu^{*G_2}_{G_1} u = \lambda^{*G_2}_{R} S_{G_1}^{G_2} T_{G_1}^{G_2} u = 0.$$

Further we have

$$\mu^{*G_2}_{R} u = S_{R}^{G_2} T_{R}^{G_2} u = S_{R}^{G_2} T_{R}^{G_1} T_{G_1}^{G_2} u = 0.$$

This implies the first part of the desired fact. In this part we cannot replace the P-niceness of G_1 in G_2 by the onto P-niceness of G_1 in G_2. This is shown by an example which is easy to construct.

By (7) we have

$$\mu^{*G_1}_{R} u = T_{G_1}^{G_2} \mu^{*G_2}_{R} S_{G_1}^{G_2} u \quad \text{and} \quad \lambda^{*G_1}_{R} u \leqq T_{G_1}^{G_2} \lambda^{*G_2}_{R} S_{G_1}^{G_2} u$$

for any $u \in PH(G_1)$, since $S_{G_1}^{G_2} u < \infty$ by the onto P-niceness of G_1 in G_2. Since R is P-nice in G_2, we have $\mu^{*G_2}_{R} = \lambda^{*G_2}_{R}$. Thus we have the desired fact:

$$\mu^{*G_1}_{R} u = \lambda^{*G_1}_{R} u.$$

THEOREM 8. *If R is most P-nice in G_2 and G_1 is onto P-nice in G_2, $G_1 \supset R$, then R is most P-nice in G_1. If R is most P-nice in G_1 and G_1 is most P-nice in G_2, then R is most P-nice in G_2.*

Proof. By the assumption we have

$$\bar{\mu}_{R}^{G_1} u = T_{G_1}^{G_2} \bar{\mu}_{R}^{G_2} S_{G_1}^{G_2} u = T_{G_1}^{G_2} \bar{\lambda}_{R}^{G_2} S_{G_1}^{G_2} u \geqq \bar{\lambda}_{R}^{G_1} u$$

if $S_{G_1}^{G_2} u < \infty$. On the other hand we have $S_{G_1}^{G_2} u < \infty$ for any $u \in PH(G_1)$ if G_1 is onto P-nice in G_2. Thus we have the desired fact $\bar{\mu}_{R}^{G_1} u = \bar{\lambda}_{R}^{G_1} u$.

By (15) we have

$$\bar{\mu}_{R}^{G_2} u = S_{G_1}^{G_2} \bar{\mu}_{R}^{G_1} T_{G_1}^{G_2} u = S_{R_1}^{G_2} \bar{\lambda}_{R}^{G_1} T_{G_1}^{G_2} u = \bar{\lambda}_{R}^{G_2} u$$

for any $u \in PH(G_2)$, $T_{G_1}^{G_2} u > 0$. If $T_{G_1}^{G_2} u = 0$, then

$$\bar{\lambda}_{R}^{G_2} u = \bar{\lambda}_{R}^{G_2} \bar{\lambda}_{R}^{G_2} u = \bar{\lambda}_{R}^{G_2} \mu^{*G_2}_{G_1} u = 0$$

and

$$\bar{\mu}_{R}^{G_2} u \leqq \mu^{*G_2}_{R} u = 0.$$

This implies the desired fact.

The later half of the above theorem can be extended in the following manner: *If R is most P-nice in G_1 and G_1 is exterior P-nice and P-nice in G_2, then R is most P-nice in G_2.*

If two open sets R_1 and R_2 with two-sidedness condition satisfy the conditions $\bar{\lambda}^G_{R_1} = \bar{\lambda}^G_{R_2}$ and $\bar{\mu}^G_{R_1} = \bar{\mu}^G_{R_2}$, then we say R_1 and R_2 are *1st P-equivalent* in G. If R_1 and R_2 satisfy the condition $\bar{\lambda}^G_{R_1} = \lambda^{*G}_{R_2}$ and $\mu^{*G}_{R_1} = \mu^{*G}_{R_2}$, then R_1 and R_2 are said to be *2nd P-equivalent* in G. If R_1 and R_2 are 1st and 2nd P-equivalent in G simultaneously, then they are said to be *most P-equivalent* in G. If $\bar{\lambda}^G_{R_1} = \bar{\lambda}^G_{R_2}$ and $\lambda^{*G}_{R_1} = \lambda^{*G}_{R_2}$, then they are said to be *exterior P-equivalent* in G and if $\mu^{*G}_{R_1} = \mu^{*G}_{R_2}$ and $\bar{\mu}^G_{R_1} = \bar{\mu}^G_{R_2}$, then they are said to be *interior P-equivalent* in G. These are also relative notions.

THEOREM 9. *If R_1 and R_2 are interior P-equivalent in G_1 and $G_2 \supset G_1$, then they are also interior P-equivalent in G_2. If R_1 and R_2 are interior P-equivalent in G_2 and G_1 is onto P-nice in G_2, then they are interior P-equivalent in G_1.*

Proof. For any $u \in PH(G_2)$ with $T^{G_2}_{G_1}u < 0$ we have

$$u^{*G_2}_{R_1}u = S^{G_2}_{G_1}\mu^{*G_1}_{R_1}T^{G_2}_{G_1}u = S^{G_2}_{G_1}\mu^{*G_1}_{R_2}T^{G_2}_{G_1}u = \mu^{*G_2}_{R_2}u$$

and

$$\bar{\mu}^{G_2}_{R_1}u = S^{G_2}_{G_1}\bar{\mu}^{G_1}_{R_1}T^{G_2}_{G_1}u = S^{G_2}_{G_1}\bar{\mu}^{G_1}_{R_2}T^{G_2}_{G_1}u = \bar{\mu}^{G_2}_{R_2}u.$$

This shows the first desired result for $T^{G_2}_{G_1}u > 0$. If $T^{G_2}_{G_1}u = 0$, then $\mu^{*G_2}_{R_1}u = \mu^{*G_2}_{R_1}\mu^{*G_2}_{G_1}u = 0$ and $\mu^{*G_2}_{R_2}u = \mu^{*G_2}_{R_2}\mu^{*G_2}_{G_1}u = 0$. This implies $\bar{\mu}^{G_2}_{R_1}u = \bar{\mu}^{G_2}_{R_2}u = 0$ for $T^{G_2}_{G_1}u = 0$.

By the onto P-niceness of G_1 in G_2 any $u \in PH(G_1)$ satisfies $S^{G_2}_{G_1}u < \infty$. Then we have

$$\bar{\mu}^{G_1}_{R_1}u = T^{G_2}_{G_1}\bar{\mu}^{G_2}_{R_1}S^{G_2}_{G_1}u = T^{G_2}_{G_1}\bar{\mu}^{G_2}_{R_2}S^{G_2}_{G_1}u = \bar{\mu}^{G_2}_{R_2}u$$

and

$$\mu^{*G_1}_{R_1}u = T^{G_2}_{G_1}\mu^{*G_2}_{R_1}S^{G_2}_{G_1}u = T^{G_2}_{G_1}\mu^{*G_2}_{R_2}S^{G_2}_{G_1}u = \mu^{*G_2}_{R_2}u$$

for any $u \in PH(G_1)$.

THEOREM 10. *If R_1 and R_2 are exterior P-equivalent in G_1 and G_1 is exterior P-nice and P-nice in G_2, then R_1 and R_2 are exterior P-equivalent in G_2. If R_1 and R_2 are exterior P-equivalent in G_2 and G_1 is exterior P-nice and P-nice and further onto P-nice in G_2, then R_1 and R_2 are exterior P-equivalent in G_1.*

THEOREM 11. *If R_1 and R_2 are second P-equivalent in G_1 and G_1 is P-nice in G_2, then they are second P-equivalent in G_2. If R_1 and R_2 are second P-equivalent in G_2 and G_1 is onto P-nice and P-nice in G_2 and $G_1 \supset R_1 \cup R_2$, then R_1 and R_2 are second P-equivalent in G_1. If R_1 and R_2 are first P-equivalent in G_1 and G_1 is* *exterior P-nice or more strongly most P-nice in G_2, then they are first P-equivalent in G_2. If R_1 and R_2 are first P-equivalent in G_2 and G_1 is onto P-nice, exterior P-nice in G_2, then they are first P-equivalent in G_1. If R_1 and R_2 are most P-equivalent in G_1 and G_1 is exterior P-nice and P-nice in G_2, then they are most P-equivalent in G_2. If R_1 and R_2 are most P-equivalent in G_2 and G_1 is onto P-nice, exterior P-nice and P-nice in G_2, then they are most P-equivalent in G_1.*

Proof of theorem 10. By the P-niceness of G_1 in G_2 we can apply (15) and we have $\bar{\lambda}^{G_2}_{R_1}u = S^{G_2}_{G_1}\bar{\lambda}^{G_1}_{R_1}T^{G_2}_{G_1}u = S^{G_2}_{G_1}\bar{\lambda}^{G_1}_{R_2}T^{G_2}_{G_1}u = \bar{\lambda}^{G_2}_{R_2}u$ and $\lambda^{*G_2}_{R_1}u = S^{G_2}_{G_1}\lambda^{*G_1}_{R_1}T^{G_2}_{G_1}u = S^{G_2}_{G_1}\lambda^{*G_1}_{R_2}T^{G_2}_{G_1}u = \lambda^{*G_2}_{R_2}u$ for any $u \in PH(G_2)$ with $T^{G_2}_{G_1}u > 0$. If $T^{G_2}_{G_1}u = 0$, then $\bar{\lambda}^{G_2}_{R_1}u = \bar{\lambda}^{G_2}_{R_1}\bar{\lambda}^{G_2}_{G_1}u = \bar{\lambda}^{G_2}_{G_1}\mu^{*G_2}_{G_1}u = 0 = \bar{\lambda}^{G_2}_{R_2}\mu^{*G_2}_{G_1}u = \bar{\lambda}^{G_2}_{R_2}\bar{\lambda}^{G_2}_{G_1}u = \bar{\lambda}^{G_2}_{R_2}u$. This implies $\lambda^{*G_2}_{R_1}u = 0 = \lambda^{*G_2}_{R_2}u$. Thus we have the desired fact. The remaining half is quite similarly proved.

Proof of theorem 11 is quite similar as in theorem 9 and 10.

THEOREM 12. *Let R_1 and R_2 be two open sets in G with two-sidedness condition, then*

$$\mu^{*G}_{R_1\cup R_2}u + \mu^{*G}_{R_1\cap R_2}u \geqq \mu^{*G}_{R_1}u + \mu^{*G}_{R_2}u$$

and

$$\lambda^{*G}_{R_1\cup R_2}u + \lambda^{*G}_{R_1\cap R_2}u \leqq \lambda^{*G}_{R_1}u + \lambda^{*G}_{R_2}u$$

for any $u \in PH(G)$.

Proof. We construct a positive harmonic function $u_n(X)$ in $W_n \frown X$ so that $u_n(X) = u$ on $X \frown \partial W_n$, $= 0$ on $\partial X \frown W_n$. Then we have $u_n(R_1 \smile R_2) + u_n(R_1 \frown R_2) = u_n(R_1 \smile R_2)$ on $\partial R_2 \frown R_1 \frown W_n + \partial R_1 \frown R_2 \frown W_n$, $=2u$ on $\partial W_n \frown R_1 \frown R_2$ and $u_n(R_1) + u_n(R_2) = u_n(R_1)$ on $\partial R_2 \frown R_1 \frown W_n$, $= u_n(R_2)$ on $\partial R_1 \frown R_2 \frown W_n$, $= 2u$ on $\partial W_n \frown R_1 \frown R_2$, and further $u_n(R_1 \smile R_2) \geqq u_n(R_1)$ on $\partial R_2 \frown R_1 \frown W_n$ and $u_n(R_1 \frown R_2) \geqq u_n(R_2)$ on $\partial R_1 \frown R_2 \frown W_n$. These imply an inequality

$$u_n(R_1 \smile R_2) + u_n(R_1 \frown R_2) \geqq u_n(R_1) + u_n(R_2)$$

on $R_1 \frown R_2 \frown W_n$. Let n tend to ∞, then there holds

$$T^G_{R_1\cup R_2}u + T^G_{R_1\cap R_2}u \geqq T^G_{R_1}u + T^G_{R_2}u$$

in $R_1 \frown R_2$. By the standard subharmonic extension of the respective members in the above inequality we have the same inequality which holds in the whole G. In fact the left hand side member is equal to 0 on $G - \overline{R_1 \smile R_2}$ and $T^G_{R_1\cup R_2}u$ on $(R_1 - \overline{R_1 \frown R_2}) \smile (R_2 - \overline{R_1 \frown R_2})$ and the right hand side member is equal to 0 on $G - \overline{R_1 \smile R_2}$ and $T^G_{R_1}u$ on $R_1 - \overline{R_1 \frown R_2}$, $T^G_{R_2}u$ on $R_2 - \overline{R_1 \frown R_2}$. Thus we have the desired inequality. We now construct two least harmonic majorants of both sides and then we have

$$S^G_{R_1\cup R_2}T^G_{R_1\cup R_2}u + S^G_{R_1\cap R_2}T^G_{R_1\cap R_2}u \geqq S^G_{R_1}T^G_{R_1}u + S^G_{R_2}T^G_{R_2}u$$

in G. This is the first desired result.

Since $(G-\bar{R}_1)\frown(G-\bar{R}_2) = G-\overline{R_1\smile R_2}$ and $(G-\bar{R}_1)\smile(G-\bar{R}_2)\subset G-\overline{R_1\frown R_2}$, we have

$$\lambda^{*G}_{R_1\cup R_2}u + \lambda^{*G}_{R_1\cap R_2}u = 2u - \mu^{*G}_{G-\overline{R_1\cup R_2}}\,u - \mu^{*G}_{G-\overline{R_1\cap R_2}}u$$
$$\leqq 2u - \mu^{*G}_{(G-\bar{R}_1)\cap(G-\bar{R}_2)}u - \mu^{*G}_{(G-\bar{R}_1)\cup(G-\bar{R}_2)}u$$
$$\leqq 2u - \mu^{*G}_{G-\bar{R}_1}u - \mu^{*G}_{G-\bar{R}_1}u = \lambda^{*G}_{R_1}u + \lambda^{*G}_{R_2}u.$$

This is the second desired result.

Let u be a minimal function in $PH(G)$ for which $\mu^{*G}_{R_1}u = \mu^{*G}_{R_2}u = u$, then $\mu^{*G}_{R_1\cup R_2}u = u$. This implies that $T^G_{R_1\cap R_2}u > 0$ and $\Delta^G_1(R_1)\frown\Delta^G_1(R_2) = \Delta^G_1(R_1\frown R_2)$. Further we have

$$\mu^{*G}_{R_1\cup R_2}u = \mu^{*G}_{R_1}u + \mu^{*G}_{R_2}u$$

for any $u \in PH(G)$ if $R_1\frown R_2 = \phi$. In fact we have

$$T^G_{R_1+R_2}u = T^G_{R_1}u + T^G_{R_2}u$$

for any $u \in PH(G)$ if $R_1\frown R_2 = \phi$. This implies the desired fact.

If $(G - \bar{R}_1)\frown(G - \bar{R}_2) = \phi$, then we have

$$\lambda^{*G}_{R_1\cup R_2}u + \lambda^{*G}_{R_1\cap R_2}u = \lambda^{*G}_{R_1}u + \lambda^{*G}_{R_2}u$$

for any $u \in PH(G)$.

THEOREM 13. *For any* $u\in PH(G)$

$$\bar{\lambda}^G_{R_1\cup R_2}u + \bar{\lambda}^G_{R_1\cap R_2}u \leqq \bar{\lambda}^G_{R_1}u + \bar{\lambda}^G_{R_2}u,$$

and

$$\bar{\mu}^G_{R_1\cup R_2}u + \bar{\mu}^G_{R_1\cap R_2}u \geqq \bar{\mu}^G_{R_1}u + \bar{\mu}^G_{R_2}u.$$

Proof. In general we can say that $\Delta^G_{1,R_1}\smile\Delta^G_{1,R_2} = \Delta^G_{1,R_1\cup R_2}$ and $\Delta^G_{1,R_1\cap}\Delta^G_{1,R_2}\supset\Delta^G_{1,R_1\cap R_2}$. Therefore we have

$$\bar{\lambda}^G_{R_1\cup R_2}u + \bar{\lambda}^G_{R_1\cap R_2}u = \int_{\Delta^G_{1R_1\cup R_2}} K_G(p, q)d\sigma_u(q) + \int_{\Delta^G_{1,R_1\cap R_2}} K_G(p, q)d\sigma_u(q)$$
$$\leqq \int_{\Delta^G_{1,R_1}\cup\,\Delta^G_{1,R_2}} K_G(p, q)d\sigma_u(q) + \int_{\Delta^G_{1,R_1}\cap\,\Delta^G_{1,R_2}} K_G(p, q)d\sigma_u(q)$$
$$= \int_{\Delta^G_{1,R_1}} K_G(p, q)d\sigma_u(q) + \int_{\Delta^G_{1,R_2}} K_G(p, q)d\sigma_u(q)$$
$$= \bar{\lambda}^G_{R_1}u + \bar{\lambda}^G_{R_2}u,$$

if we put

$$u(p) = \int_{\Delta^G_1} K_G(p, q)d\sigma_u(q).$$

By the integral representation of $\bar{\mu}^G_X$ it is the matter of the set of Martin minimal points which do not belong to $\Delta_{1,G-\bar{X}}^G$. If p belongs to $(\Delta_1^G - \Delta_{1,G-\bar{R}_1}^G)\frown(\Delta_1^G - \Delta_{1,G-\bar{R}_2}^G)$, then $d(p, G-\bar{R}_1) > 0$ and $d(p, G-\bar{R}_2) > 0$. Thus there is a suitable neighborhood $N(p)$ of p such that $N(p) \subset R_1$ and $N(p) \subset R_2$. This implies that $N(p) \subset R_1 \frown R_2$ and hence $d(p, G-\overline{R_1 \frown R_2}) > 0$. Further we see $0 < d(p, G-\overline{R_1\frown R_2}) \leqq d(p, G-\overline{R_1 \smile R_2})$. Therefore p belongs to $\Delta_1^G - \Delta_{1,G-\overline{R_1\cup R_2}}^G$ and $\Delta_1^G - \Delta_{1,-\overline{R_1\cap R_2}}^G$, simultaneously. If p belongs to $(\Delta_1^G - \Delta_{1,G-\bar{R}_1}^G)\smile(\Delta_1^G - \Delta_{1,G-\bar{R}_2}^G)$ but does not belong to $(\Delta_1^G - \Delta_{1,G-\bar{R}_1}^G)\frown(\Delta_1^G - \Delta_{1,G-\bar{R}_2}^G)$, then either $d(p, G-\bar{R}_1)$ or $d(p, G-\bar{R}_2)$ is positive. Thus we see $d(p, G-\overline{R_1\smile R_2}) > 0$. This shows that p belongs to $\Delta_1^G - \Delta_{1,G-\overline{R_1\cup R_2}}^G$. Let u be canonically represented as in the above, then

$$\bar{\mu}^G_{R_1}u + \bar{\mu}^G_{R_2}u = \int_{\Delta_1^G - \Delta_{1,G-\bar{R}_1}^G} K_G(p, q)\, d\sigma_u(q) + \int_{\Delta_1^G - \Delta_{1,G-\bar{R}_2}^G} K_G(p, q) d\sigma_u(q)$$

$$\leqq \left(\int_{\Delta_1^G - \Delta_{1,G-\overline{R_1\cup R_2}}^G} + \int_{\Delta_1^G - \Delta_{1,G-\overline{R_1\cup R_2}}^G}\right) K_G(p, q) d\sigma_u(q)$$

$$= \bar{\mu}^G_{R_1\cup R_2}u + \bar{\mu}^G_{R_1\cap R_2}u.$$

THEOREM 14. *Let R_1 and R_2 be two open sets in G and satisfy two conditions $\mu^{*G}_{R_1}u > 0$ and $\mu^{*G}_{R_2}u > 0$ for a same minimal function $u \in PH(G)$, then $\mu^{*G}_{R_1\cap R_2}u > 0$ and $R_1 \frown R_2 \neq \phi$. If R_1 and R_2 are disjoint, then for any $u \in PH(G)$*

$$\mu^{*G}_{R_1\cup R_2}u = \mu^{*G}_{R_1}u + \mu^{*G}_{R_2}u.$$

If R_1 and R_2 are two open sets in G satisfying two conditions $\bar{\mu}^G_{R_1}u > 0$ and $\bar{\mu}^G_{R_2}u > 0$ for a same minimal function $u \in PH(G)$, then $\bar{\mu}^G_{R_1\cap R_2}u > 0$ and $R_1 \frown R_2 \neq \phi$. If R_1 and R_2 are disjoint, then for any $u \in PH(G)$

$$\bar{\mu}^G_{R_1\cup R_2}u = \bar{\mu}^G_{R_1}u + \bar{\mu}^G_{R_2}u.$$

Proof. Let u be a minimal in $PH(G)$ for which $\mu^{*G}_{R_1}u > 0$ and $\mu^{*G}_{R_2}u > 0$, then $\mu^{*G}_{R_1}u = \mu^{*G}_{R_2}u = u$ and $\mu^{*G}_{R_1\cup R_2}u = u$ Thus there holds $\mu^{*}_{R_1\cap R_2}u = u$. This shows that $R_1 \frown R_2 \neq \phi$. These facts were already stated in [1]. If $R_1 \frown R_2 = \phi$, then

$$T^G_{R_1+R_2}u = T^G_{R_1}u + T^G_{R_2}u$$

by the definition of T mapping and hence there holds

$$S^G_{R_1+R_2}T^G_{R_1+R_2}u = S^G_{R_1}T^G_{R_1}u + S^G_{R_2}T^G_{R_2}u,$$

which is the desired result.

Let u be a minimal in $PH(G)$ for which $\bar{\mu}^G_{R_1}u > 0$ and $\bar{\mu}^G_{R_2}u > 0$, then $\bar{\mu}^G_{R_1}u = \bar{\mu}^G_{R_2}u = u$ and $\bar{\mu}^G_{R_1\cup R_2}u = u$. Thus there holds $\bar{\mu}^G_{R_1\cap R_2}u = u$. The corresponding Martin's minimal point belongs to a common set of R_1 and R_2, that is, $R_1 \frown R_2 \neq \phi$. This implies

$$\Delta_1{}^G - \Delta^G_{1,G-\bar{R}_1} \cup \Delta^G_{1,G-R_2} = (\Delta^G_1 - \Delta_{1,G-\bar{R}_1}^{G}) \frown (\Delta_1{}^G - \Delta_{1,G-\bar{R}_2}^{G})$$

$$\leqq \Delta_1{}^G - \Delta_{1,\, G-\overline{R_1 \cap R_2}}^{G} \leqq \Delta_1{}^G - \Delta_{1,\, (G-\bar{R}_1)\cup(G-\bar{R}_2)}^{G}.$$

We now assume that $R_1 \frown R_2 = \phi$. Let u be any minimal function in $PH(G)$, then we can say either $\bar{\mu}^G_{R_1 \cup R_2} u = u$ or $\bar{\mu}^G_{R_1 \cup R_2} u = 0$. If the first is the case, then $\bar{\mu}^G_{R_1} u = u$ and $\bar{\mu}^G_{R_2} u = 0$ or $\bar{\mu}^G_{R_1} u = 0$ and $\bar{\mu}^G_{R_2} u = u$. If the second case occurs, then $\bar{\mu}^G_{R_1} u = \bar{\mu}^G_{R_2} u = 0$. This implies that a minimal point q belonging to $\Delta_1{}^G - \Delta_{1,G-\overline{R_1 \cup R_2}}^{G}$ belongs to either $\Delta_1{}^G - \Delta_{1,G-\bar{R}_1}^{G}$ or $\Delta_1{}^G - \Delta_{1,G-\bar{R}_2}^{G}$ and $(\Delta_1{}^G - \Delta_{1,G-\bar{R}_1}^{G}) \frown (\Delta_{1G} - \Delta_{1,G-\bar{R}_2}^{G}) = \phi$ if $R_1 \frown R_2 = \phi$. Therefore putting

$$u(p) = \int_{\Delta_1{}^G} K_G(p, q) d\sigma_u(q)$$

we have

$$\bar{\mu}^G_{R_1 \cup R_2} u = \int_{\Delta^G_1 - \Delta_{1,G-\overline{R_1 \cup R_2}}^{G}} K_G(p, q) d\sigma_u(q)$$

$$= \int_{\Delta_1{}^G - \Delta_{1,G-\bar{R}_1}^{G}} K_G(p, q) d\sigma_u(q) + \int_{\Delta_1{}^G - \Delta_{1,G-\bar{R}_2}^{G}} K_G(p, q) d\sigma_u(q)$$

$$= \bar{\mu}^G_{R_1} u + \bar{\mu}^G_{R_2} u.$$

THEOREM 15. *If R_1 and R_2 are two open sets in G and $R_1 \subset R_2$, then*

$$\lambda^{*G}_{R_2} u \leqq \lambda^{*G}_{R_2 - \bar{R}_1} u + \lambda^{*G}_{R_1} u, \quad \mu^{*G}_{R_2} u \geqq \mu^{*G}_{R_2 - \bar{R}_1} u + \mu^{*G}_{R_1} u,$$

$$\bar{\lambda}^G_{R_2} u \leqq \bar{\lambda}^G_{R_2 - \bar{R}_1} u + \bar{\lambda}^G_{R_1} u, \qquad \bar{\mu}^G_{R_2} u \geqq \bar{\mu}^G_{R_2 - \bar{R}_1} u + \bar{\mu}^G_{R_1} u$$

for any $u \in PH(G)$.

Proof. This is easily concluded by theorem 12 and 13.

THEOREM 16. *If R_1 and $R_2 - \bar{R}_1$ are (most) P-nice in G, then R_2 is also so in G. If R_1 and $R_2 - \bar{R}_1$ are exterior P-nice and P-nice in G, then R_2 is also so in G.*

Proof. If R_1 and $R_2 - \bar{R}_1$ are P-nice in G, then $\lambda^{*G}_{R_1} = \mu^{*G}_{R_1}$, $\lambda^{*G}_{R_2 - \bar{R}_1} = \mu^{*G}_{R_2 - \bar{R}_1}$. Thus we have

$$\lambda^{*G}_{R_2} u \leqq \lambda^{*G}_{R_2 - \bar{R}_1} u + \lambda^{*G}_{R_1} u = \mu^{*G}_{R_2 - \bar{R}_1} u + \mu^{*G}_{R_1} u \leqq \mu^{*G}_{R_2} u,$$

which implies $\lambda^{*G}_{R_2} u = \lambda^{*G}_{R_2} u$ for any $u \in PH(G)$.
The remaining parts can be proved quite similarly.

Let R_1 and R_2 be two disjoint open sets in G. We define two *ideal intersection operators* as follows:

$$I_1(R_1, R_2; G) = \lambda^{*G}_{R_1} + \lambda^{*G}_{R_2} - \lambda^{*G}_{R_1 + R_2} \text{ and } I_2(R_1, R_2; G) = \bar{\lambda}^G_{R_1} + \bar{\lambda}^G_{R_2} - \bar{\lambda}^G_{R_1 + R_2}.$$

We shall now prove the following lemmas.

LEMMA 1. *Let R_1 and R_2 be two open sets in G, then* $\mu^{*G}_{R_1}\mu^{*G}_{R_2} = \mu^{*G}_{R_2}\mu^{*G}_{R_1} = \mu^{*G}_{R_1\cap R_2}$ *and* $\bar{\mu}^G_{R_1}\bar{\mu}^G_{R_2} = \bar{\mu}^G_{R_2}\bar{\mu}^G_{R_1} = \bar{\mu}^G_{R_1\cap R_2}$.

Proof. $\mu^{*G}_{R_2} \geqq \mu^{*G}_{R_1\cap R_2}$ is evident. Thus we have

$$\mu^{*G}_{R_1}\mu^{*G}_{R_2} \geqq \mu^{*G}_{R_1}\mu^{*G}_{R_1\cap R_2} = \mu^{*G}_{R_1\cap R_2}.$$

On the other hand by theorem 12 we have

$$\mu^{*G}_{R_1}\mu^{*G}_{R_1\cup R_2} + \mu^{*G}_{R_1}\mu^{*G}_{R_1\cap R_2} \geqq \mu^{*G}_{R_1}\mu^{*G}_{R_1} + \mu^{*G}_{R_1}\mu^{*G}_{R_2},$$

whence follows that

$$\mu^{*G}_{R_1} + \mu^{*G}_{R_1\cap R_2} \geqq \mu^{*G}_{R_1} + \mu^{*G}_{R_1}\mu^{*G}_{R_2},$$

that is,

$$\mu^{*G}_{R_1\cap R_2} \geqq \mu^{*G}_{R_1}\mu^{*G}_{R_2}.$$

Another part of the first identity can be proved similarly. For $^-$ operator we can proceed quite similarly using theorem 13 instead of theorem 12.

LEMMA 2. $\lambda^{*G}_{R_1}\lambda^{*G}_{R_2} = \lambda^{*G}_{R_2}\lambda^{*G}_{R_1} = \lambda^{*G}_{R_1} + \lambda^{*G}_{R_2} - \lambda^{*G}_{R_1\cup R_2}$ *and* $\bar{\lambda}^G_{R_1}\bar{\lambda}^G_{R_2} = \bar{\lambda}^G_{R_2}\bar{\lambda}^G_{R_1} = \bar{\lambda}^G_{R_1} + \bar{\lambda}^G_{R_2} - \bar{\lambda}^G_{R_1\cup R_2}$.

Proof. By lemma 1 we can say that

$$\begin{aligned}\lambda^{*G}_{R_1}\lambda^{*G}_{R_2} &= (I - \mu^{*G}_{G-\bar{R}_1})(I - \mu^{*G}_{G-\bar{R}_2})\\ &= I - \mu^{*G}_{G-\bar{R}_2} - \mu^{*G}_{G-\bar{R}_1} + \mu^{*G}_{G-\bar{R}_1}\mu^{*G}_{G-\bar{R}_2}\\ &= I - \mu^{*G}_{G-\bar{R}_1} - \mu^{*G}_{G-\bar{R}_2} + \mu^{*G}_{(G-\bar{R}_1)\cap(G-\bar{R}_2)}\\ &= I - \mu^{*G}_{G-\bar{R}_1} - \mu^{*G}_{G-\bar{R}_2} + \mu^{*G}_{G-\overline{R_1\cup R_2}} = \lambda^{*G}_{R_1} + \lambda^{*G}_{R_2} - \lambda^{*G}_{R_1\cup R_2}.\end{aligned}$$

The remaining ones are quite similarly proved as in the above.

By lemma 2 we have the following representations

$$I_1(R_1, R_2; G) = \lambda^{*G}_{R_1}\lambda^{*G}_{R_2} = \lambda^{*G}_{R_2}\lambda^{*G}_{R_1},\ I_2(R_1, R_2; G) = \bar{\lambda}^G_{R_1}\bar{\lambda}^G_{R_2} = \bar{\lambda}^G_{R_2}\bar{\lambda}^G_{R_1}.$$

THEOREM 17. *If G_1 is P-nice in G_2 and contains two disjoint open sets R_1 and R_2, then the vanishing of $I_1(R_1, R_2; G_1)$ implies that of $I_1(R_1, R_2; G_2)$. If G_1 is onto P-nice in G_2, then the vanishing of $I_1(R_1, R_2; G_2)$ implies that of $I_1(R_1, R_2; G_1)$. If G_1 is most P-nice in G_2, then the vanishing of $I_2(R_1, R_2; G_1)$ implies that of $I_2(R_1, R_2; G_2)$. If G_1 is onto P-nice in G_2, then the vanishing of $I_2(R_1, R_2; G_2)$ implies that of $I_2(R_1, R_2; G_1)$.*

Proof. If $T^{G_2}_{G_1}u \neq 0$, then we have $I_1(R_1, R_2; G) = S^{G_2}_{G_1} I_1(R_1, R_2; G_1)\, T^{G_2}_{G_1}$. If $T^{G_2}_{G_1}u = 0$, then we have $I_1(R_1, R_2; G_2)u = \lambda^{*G_2}_{R_1}\lambda^{*G_2}_{R_2}u = \lambda^{*G_2}_{R_1}\lambda^{*G_2}_{R_2}\mu^{*G_2}_{G_1}u = 0$. By the onto

P-niceness of G_1 in G_2, $S_{G_1}^{G_2}u<\infty$ for any $u \in PH(G_1)$ and further $I_1(R_1, R_2; G_1) \leqq T_{G_1}^{G_2} I_1(R_1, R_2; G_2) S_{G_1}^{G_2}$ holds. For the remaining two results a similar method is available.

If $R_1 \frown R_2 = \phi$ and R_1 and R_2 are P-nice in G, then $I_1(R_1, R_2; G) = 0$ and if R_1 and R_2 are most P-nice in G, then $I_2(R_1, R_2; G) = 0$. If $R_1 \frown R_2 = \phi$ and $I_2(R_1, R_2; G) = 0$ and $R_1 \smile R_2$ is exterior P-nice and P-nice (resp. exterior P-nice, P-nice and interior P-nice) in G, then R_1 and R_2 are exterior P-nice and P-nice (resp. most P-nice) in G.

THEREOM 18. *If R_1 and R_2 are both P-nice in G, then $R_1 \smile R_2$ and $R_1 \frown R_2$ are also both P-nice in G. If R_1 and R_2 are both most P-nice in G, then $R_1 \smile R_2$ and $R_1 \frown R_2$ are also so. If R_1 and R_2 are exterior (interior) P-nice and P-nice in G, then $R_1 \smile R_2$ and $R_1 \frown R_2$ are also so.*

If R_1 and R_2 are two open sets in G which satisfy

$$\lambda^{*G}_{R_1}u + \lambda^{*G}_{R_2}u = \lambda^{G}_{R_1 \cup R_2}u + \lambda^{*G}_{R_1 \cap R_2}u$$

and

$$\lambda^{*G}_{R_1}u + \mu^{*G}_{R_2}u = \mu^{*G}_{R_1 \cup R_2}u + \mu^{*G}_{R_1 \cap R_2}u$$

for any $u \in PH(G)$, then we say that R_1 and R_2 are *situated P-regularly* in G. If R_1 and R_2 are P-nice in G, then they are situated P-regularly in G. In the above definition of P-regular situation of R_1 and R_2 in G we replace *operators by $^-$operators. Then we say that R_1 and R_2 are *most P-regularly situated* in G. If R_1 and R_2 are most P-nice in G, then they are situated most P-regularly in G.

THEOREM 19. *If R_1 and R_2 are situated P-regularly in G_1 which is P-nice in G_2, then R_1 aud R_2 are situated P-regularly in G_2. If R_1 and R_2 are situated P-regularly in G_2 and G_1 is onto P-nice and P-nice in G_2, then R_1 and R_2 are situated P-regularly in G_1. If R_1 and R_2 are situated most P-regularly in G_1 and is most P-nice in G_2, then they are situated most P-regularly in G_2. If R_1 and R_2 are situated most P-regularly in G_2 and G_1 is onto P-nice and P-nice in G_2, then they are situated most P-regularly in G_1.*

Let R_1 and R_4 be two opan sets in G. If $\lambda^{*G}_{R_1 - \overline{R_1 \cap R_2}}u = \lambda^{*G}_{R_2 - \overline{R_1 \cap R_2}}u = 0$ for any $u \in PH(G)$, then we say R_1 and R_2 *truely P-equivalent* in G. If $\bar{\lambda}^{G}_{R_1 - \overline{R_1 \cap R_2}}u = \bar{\lambda}^{G}_{R_2 - \overline{R_1 \cap R_2}}u = 0$ for any $u \in PH(G)$, then we say R_1 and R_2 *most truely P-equivalent* in G_1.

THEOREM 20. *If R_1 and R_2 are truely P-equivalent in G, then R_1 and R_2 are second P-equivalent in G and they are P-regularly situated in G. If R_1 and R_2 are most truely P-equivalent in G, then they are most P-equivalent in G and further they are situated P-regularly and most P-regularly in G_1.*

Proof. By the assumption $\mu^{*G}_{R_1 - \overline{R_1 \cap R_2}}u = 0$ we have

$$\lambda^{*G}_{R_1\cap R_2}u = \lambda^{*G}_{R_1-\overline{R_1\cap R_2}}u + \lambda^{*G}_{R_1\cap R_2}u \geqq \lambda^{*G}_{R_1}u \geqq \lambda^{*G}_{R_1\cap R_2}u$$

for any $u \in PH(G)$, which leads us to a fact $\lambda^{*G}_{R_1}u = \lambda^{*G}_{R_1\cap R_2}u$. Similarly we have $\lambda^{*G}_{R_2}u = \lambda^{*G}_{R_1\cap R_2}u$ and hence $\lambda^{*G}_{R_1}u = \lambda^{*G}_{R_1\cup R_2}u$. By a simple calculation

$$0 = \lambda^{*G}_{R_1-\overline{R_1\cap R_2}}u = \lambda^{*G}_{G-\bar{R}_2-\overline{(G-\bar{R}_1)\cap(G-\bar{R}_2)}}u$$

$$= \lambda^{*G}_{R_2-\overline{R_1\cap R_2}}u = \lambda^{*G}_{G-\bar{R}_1-\overline{(G-\bar{R}_1)\cap(G-\bar{R}_2)}}u,$$

which imply that

$$\lambda^{*G}_{G-\bar{R}_1}u = \lambda^{*G}_{G-\bar{R}_2}u = \lambda^{*G}_{(G-\bar{R}_1)\cap(G-\bar{R}_2)}u = \lambda^{*G}_{(G-\bar{R}_1)\cup(G-\bar{R}_2)}u$$

and hence

$$\mu^{*G}_{R_1}u = \mu^{*G}_{R_2}u = \mu^{*G}_{R_1\cup R_2}u = \mu^{*G}_{R_1\cap R_2}u$$

for any $u \in PH(G)$. This is the first desired result. The proof of the remaining part is quite similar and then we use $\bar{\lambda}^G_R u \geqq \lambda^{*G}_R u$.

THEOREM 21. *If R_1 and R_2 are P-nice and second (or interior) P-equivalent in G, then they are truely P-equivalent in G. If R_1 is P-nice in G and if R_1 and R_2 are truely P-equivalent in G, then R_2 is P-nice in G.*

Proof. If A and B are disjoint open sets in G and $\mu^{*G}_A u = \mu^{*G}_B u$ for any $u \in PH(G)$, then $\mu^{*G}_A u \equiv 0$. Indeed, if $\mu^{*G}_A u \neq 0$ for some $u \in PH(G)$, then we have $\mu^{*G}_A \mu^{*G}_A u = \mu^{*G}_A u \neq 0$ but $\mu^{*G}_B \mu^{*G}_A u = \mu^{*G}_{A\cap B} u = 0$. This is untenable.

By the P-niceness of R_1 and R_2 in G and hence that of $G-\bar{R}_1$ and $R_1 \frown R_2$ in G, we can prove the P-niceness of $R_1-\overline{R_1\frown R_2}$ in G. Thus we have by a simple calculation

$$\mu^{*G}_{R_1-\overline{R_1\cap R_2}}u \geqq \mu^{*G}_{R_2-\overline{R_1\cap R_2}}u.$$

Similarly we have an inversely directed inequality and hence we have an equality

$$\mu^{*G}_{R_1-\overline{R_1\cap R_2}}u = \mu^{*G}_{R_2-\overline{R_1\cap R_2}}u.$$

On the other hand $(R_1-\overline{R_1\frown R_2})\frown(R_2-\overline{R_1\frown R_2}) = \phi$, therefore we can apply a fact stated firstly. Then we have the vanishing of each quantity. By the P-niceness of $R_1-\overline{R_1\frown R_2}$ and $R_2-\overline{R_1\frown R_2}$ in G, we can say

$$\lambda^{*G}_{R_1-\overline{R_1\cap R_2}}u = \lambda^{*G}_{R_2-\overline{R_1\cap R_2}}u = 0$$

for any $u \in PH(G)$. The remaining part is evident.

THEOREM 22. *If R_1 and R_2 are most P-nice and first (or interior) P-equivalent in G, then they are most truely P-equivalent (and hence most P-equivalent) in G.*

Proof. If A and B are disjoint open sets in G and $\bar{\mu}^G_A u = \bar{\mu}^G_B u$ for any $u \in PH(G)$, then $\bar{\mu}^G_A u = 0$. Indeed, if $\bar{\mu}^G_A u \neq 0$ for some $u \in PH(G)$, then we have $\bar{\mu}^G_A \bar{\mu}^G_A u = \bar{\mu}^G_A u \neq 0$, $\bar{\mu}^G_B \bar{\mu}^G_A u = \bar{\mu}^G_{A\cap B} u = 0$, which is untenable. By this fact we can proceed quite similarly as in the preceding theorem.

THEOREM 23. *If R_1 and R_2 are truely P-equivalent in G_1 and G_1 is P-nice in G_2, then R_1 and R_2 are also truely P-equivalent in G_2. If R_1 and R_2 are truely P-equivalent in G_2 and G_1 is onto P-nice and P-nice in G_2, then R_1 and R_2 are also truely P-equivalent in G_1. If R_1 and R_2 are most truely P-equivalent in G_1 and G_1 is most P-nice in G_2, then R_1 and R_2 are also most truely P-equivalent in G_2. If R_1 and R_2 are most truely P-equivalent in G_2 and G_1 is exterior P-nice and P-nice and onto P-nice in G_2, then R_1 and R_2 are most truely P-equivalent in G_1.*

§6. Definition of paths We shall define and discuss two sorts of paths.

Let q be a minimal point in Δ_1^G and let γ be a curve which determines the q. Let D be a domain in G for which γ is a part of ∂D. If $T_D^G K_G(p, q) = 0$ for any D, then we say γ a *non-tangential path* to q. If there is a domain D for which $T_D^G K_G(p, q) > 0$, then we say γ a *tangential path* to q.

If q is an isolated minimal point in Δ^G or Δ_1^G, then to construct a tangential path to q is a very important problem, that is, in this case it is not yet solved to guarantee the existence of the path.

If $q \in \Delta_1^G$ is not isolated in Δ_1^G, then there is an infinite number of minimal points in any ε-neighborhood N_ε of q, where N_ε is a point set on which $d_G(p, q) < \varepsilon$, $p \in \Delta^G$. Let $R(\gamma, p)$ be a domain bounded by γ and by a defining tail of $p \in N_\varepsilon$. We can use this $R(\gamma, p)$ as a testing domain D in the definition of tangential path to q.

If there is a testing domain D in the right hand side of γ for which

$$T_D^G K_G(p, q) > 0,$$

then we say γ a *left tangential path* to q. Similarly we define a *right tangential path* to q. It may occur that γ is simultaneously a left and right tangential path to q, then we cannot choose two disjoint testing domains D_1 and D_2 one of which lies in the right hand side of γ and another of which lies in the left hand side of γ and for which

$$T_{D_1}^G K_G(p, q) > 0 \text{ and } T_{D_2}^G K_G(p, q) > 0.$$

Indeed, if it is possible, then $D_1 \frown D_2 \neq \phi$. Therefore if γ is simultaneously left and right tangential to q, then any testing domain D for which

$$T_D^G K_G(p, q) > 0$$

lies in both sides of the curve γ.

Let $\{D(t)\}$ be a sequence of non-compact domains for which

$$u(t) \equiv T_{D(t)}^G u > 0 \text{ and } T_{D(s)}^{D(t)} u(t) = u(s)$$

for a minimal $u \in PH(G)$ and $\lim_{t\to\infty}$ (diameter of $D(t)$) $= 0$, then we say $\{D(t)\}$ a defining sequence of a minimal point of G. And this sequence defines only one point in Δ_1^G which corresponds to the function u.

Let $D(t)$ be a domain defined by $u > t$ if u is a singular minimal and $u > M(1-1/t)$ if u is a bounded minimal and $\sup_G u = M$. Then $\{D(t)\}$ is a defining sequence of a minimal point of G, since $T_{D(s)}^{D(t)}(u-t)=u-s$ and $T_{D(s)}^{D(t)}(u-M(1-1/t)) = u\,M(1-1/s)$ in the respective cases.

THEOREM 24. *If γ is a tangential path to q in the above sense, then there is a defining sequence $\{D(t)\}$ of q for which $D(t)\frown\gamma = \phi$ for any $t > t_0$ and vice versa.*

Proof. Let R be a domain for which $K_D(p, q) = T_D^G K_G(p, q) > 0$. Let $D(t)$ be a domain defined by $K_D(p, q) > t$ if $K_D(p, q)$ is singular minimal or $K_D(p, q) > M(1-1/t)$ if $K_D(p, q)$ is bounded minimal and $\sup_D K_D(p, q) = M$. Then we have

$$T_{D(t)}^G K_G(p, q) = T_{D(t)}^D T_D^G K_G(p, q) = T_{D(t)}^D K_D(p, q)$$

$$= \begin{cases} K_D(p, q) - t, \text{ if } K_G \text{ is singular,} \\ K_D(p, q)-M(1-1/t), \text{ if } K_G \text{ is bounded and } M=\sup_D K_D(p, q)<0. \end{cases}$$

Thus $\{D(t)\}$ is a defining sequence of q. Further $D(t)\frown\gamma = \phi$. since $K_D(p, q)=0$ on γ. This is the desired conclusion.

Inversely if there exists a defining sequence $\{D(t)\}$ of q for which $\gamma\frown D(t) = \phi$ for any $t > t_0$, then there is a domain R for which $R\frown\gamma = \phi$ and $\bar{R}\frown\gamma = \gamma$ and $G \supset R\supset D(t)$ for $t > t_0$. Then

$$0 < T_{D(t)}^G K_G(p, q) = T_{D(t)}^R T_R^G K_G(p, q),$$

which shows that $T_R^G K_G(p, q) > 0$. This is the desired result.

THEOREM 25. *If R is a domain such that $\Delta_{1,R}^G = q$ and ∂R are two non-tangential paths ending at q, then any non-compact curve lying in R is also a non-tangential path to q.*

Proof. If there is a non-compact curve γ lying in R and being tangential to q, then there is an R_1 for which $T_{R_1}^G K_G(p, q) > 0$. Then R_1 contains a part of ∂R and does not contain another part of ∂R. Thus $R_2 = R_1 \smile R$ is a domain for which $T_{R_2}^G K_G(p, q) > 0$. This shows that a part of ∂R is not a non-tangential path to q, which is untenable.

THEOREM 26. *Let q be any point in Δ_1^G, then any level lines of the corresponding minimal ending at q is a tangential path to q.*

Proof. Let u_q be the corresponding minimal to q and $u_q = \alpha$ be the level lines ending at q. Then the domain $D = \{u_q > \beta > \alpha\}$ is a member of a definding sequence of the q, since $T_{D_\beta}^G u_q = u_q - \beta$ and $D_\beta \cap \{u_q = \alpha\} = \phi$.

THEOREM 27. *Any non-tangential path γ in G ending at a minimal point q intersects with all the level curves of the corresponding minimal function $K_G(p, q)$.*

Further this implies $\overline{\lim}\ K_G(p, q) = \sup_G K_G(p, q)$.

Proof. If not so, then there is a positive member δ such that the domain D_δ satisfying $K(p, q) > \delta$ and the curve γ have no common point. Then $T^G_{D_\delta} K_G(p, q) = K_G(p, q) - \delta > 0$. This shows that γ is a tangential path to q.

In this theorem we say that γ intersects with all the level curves if it intersects with all the level curves $\{K_G(p, q) = \alpha\}$ for $\sup_G K_G(p, q) > \alpha > \alpha_0$.

By the same reasoning we can say that any non-tangential path γ to q intersects with all the level curves of $T^G_D K_G(p, q)$ if it is positive and

$$\overline{\lim_{\gamma}}\ T^G_D K_G(p, q) = \sup_D T^G_D K_G(p, q).$$

If for a suitable continuous function $\varepsilon(t)$ with $\lim_{t\to\infty} \varepsilon(t) = 0$ two curves $\gamma_1 = \gamma_1(t)$ and $\gamma_2 = \gamma_2(t)$ satisfy $d_G(\gamma_1(t), \gamma_2(t)) < \varepsilon(t)$, then we say that two curves are $\varepsilon(t)$-near. This depends on the parameter t, however it is a matter of sufficiently large $t \geqq t_0$.

THEOREM 28. *If γ is a tangential path to $q \in \Delta^G_1$, then there is a curve γ' which is $\varepsilon(t)$-near to γ and defines q and is also tangential to q.*

Proof. By the assumption there is a domain D for which $K_D(p, q) \equiv T^G_D K_G(p, q) > 0$ and whose boundary ∂D contains γ as a part. Then any level curve $= \{K_D(p, q) = \alpha\}$ does not intersect with γ. Further l_α lies in D and is a tangential path to q. Then between l_α and γ there is a curve γ' having desired properties.

THEOREM 29. *If a defining tail γ of $q \in \Delta^G_1$ intersects with all levels l_α of any $T^G_D K_G(p, q)$ with height $\alpha (> \alpha_0)$, then γ is a non-tangential path to q.*

THEOREM 30. *If γ is a non-tangential path to $q \in \Delta^G_1$, then any curve lying $\varepsilon(t)$-neighborhood of γ with sufficiently small $\varepsilon(t)$ is also a non-tangential path to q.*

Proof. If in any sufficiently small $\varepsilon(t)$-neighborhood of γ there is a tangential path γ' to q, then γ is also tangential to q by the above theorem.

A method choosing a suitable $\varepsilon(t)$-neighborhood is determined in the following section.

§ 7. Modification theorems.

THEOREM 31. *Let R be an open set in G with two-sidedness condition. Then there are two open sets R' and R'' with two-sidedness condition satisfying the following conditions:*

(i) *$R' \supset R \supset R''$ and $\partial R' \frown \partial R = \phi$, $\partial R'' \frown \partial R = \phi$ with the exception of any boundary point belonging to $\partial R \frown \partial G$ if it exists,*

(ii) $\Delta_{1,R}^{\ G} = \Delta_{1,R'}^{\ G} = \Delta_{1,R''}^{\ G}$, $\Delta_{1,G-\bar{R}}^{\ G} = \Delta_{1,G-\bar{R}'}^{\ G} = \Delta_{1,G-\bar{R}''}^{\ G}$, *and*

(iii) $\Delta^G_1(R) = \Delta^G_1(R') = \Delta^G_1(R'')$, $\Delta^G_1(G - \bar{R}) = \Delta^G_1(G - \bar{R}') = \Delta^G_1(G - \bar{R}'')$.

In order to prove this theorem it should be noted the following trivial facts:

If D is different from R only in a compact part, then the conclusions of our theorem are evidently true with the exception of the condition (i).

If a sequence of open sets D tends to R and each member D is different from R in a fixed compact part, then any member of $PH(D)$ tends to a member of $PH(R)$.

Proof of theorem 31. It is sufficient to prove our theorem for any normalized family in each connected component of R. Such a family is denoted by $Q(R)$. Let R_1 be an open set contained in R and defined by the following conditions: The boundary ∂R_1 consists of three parts. The first one coincides with ∂R and lies in $W - \overline{W}_1$. The second one is a collection of curves every of which joins two points on ∂W and lies in $R \frown W_1$ and lies in a small neighborhood of ∂R. The last one is a collection of two curves which are two parts joining the points $\partial W_1 \frown \partial R$ and the end points of the second one along ∂W_1. In this case we can construct $\partial(\overline{W}_1 \frown \overline{R}_1)$ so near to $\partial R \frown \overline{W}_1$ that

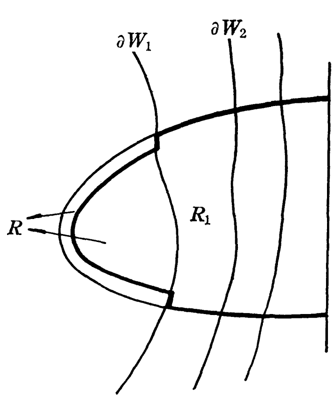


Fig. 11

$$\sup_{u \in Q(R)} \max_{p \in \partial(\overline{W}_1 \frown \overline{R}_1)} u(p) \leqq \varepsilon.$$

Then we have

$$0 \leqq u(p) - T^R_{R_1} u(p) \leqq \varepsilon$$

in R_1. Next we construct an open subset R_2 of R_1 by a similar process which is done in $(W_2 - \overline{W}_1) \frown R_1$. In this process we can construct R_2 so near to R_1 that

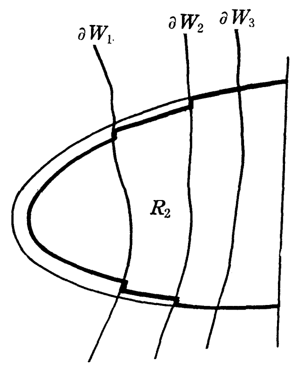


Fig. 12

$$0 \leqq T^R_{R_1} u(p) - T^{R_1}_{R_2} T^R_{R_1} u(p) \leqq \frac{\varepsilon}{2}$$

in R_2. We continue this construction process ad infinitum, then we have at the n th step

$$0 \leqq T^R_{R_{n-1}} u(p) - T^{R_{n-1}}_{R_n} T^R_{R_{n-1}} u(p) \leqq \frac{\varepsilon}{2^{n-1}}.$$

By the method of construction of $\{R_n\}$, R_n decreases monotonically and hence tends to some set $X_\varepsilon \neq \phi$. Further we can say that

$$u(p) - \varepsilon \sum_{\nu=0}^{n-1} \frac{1}{2^\nu} \leqq T^R_{R_n} u(p) \leqq u(p)$$

in $X_\varepsilon \subset R_n$. $T^R_{R_n} u(p)$ then decreases monotonically and hence there exists a limit function

$$\lim_{n\to\infty} T^R_{R_n} u(p),$$

which is also harmonic in X_ε being positive or not. Since $R_n \supset X_\varepsilon$, there holds an inequality

$$\lim_{n\to\infty} T^R_{R_n} u(p) \geqq T^R_{X_\varepsilon} u(p)$$

in X_ε. On the other hand it is evident that $T^R_{R_n}\, u \leqq u$ in R_n and especially on $\partial W_n \frown R_n = \partial W_n \frown X_\varepsilon$. Further $T^R_{R_n} u = 0$ on $\partial R_n \frown W_n = \partial X_\varepsilon \frown W_n$. Therefore there holds an inequality $u_n \geqq T^R_{R_n} u$ in $R_n \frown W_n$, where u_n is a positive harmonic function in $R_n \frown W_n$ defined by the boundary condition $u_n = u$ on $\partial W_n \frown R_n = \partial W_n \frown X_\varepsilon, = 0$ on $\partial R_n \frown W_n = \partial X_\varepsilon \frown W_n$. This shows that in X_ε

$$T^R_{X_\varepsilon} u \geqq \lim_{n\to\infty} T^R_{R_n} u,$$

since $T^R_{X_\varepsilon} u = \lim_{n\to\infty} u_n$ there. Thus we have

$$u - 2\varepsilon \leqq \lim_{n\to\infty} T^R_{R_n} u = T^R_{X_\varepsilon} u \leqq u$$

in X_ε, whence follows that $T^R_{X_\varepsilon} u \in PH(X_\varepsilon)$ for any $u \in PH(R)$. If V belongs to $PH(G)$ for which $T^G_R V > 0$, then by the above fact $T^G_{X_\varepsilon} V = T^G_{X_\varepsilon} T^G_R V > 0$. If $T^G_{X_\varepsilon} V > 0$ for a $V \in PH(G)$, then $0 < T^G_{X_\varepsilon} V = T^R_{X_\varepsilon} T^G_R V$ implies $T^G_R V > 0$. This shows that $\Delta^G_1(X_\varepsilon) = \Delta^G_1(R)$.

Next we shall construct another set Y_ε contained in R in the following manner. Let R^1 be an open set having the same structure as R_1 for which we impose the following conditions: for any $v \in Q(R)$

$$\sup_{T^G_{G-\bar R} v > 0} \; \max_{p \in \partial R \frown W_1} S^{G-\bar R^1}_{G-\bar R} T^G_{G-\bar R} v(p) \leqq \varepsilon.$$

Then we have for any v

$$0 \leqq S^{G-\bar R^1}_{G-\bar R} T^G_{G-\bar R} v - T^G_{G-\bar R}\, v \leqq \varepsilon$$

in $G - \bar R$. Next we construct an open set R^2 having the same structure as R_2 for which we have

$$0 \leqq S^{G-\bar R^2}_{G-\bar R^2} S^{G-\bar R^1}_{G-\bar R} T^G_{G-\bar R} v - S^{G-\bar R^1}_{G-\bar R} T^G_{G-\bar R} v \leqq \frac{\varepsilon}{2}$$

in $G - \bar R$. We continue this process ad infinitum, then R^n decreases and hence $G - \bar R^n$ increases monotonically and tends to a set Y_ε. Further

$$S^{G-\bar R^n}_{G-\bar R} T^G_{G-\bar R} v = \prod_{\nu=1}^{n} \; S^{G-\bar R^\nu}_{G-\bar R^{\nu-1}}\, T^G_{G-\bar R} v$$

increases monotonically and satisfies

$$0 \leqq S^{G-\bar R^n}_{G-\bar R} T^G_{G-\bar R}\, v - T^G_{G-\bar R}\, v \leqq \varepsilon \sum_{\nu=0}^{n-1} \frac{1}{2^\nu} < 2\varepsilon$$

in $G-\bar{R}$. Thus there exists a limit function

$$\lim_{n\to\infty} S_{G-\bar{R}}^{G-\bar{R}^n} T_{G-\bar{R}}^{G} v$$

for any v with $T_{G-\bar{R}}^{G} v > 0$ and there holds

$$0 < T_{G-\bar{R}}^{G} v \leqq \lim_{n\to\infty} S_{G-\bar{R}}^{G-\bar{R}^n} T_{G-\bar{R}}^{G} v \leqq T_{G-\bar{R}}^{G} v + 2\varepsilon .$$

Evidently we have

$$S_{G-\bar{R}}^{G-\bar{X}_\varepsilon} T_{G-\bar{R}}^{G} v \geqq \lim_{n\to\infty} S_{G-\bar{R}}^{G-\bar{R}^n} T_{G-\bar{R}}^{G} v$$

in $G-\bar{R}$. On the other hand there holds

$$S_{G-\bar{R}}^{G-\bar{R}^n} T_{G-\bar{R}}^{G} v \geqq T_{G-\bar{R}}^{G} v$$

on $\partial W_n \frown (G-\bar{R})$ and hence

$$S_{G-\bar{R}}^{G-\bar{R}^n} T_{G-\bar{R}}^{G} v \geqq u_n$$

in $W_n \frown (G-R^n)$, where u_n is a positive harmonic function in $W_n \frown (G-R^n)$ defined by the boundary condition: $u_n = T_{G-\bar{R}}^{G} v$ on $\partial W_n \frown (G-\bar{R})$, $=0$ on any other boundary of $W_n \frown (G-\bar{R}^n)$. Since u_n tends to a function

$$S_{G-\bar{R}}^{G-\bar{Y}_\varepsilon} T_{G-\bar{R}}^{G} v$$

in $G-\bar{Y}_\varepsilon$, we have

$$\lim_{n\to\infty} S_{G-\bar{R}}^{G-\bar{R}^n} T_{G-\bar{R}}^{G} v \geqq S_{G-R}^{G-\bar{Y}_\varepsilon} T_{G-\bar{R}}^{G} v$$

in $G-\bar{Y}_\varepsilon$ for any v with $T_{G-\bar{R}}^{G} v > 0$. Therefore there holds

$$T_{G-\bar{R}}^{G} v + 2\varepsilon \geqq \lim_{n\to\infty} S_{G-\bar{R}}^{G-\bar{R}^n} T_{G-\bar{R}}^{G} v = S_{G-\bar{R}}^{G-\bar{Y}_\varepsilon} T_{G-\bar{R}}^{G} v \geqq T_{G-\bar{R}}^{G} v > 0$$

in $G-\bar{R}$. Thus we can say that

$$\Delta_1^G(G-\bar{R}) = \Delta_1^G(G-\bar{Y}_\varepsilon) .$$

In $R-X_\varepsilon \smile Y_\varepsilon$ we can write a collection of curves $\partial R''$ such that $X_\varepsilon \smile Y_\varepsilon \subset R'' \subset R$ and $\partial R'' \frown \partial R = \phi$ and R'' satisfies the two-sidedness condition. For this open set R'' we have

$$\Delta_1^G(R'') = \Delta_1^G(R) \text{ and } \Delta_1^G(G-\bar{R}'') = \Delta_1^G(G-\bar{R}) .$$

We further modify the above R'' as follows: For any point p on ∂R there exists a $\delta(n)$-neighborhood $N(p)$, that is, the set satisfying $d_G(p, q) < \delta(n)$, where $\delta(n)$ satisfies

$$\delta(n) \leqq d_G((W_n - \overline{W}_{n-1}) \frown G, \Delta^G)/n .$$

Let Z_δ be a set $\{N(p)\}$, $p \in \partial R$. In $(R-X_\varepsilon \smile Y_\varepsilon) \frown Z_\delta$ we can write a collection of curves $\partial R''$ as before. By its construction we have further for this R''

$$\Delta_{1,R''}^{G} = \Delta_{1,R}^{G} \text{ and } \Delta_{1,G-\bar{R}''}^{G} = \Delta_{1,G-\bar{R}}^{G}.$$

Similarly we can construct an open set $R'(\supset R)$ with the two-sidedness condition satisfying the desired facts.

In a way we can prove the following another modification theorem.

THEOREM 32. *Let R be an open set in G with two-sidedness condition. Then there is an open set R' in G with two-sidedness condition satisfying the following conditions:*

(i) *$R'\supset R$ and $\partial R'\frown\partial R=\phi$ with the exception of any boundary point belonging to $\partial R\frown\partial G$ if it exists, and*

(ii) $\Delta_1^R = \Delta_1^{R'}(R)$.

Proof. Let R_1 be an open set containing R for which $R_1\frown(W-\overline{W}_1) = R\frown(W-\overline{W}_1)$. We impose the following condition to R_1:

$$\sup_{u\in Q(R)} \sup_{p\in\partial R\frown W_1} S_R^{R_1} u(p) \leqq \varepsilon .$$

The method of construction of R_1 is quite similar as in the former theorem. Then we have

$$0 \leqq S_R^{R_1} u - u \leqq \varepsilon$$

in R for any $u\in Q(R)$. Similarly we construct an open set $R_2(\supset R_1)$ such that $R_2\frown W_1 = R_1\frown W_1$ and $R_2\frown(W-\overline{W}_2) = R_1\frown(W-\overline{W}_2)$ and

$$0 \leqq S_{R_1}^{R_2} S_R^{R_1} u - S_R^{R_1} u \leqq \frac{\varepsilon}{2}$$

for any $u\in Q(R)$ in R_1. We continue this process ad infinitum, then $\{R_n\}$ is an increasing sequence and hence $X_\varepsilon = \lim_{n\to\infty} R_n$ exists. Further we have

$$0 \leqq S_R^{R_n} u - u \leqq \varepsilon \sum_{\nu=0}^{n-1} \frac{1}{2^\nu}$$

in R and $S_R^{R_n} u$ increases monotonically in R. Thus $\lim_{n\to\infty} S_R^{R_n} u$ exists and satisfies an inequality

$$S_R^{X_\varepsilon} u \geqq \lim_{n\to\infty} S_R^{R_n} u$$

in R for any $u\in Q(R)$. On the other hand we have

$$S_R^{R_n} u \geqq u_n$$

in $R_n\frown W_n$, where u_n is a positive harmonic function in $R_n\frown W_n$ with boundary value $u_n = u$ on $\partial W_n\frown R_n = \partial W_n\frown X_\varepsilon = 0$ on $\partial R_n\frown W_n = \partial X_\varepsilon\frown W_n$. This implies that

$$S_R^{X_\varepsilon} u \leqq \lim_{n\to\infty} S_R^{R_n} u$$

in R, since $\lim_{n\to\infty} u_n = S_R^{X_\varepsilon} u$. Therefore we have $u \leqq S_R^{X_\varepsilon} u = \lim_{n\to\infty} S_R^{R_n} u \leqq u + 2\varepsilon$ in R for any $u \in Q(R)$. This shows that Δ_1^R can be imbedded into $\Delta_1^{X_\varepsilon}$ by $S_R^{X_\varepsilon}$. We can construct a collection of curves lying in $X_\varepsilon - \bar{R}$. Then an open set R' bounded by this collection of curves is the desired one. Since we have $S_R^{R'} \Delta_1^R \subset \Delta_1^{R'}$, $S_R^{R'} \Delta_1^R = \Delta_1^{R'}(R)$, which is the desired result.

Chapter II. Extensions and other linear mappings in $PH(G)$.

In this chapter we shall extend our linear mappings in $PH(G)$ in order to be able to apply them to a more general set R in G and further we shall define several other linear mappings in $PH(G)$ making use of our earlier mappings in order to reveal several other features of the given sets. In this chapter, save when the contrary is explicitely mentioned, we do not assume that any given sets satisfy the two-sidedness condition. We have already explained that in our general case we cannot say an inequality $\mu^{*G}_R \geqq \bar{\mu}^G_R$. Therefore many situations become very complicated and troublesome. Thus we start from a new view-point and redefine several operators in such a manner that they have various geometric or potential-theoretic meanings.

§1. Linear mappings in a general sets. Let G be a subdomain of W satisfying the two-sidedness condition. Let R be a subset of G whose connected components cluster at the ideal boundary of G. Let U be any open set with two-sidedness condition containing R and V be any open set with two-sidedness condition contained in R. Then we define outer operators and inner operators in the following manner:

$$\bar{\Lambda}^G_R = \inf_U \bar{\lambda}^G_U,\ \Lambda^{*G}_R = \inf_U \lambda^{*G}_U,\ M^{*G}_R = \inf_U \mu^{*G}_U,\ \bar{M}^G_R = \inf_U \bar{\mu}^G_U$$

and

$$\underline{\Lambda}^G_R = \sup_V \bar{\lambda}^G_V,\ \Lambda_{*R}^G = \sup_V \lambda^{*G}_V,\ M_{*R}^G = \sup_V \mu^{*G}_V,\ \underline{M}^G_R = \sup_V \bar{\mu}^G_V .$$

Evidently we have their existence and the inequalities

$$\begin{array}{ccccccc} \bar{\Lambda}^G_R & \geqq & \Lambda^{*G}_R & \geqq & M^{*G}_R & \geqq & \bar{M}^G_R \\ \text{VII} & & \text{VII} & & \text{VII} & & \text{VII} \\ \underline{\Lambda}^G_R & \geqq & \Lambda_{*R}^G & \geqq & M_{*R}^G & \geqq & \underline{M}^G_R . \end{array}$$

THEOREM 33. *If R satisfies the two-sidedness condition and is open in G, then there hold $\bar{\Lambda}^G_{\bar{R}} = \bar{\Lambda}^G_R = \bar{\lambda}^G_R$, $\Lambda^{*G}_{\bar{R}} = \Lambda^{*G}_R = \lambda^{*G}_R$, $M^{*G}_{\bar{R}} = M^{*G}_R = \mu^{*G}_R$, $\bar{M}^G_{\bar{R}} = \bar{M}^G_R = \bar{\mu}^G_R$.*

Proof. Any open set U with two-sidedness condition containing R is a member of open sets with two-sidedness condition containing R. Thus we have

$$\bar{\Lambda}^G_{\bar{R}} = \inf_{U \supset \bar{R}} \bar{\lambda}^G_U \geqq \inf_{U \supset R} \bar{\lambda}^G_R = \bar{\Lambda}^G_R .$$

Further R is an open set with two-sidedness condition containing R. Therefore there holds $\bar{\Lambda}^G_R = \bar{\lambda}^G_R$. By the modification theorem we can construct an open set U

with two-sidedness condition containing $\bar{R}$ for which $\varDelta_{1,\bar{R}}^{G}=\varDelta_{1,\mathring{\bar{R}}}^{G}=\varDelta_{1,\bar{U}}^{G}$ and $\varDelta_{1,G-\bar{R}}^{G}=\varDelta_{1,G-R}^{G}=\varDelta_{1,G-\bar{U}}^{G}$. This implies that $\bar{\lambda}_{U}^{G}=\bar{\lambda}_{R}^{G}$ and $\bar{\lambda}_{G-\bar{U}}^{G}=\bar{\lambda}_{G-\bar{R}}^{G}$, and hence $\bar{\varLambda}_{\bar{R}}^{G}\leqq\bar{\lambda}_{R}^{G}$ and $\bar{M}_{\bar{R}}^{G}\leqq\bar{\mu}_{R}^{G}$. Thus we have the desired result:

$$\bar{\varLambda}_{\bar{R}}^{G}=\bar{\varLambda}_{R}^{G}=\bar{\lambda}_{R}^{G}.$$

Further evidently there holds $\bar{M}_{\bar{R}}^{G}\geqq\bar{\mu}_{R}^{G}=\bar{M}_{R}^{G}$, which implies the desired result:

$$\bar{M}_{\bar{R}}^{G}=\bar{M}_{R}^{G}=\bar{\mu}_{R}^{G}.$$

In our case we have $T_{\bar{R}}^{G}u=T_{R}^{G}u$ and hence $\mu^{*}{}_{\bar{R}}^{G}u=\mu^{*}{}_{R}^{G}u$ for any $u\in PH(G)$. Then we can construct an open set U with two-sidedness condition containing R for which $\varDelta_{1}^{G}(\bar{R})=\varDelta_{1}^{G}(U)$ and $\varDelta_{1}^{G}(G-R)=\varDelta_{1}^{G}(G-\bar{U})$. This implies that $\mu^{*}{}_{U}^{G}=\mu^{*}{}_{\bar{R}}^{G}$ and $\mu^{*}{}_{G-R}^{G}=\mu^{*}{}_{G-\bar{U}}^{G}$ and hence

$$M^{*}{}_{\bar{R}}^{G}\leqq\mu^{*}{}_{\bar{R}}^{G}=\mu^{*}{}_{R}^{G}\leqq M^{*}{}_{R}^{G} \text{ and } \varLambda^{*}{}_{\bar{R}}^{G}\leqq\lambda^{*}{}_{\bar{R}}^{G}=\lambda^{*}{}_{R}^{G}\leqq\varLambda^{*}{}_{R}^{G}.$$

Thus we have the desired results:

$$M^{*}{}_{\bar{R}}^{G}=\mu^{*}{}_{R}^{G}=M^{*}{}_{R}^{G},\qquad \varLambda^{*}{}_{\bar{R}}^{G}=\lambda^{*}{}_{R}^{G}=\varLambda^{*}{}_{R}^{G}.$$

THEOREM 34. *If R satisfies the two-sidedness condition, then there hold* $\bar{\varLambda}_{R}^{G}=\underline{\varLambda}_{R}^{G}\geqq\varLambda^{*}{}_{R}^{G}=\varLambda_{*}{}_{R}^{G}\geqq M^{*}{}_{R}^{G}=M_{*}{}_{R}^{G}\geqq\bar{M}_{R}^{G}=\underline{M}_{R}^{G}$.

Proof. If R satisfies the two-sidedness condition, then the closure $\bar{R}$ and the open kernel $\mathring{R}$ of R satisfy the same condition. And we have

$$\bar{\varLambda}_{\bar{R}}^{G}\geqq\bar{\varLambda}_{R}^{G}\geqq\underline{\varLambda}_{R}^{G}\geqq\bar{\varLambda}_{\mathring{R}}^{G}=\bar{\lambda}_{\mathring{R}}^{G}.$$

On the other hand there holds $\bar{\varLambda}_{\bar{R}}^{G}=\bar{\varLambda}_{\mathring{R}}^{G}$ by the preceding theorem. Therefore

$$\bar{\varLambda}_{\bar{R}}^{G}=\bar{\varLambda}_{R}^{G}=\underline{\varLambda}_{R}^{G}=\bar{\varLambda}_{\mathring{R}}^{G}.$$

The remaining equalities are quite similarly established and hence the inequalities reduce to the trivial ones.

By this theorem it is reasonable to define the following notion: If $\bar{\varLambda}_{R}^{G}=\underline{\varLambda}_{R}^{G}$, $\varLambda^{*}{}_{R}^{G}=\varLambda_{*}{}_{R}^{G}$, $M^{*}{}_{R}^{G}=M_{*}{}_{R}^{G}$ and $\bar{M}_{R}^{G}=\underline{M}_{R}^{G}$, then we say that R is *almost two-sided* in G.

We shall here formulate two more general modification theorems.

THEOREM 35. *Let R be a subset in G such that $\mathring{\bar{R}}=\mathring{R}$ in G. Then there is an open set R' with two-sidedness condition in G which satisfies the following conditions*:

(1) *$R'\supset R$ and $\partial R'\frown\partial R=\phi$ excepting the points on $\partial G\frown\partial R$,*

(2) $\varDelta_{1,R'}^{G}=\varDelta_{1,\mathring{R}}^{G}=\varDelta_{1,R}^{G}$, $\varDelta_{1,G-\bar{R}'}^{G}=\varDelta_{1,G-\bar{R}}^{G}=\varDelta_{1,G-R}^{G}$ *and*

(3) $\varDelta_{1}^{G}(R')=\varDelta_{1}^{G}(\bar{R})=\varDelta_{1}^{G}(R)$, $\varDelta_{1}^{G}(G-\bar{R}')=\varDelta_{1}^{G}(G-\bar{R})=\varDelta_{1}^{G}(G-R)$.

THEOREM 36. *Let R be a subset in G such that $\bar{R}=\bar{X}$, $X=\mathring{\bar{R}}$ in G. Then there is an open set R'' with two-sidedness condition in G which satisfies the following conditions*:

(1) *$R''\subset R$ and $\partial R''\frown\partial R=\phi$ excepting the points on $\partial G\frown\partial R$,*

(2) $\Delta_{1,R''}^{G}=\Delta_{1,R}^{G}=\Delta_{1,\mathring{R}}^{G}$, $\Delta_{1,G-\bar{R}''}^{G}=\Delta_{1,G-\bar{R}}^{G}=\Delta_{1,G-\bar{\mathring{R}}}^{G}$, *and*

(3) $\Delta_1^G(R'')=\Delta_1^G(R)=\Delta_1^G(\mathring{R})$, $\Delta_1^G(G-\bar{R}'')=\Delta_1^G(G-\bar{R})=\Delta_1^G(G-\bar{\mathring{R}})$.

In order to do the proof of these it should be here remarked that a fact $\bar{\mathring{R}}=\mathring{R}$ in G is equivalent to a fact that there is no inner slit boundary of R, that is, the part of ∂R which does not contain any accumulation point of the set $(G-R)^\circ$ and a fact $\bar{R}=\bar{X}$, $X=\bar{\mathring{R}}$ in G is equivalent to a fact that there is no outer slit part of ∂R, that is, the part of ∂R which contains no accumulation point of the set $\mathring{R}$. Thus we can proceed to the proof in a quite similar manner as in the earlier modification theorem.

In the sequel we always denote X_R and Y_R as the open kernel of the closure of R and the closure of the open kernel of R, respectively. Now we define the first open kernel operators as follows:

$$\underset{\circ}{\Lambda}{}_R^G=\bar{\Lambda}_{\mathring{R}}^G,\ \underset{\circ}{\Lambda}{}^{*G}_R=\Lambda^{*G}_{\mathring{R}},\ \underset{\circ}{M}{}^{*G}_R=M^{*G}_{\mathring{R}},\ \underset{\circ}{M}{}_R^G=\ {}^G_{\mathring{R}}.$$

Further we define the second open kernel operators as follows:

$$\mathring{\Lambda}_R^G=\underline{\Lambda}_X^G,\ \mathring{\Lambda}^{*G}_R=\Lambda^{*G}_{X_R},\ \mathring{M}^{*G}_R=M^{*G}_{X_R},\ \mathring{M}_R^G=\bar{M}_{X_R}^G.$$

Then we have evidently the following inequalities:

$$\begin{array}{ccccccc}\bar{\Lambda}_R^G & \geqq & \Lambda^{*G}_R & \geqq & M^{*G}_R & \geqq & \bar{M}_R^G\\ \text{\rotatebox{90}{$\geqq$}} & & \text{\rotatebox{90}{$\geqq$}} & & \text{\rotatebox{90}{$\geqq$}} & & \text{\rotatebox{90}{$\geqq$}}\\ \mathring{\Lambda}_R^G & \geqq & \mathring{\Lambda}^{*G}_R & \geqq & \mathring{M}^{*G}_R & \geqq & \mathring{M}_R^G\\ \text{\rotatebox{90}{$\geqq$}} & & \text{\rotatebox{90}{$\geqq$}} & & \text{\rotatebox{90}{$\geqq$}} & & \text{\rotatebox{90}{$\geqq$}}\\ \underset{\circ}{\Lambda}{}_R^G & \geqq & \underset{\circ}{\Lambda}{}^{*G}_R & \geqq & \underset{\circ}{M}{}^{*G}_R & \geqq & \underset{\circ}{M}{}_R^G.\end{array}$$

THEOREM 37. *If $X_R=\mathring{R}$, then the first open kernel operators and the second open kernel operators and the inner operators coincide with each others, respectively, that is,* $\mathring{\Lambda}_R^G=\underline{\Lambda}_R^G=\underset{\circ}{\Lambda}{}_R^G$, $\mathring{\Lambda}^{*G}_R=\Lambda_{*R}^G=\underset{\circ}{\Lambda}{}^{*G}_R$, $\mathring{M}^{*G}_R=M_{*R}^G=\underset{\circ}{M}{}_{*R}^G$, $\mathring{M}_R^G=\underline{M}_R^G=\underset{\circ}{M}{}_R^G$.

Proof. As is already remarked, the condition $X_R=\mathring{R}$ is equivalent to a fact that there is no inner slit boundary of R. Further $\mathring{R}$ is an open set with no outer slit boundary. Therefore any open set contained in R is always contained in the closure of $\mathring{R}$ in this case. Then we have

$$\bar{\Lambda}_{\bar{X}_R}^G=\bar{\Lambda}_{Y_R}^G\geqq\underline{\Lambda}_R^G\geqq\bar{\Lambda}_{X_R}^G=\bar{\Lambda}_{\mathring{R}}^G,$$

$$\Lambda^{*G}_{\bar{X}_R}=\Lambda^{*G}_{Y_R}\geqq\Lambda_{*R}^G\geqq\Lambda_{X_R}^G=\Lambda^{*G}_{\mathring{R}},$$

$$M^{*G}_{\bar{X}_R}=M^{*G}_{Y_R}\geqq M_{*R}^G\geqq M^{*G}_{X_R}=M^{*G}_{\mathring{R}},$$

$$\bar{M}_{\bar{X}_R}^G=\bar{M}_{Y_R}^G\geqq M_R^G\geqq\bar{M}_{X_R}^G=\bar{M}_{\mathring{R}}^G.$$

Now we can apply the modification theorem 35 to $\mathring{R}$ and $\bar{X}_R$ and then we have the following fact: there is an open set $U\supset\mathring{R}$ with two-sidedness condition in G for which $\Delta_{1,U}^{G} = \Delta_{1,\mathring{R}}^{G}$, $\Delta_{1,G-\bar{U}}^{G} = \Delta_{1,G-Y_R}^{G}$, $\Delta_1^G(U) = \Delta_1^G(\mathring{R})$, $\Delta_1^G(G-\bar{U}) = \Delta_1^G(G-Y_R)$ and $\partial U \frown \partial\mathring{R} = \phi$ excepting the points on $\partial G \frown \partial\mathring{R}$. Since $\partial\bar{X}_R = \partial\mathring{R}$, we can say that $\mathring{R} \subset \bar{X}_R \subset U$. Thus we have $\Delta_{1,U}^{G} = \Delta_{1,\bar{X}_R}^{G} = \Delta_{1,\mathring{R}}^{G}$, $\Delta_1^G(U) = \Delta_1^G(\bar{X}_R) = \Delta_1^G(\mathring{R})$, $\Delta_1^G(G-\bar{U}) = \Delta_1^G(G-Y_R) = \Delta_1^G(G-\bar{X}_R)$, $\Delta_{1,G-\bar{U}}^{G} = \Delta_{1,G-Y_R}^{G} = \Delta_{1,G-\bar{X}_R}^{G}$. Therefore we have

$$\bar{\Lambda}_{\mathring{R}}^{G} \leqq \bar{\Lambda}_{\bar{X}_R}^{G} \leqq \bar{\lambda}_{U}^{G} \leqq \bar{\Lambda}_{\mathring{R}}^{G}, \quad \Lambda^{*G}_{\mathring{R}} \leqq \Lambda^{*G}_{\bar{X}_R} \leqq \lambda^{*G}_{U} \leqq \Lambda^{*G}_{\mathring{R}},$$

$$M^{*G}_{\mathring{R}} \leqq M^{*G}_{\bar{X}_R} \leqq \mu^{*G}_{U} \leqq M^{*G}_{\mathring{R}}, \quad \bar{M}_{\mathring{R}}^{G} \leqq \bar{M}_{\bar{X}_R}^{G} \leqq \bar{\mu}_{U}^{G} \leqq \bar{M}_{\mathring{R}}^{G}.$$

These lead to the desired results.

THEOREM 38. *If* $\bar{R} = \bar{X}_R$, *then* $\mathring{\Lambda}_{R}^{G} = \bar{\Lambda}_{R}^{G} = \bar{\Lambda}_{\bar{R}}^{G} = \underset{\circ}{\Lambda}{}_{R}^{G} \geqq \underline{\Lambda}_{R}^{G} = \underline{\Lambda}_{\mathring{R}}^{G}$, $\mathring{\Lambda}^{*G}_{R} = \Lambda^{*G}_{R} = \Lambda^{*G}_{\bar{R}} = \underset{\circ}{\Lambda}{}^{*G}_{R} \geqq \Lambda_{*R}^{G} = \Lambda_{*\mathring{R}}^{G}$, $\mathring{M}^{*G}_{R} = M^{*G}_{R} = M^{*G}_{\bar{R}} = \underset{\circ}{M}{}^{*G}_{R} \geqq M_{*R}^{G} = M_{*\mathring{R}}^{G}$, $\mathring{M}_{R}^{G} = \bar{M}_{R}^{G} = \bar{M}_{\bar{R}}^{G} = \underset{\circ}{M}{}_{R}^{G} \geqq \underline{M}_{R}^{G} = \underline{M}_{\mathring{R}}^{G}$.

Proof. As is already remarked, the condition $\bar{R} = \bar{X}_R$ is equivalent to a fact that there is no outer slit boundary part. Thus X_R and hence $\bar{R}$ are sets with no outer and inner slit boundary part. Therefore any open set with two-sidedness condition containing R contains $\bar{X}_R \supset R \supset X_R$. Thus we have

$$\bar{\Lambda}_{\bar{R}}^{G} \geqq \bar{\Lambda}_{\bar{X}_R}^{G} \geqq \bar{\Lambda}_{R}^{G} \geqq \bar{\Lambda}_{X_R}^{G}, \quad \Lambda^{*G}_{\bar{R}} \geqq \Lambda^{*G}_{\bar{X}} \geqq \Lambda^{*G}_{R} \geqq \Lambda^{*G}_{X_R},$$

$$M^{*G}_{\bar{R}} \geqq M^{*G}_{\bar{X}_R} \geqq M^{*G}_{R} \geqq M^{*G}_{X_R}, \quad \bar{M}_{\bar{R}}^{G} \geqq \bar{M}_{\bar{X}_R}^{G} \geqq \bar{M}_{R}^{G} \geqq \bar{M}_{X_R}^{G}.$$

On the other hand $\partial X_R = \partial R = \partial\bar{X}_R = \partial\bar{R}$. By the modification theorem we can construct an open set $U \supset X_R$ for which $\partial U \frown \partial X_R = \phi$ in G and $\Delta_{1,U}^{G} = \Delta_{1,X_R}^{G}$, $\Delta_{1,G-\bar{U}}^{G} = \Delta_{1,G-\bar{X}_R}^{G}$, $\Delta_1^G(U) = \Delta_1^G(X_R)$, $\Delta_1^G(G-\bar{U}) = \Delta_1^G(G-\bar{X}_R)$. Further U satisfies the two-sidedness condition. Thus we have the first half of the desired equalities. Further Y_R coincides with $\bar{R}$. Then any open set containing $Y_R(\mathring{R})$ and satisfying two-sidedness condition in G contains $\bar{R}(X_R)$. We can construct such an open set U with two-sidedness condition in G that U contains Y_R and $\mathring{R}$ and $\partial U \frown \partial\bar{R} = \phi$ and $\Delta_{1,U}^{G} = \Delta_{1,Y_R}^{G}$, $\Delta_{1,G-\bar{U}}^{G} = \Delta_{1,G-Y_R}^{G}$, $\Delta_1^G(U) = \Delta_1^G(Y_R)$, $\Delta_1^G(G-\bar{U}) = \Delta_1^G(G-Y_R)$. This implies $U \supseteqq \bar{R} = \bar{X}_R \supset R \supset X_R$ and hence we have the second half of the equalities. The last equalities are also obtained by the modification theorem similarly.

THEOREM 39. *Let R be a general set in G, then*

$$\bar{\Lambda}_{R}^{G} = \bar{\Lambda}_{\bar{R}}^{G} \geqq \mathring{\Lambda}_{R}^{G} \geqq \underset{\circ}{\Lambda}{}_{R}^{G} \geqq \underline{\Lambda}_{R}^{G} = \underline{\Lambda}_{\mathring{R}}^{G}.$$

Proof. Let q be a point of $\Delta_{1,\bar{R}}^{G}$, then there exists a point-sequence $\{q_n\}$ such that $q_n \in \bar{R}$, $d_G(q_n, q) \to 0$ $(n \to \infty)$. For each n there exists a point $q_n' \in R$ for which $d_G(q_n', q_n) < \varepsilon_n$ and $\varepsilon_n \to 0$ $(n \to \infty)$. Then by the triangular inequality

$$d_G(q_n', q) \leqq d_G(q_n', q_n) + d_G(q_n, q) \to 0 \ (n \to \infty).$$

This shows that $q \in \Delta_{1,R}^{G} \subset \Delta_{1,U}^{G}$ for any $U \supset R$ with two-sidedness condition. Therefore we have

$$\bigcap_{U \supset R} \Delta_{1,U}^{G} \supseteqq \Delta_{1,\bar{R}}^{G}.$$

This implies the following inequality:

$$\bar{\lambda}^G_R = \inf_{U \supset R} \bar{\lambda}^G_U \geqq \bar{\lambda}^G_{\bar{R}}.$$

It is evident that $\bar{\Lambda}^G_R \leqq \bar{\Lambda}^G_{\bar{R}}$. On the other hand we can construct an open set U with two-sidedness condition containing $\bar{R}$ such that $\Delta_{1,\bar{R}}^G = \Delta_{1,U}^G$ and $\partial U \frown \partial \bar{R} = \phi$ in G. This implies that $\bar{\lambda}^G_{\bar{R}} = \bar{\lambda}^G_U \geqq \bar{\Lambda}^G_R$. This shows that $\bar{\Lambda}^G_R = \bar{\Lambda}^G_{\bar{R}}$. In a way we have

$$\bigcap_{U \supset R} \Delta_{1,U}^G = \Delta_{1,\bar{R}}^G.$$

The set $R - \mathring{R}$ consists of a set of outer slits of R and a part of ∂R. Since any component of outer slits has no inner point, any open set U contained in R has not its component in that component of outer slits. Therefore any open set V containing $\mathring{R}$ contains any open set U contained in R. This implies that

$$\underset{\circ}{\underline{\Lambda}}{}^G_R = \Lambda^G_{\mathring{R}} \geqq \underline{\Lambda}^G_R$$

Further we can conclude by the modification theorem applied to R that $\underline{\Lambda}^G_R = \underline{\Lambda}^G_{\mathring{R}}$. In a way we have

$$\bigcup \Delta_{1,U}^G = \Delta_{1,\mathring{R}}^G,$$

where U runs through all the open set contained in R.

THEOREM 40. $M^{*G}_R = \mathring{M}^{*G}_R = M^{*G}_{\bar{R}} \geqq \underset{\circ}{M}{}^{*G}_R \geqq M_{*R}^G = M_{*\mathring{R}}^G$.

Proof. The set $\bar{R} - \bar{X}_R$, $X_R = \mathring{\bar{R}}$, consists of a number of outer slits lying in G. Let γ be the part of outer slits of $\bar{R}$. For X_R we can apply the modification theorem and we can say that there exists a suitable open set U with two-sidedness condition containing X_R for which $\Delta_1^G(X_R) = \Delta_1^G(U)$ and $\partial X_R \frown \partial U = \phi$ in G. Let γ_U be a part of γ not lying in U. For any $u \in PH(G)$ we construct a positive harmonic function u_n in $W_n \frown (U + \gamma_U)$ with boundary value u on $\partial W_n \frown (U + \gamma)$ and 0 on $\partial U \frown W_n + \gamma_U \frown W_n$. There is no such function if γ_U exists. So we conventionally say u_n positive harmonic. Then u_n tends to a positive function $T^G_{U+\gamma_U} u$ or zero function which is harmonic in X_R and vanishes identically on γ_U. This limit function coincides with $T^G_U u$ in X_R. This implies that γ (or γ_U) has no effect for any T mappings. Thus we have $\Delta_1^G(\bar{R}) = \Delta_1^G(\bar{X}_R) = \Delta_1^G(X_R) = \Delta_1^G(U)$. Therefore we can say that

$$\mu^{*G}_U = M^{*G}_{X_R} = M^{*G}_{\bar{X}_R} = \mu^{*G}_{\bar{R}}.$$

Similarly we can say that there exists an open set U' with two-sidedness condition containing R for which $\Delta_1^G(U') = \Delta_1^G(R)$ and $U' - U$ has no effect for any T mapping. This is done by a slightly modified method of proof of the modification theorem. Then we have

$$\mu^{*G}_{\bar{R}} = \mu^{*G}_{U'} \geqq M^{*G}_{\bar{R}} \geqq \mu^{*G}_{\bar{R}}.$$

Any open set containing $\mathring{R}$ contains any open set contained in R. Thus we have

$\underset{\circ}{M}{}^{*G}_{R} \geqq M_{*}{}^{G}_{R}$. Any open set contained in R is contained in $\overset{\circ}{R}$. Thus we have $M_{*}{}^{G}_{\overset{\circ}{R}} \geqq M_{*}{}^{G}_{R}$. Since $R \supset \overset{\circ}{R}$, we have $M_{*}{}^{G}_{\overset{\circ}{R}} \leqq M_{*}{}^{G}_{R}$. Thus we have $M_{*}{}^{G}_{\overset{\circ}{R}} = M_{*}{}^{G}_{R}$. The remaining inequality is evidently valid by the definition. In a way we have

$$\bigcap_{U\supset R} \Delta_1^G(U) = \Delta_1^G(R) = \Delta_1^G(X_R) \text{ and } \bigcup_{U\subset R} \Delta_1^G(U) = \Delta_1^G(R).$$

By the above analysis we have

$$\bigcap_{U\supset R}(\Delta_1^G - \Delta_1^G(G-\bar{U})) = \Delta_1^G - \bigcup_{U\supset R}\Delta_1^G(G-\bar{U}) = \Delta_1^G - \bigcup_{V\subset G-R}\Delta_1^G(V) = \Delta_1^G - \Delta_1^G((G-R)^\circ),$$

$$\bigcap_{U\supset \bar{R}}(\Delta_1^G - \Delta_1^G(G-\bar{U})) = \Delta_1^G - \bigcup_{V\subset G-\bar{R}}\Delta_1^G(V) = \Delta_1^G - \Delta_1^G((G-\bar{R})^\circ) = \Delta_1^G - \Delta_1^G(G-\bar{R}).$$

On the other hand we have $(G-R)^\circ = R^{cebc} = R^{bc} = G-\bar{R}$, where c and b are symbols showing $G-R=R^c$ and $\bar{R}=R^b$. Therefore we have $\Delta_1^G(G-\bar{R}) = \Delta_1^G((G-R)^\circ)$. This shows that

$$\Lambda^{*G}_{R} = I - M^{*G}_{(G-R)^\circ} = I - M_{*}{}^{G}_{(G-\bar{R})^\circ} = \Lambda^{*G}_{\bar{R}}.$$

Further we see that $R^{bc} \subset R^{bcb}$ implies $R^{bc} \subset R^{bcbcbc}$. Further $R^b \supset R$ implies $R^{bc} \subset R^{bcbcbc} \subset R^{cbcbc} \subset R^{bc}$, since $R^{cbc} = R^\circ \subset R$. Therefore we can say that $R^{bc} = R^{bcbcbc}$, that is, $G - \bar{R} = \overset{\circ}{\overline{(G-\bar{R})}} = X_{G-\bar{R}}$. This shows that

$$\Delta_1^G - \Delta_1^G(G-\bar{R}) = \Delta_1^G - \Delta_1^G(x_{G-\bar{R}}).$$

$$= \Delta_1^G - \bigcup_{V\subset X_{G-\bar{R}}} \Delta_1^G(V) = \bigcap_{U\subset X_R}(\Delta_1^G - \Delta_1^G(G-\bar{U})),$$

whence follows that $\overset{\circ}{\Lambda}{}^{*G}_{R} = \Lambda^{*G}_{X_R} = \Lambda^{*G}_{\bar{R}}$. Since there hold

$$\underset{\circ}{\Lambda}{}^{*G}_{R} = \Lambda^{*G}_{\overset{\circ}{R}} = \inf_{U\supset \overset{\circ}{R}} \lambda^{*G}_{U} = I - \sup_{U\supset \overset{\circ}{R}} \mu^{*G}_{G-\bar{U}} = I - \sup_{V\subset G-\overset{\circ}{R}} \mu^{*G}_{V}$$

$$= I - M_{*}{}^{G}_{G-\overset{\circ}{R}} = I - M_{*}{}^{G}_{\overline{G-R}},$$

$$\Lambda_{*}{}^{G}_{R} = \sup_{U\subset R} \lambda^{*G}_{U} = I - \inf_{U\subset R} \mu^{*G}_{G-\bar{U}} = I - \inf_{U\subset R} \mu^{*G}_{(G-U)^\circ}.$$

$$= I - \inf_{V\supset (G-R)^\circ} \mu^{*G}_{V} = I - \underset{\circ}{M}{}^{*G}_{G-R}$$

and

$$M_{*}{}^{G}_{G-\overset{\circ}{R}} \leqq \underset{\circ}{M}{}^{*G}_{G-\overset{\circ}{R}} \leqq \underset{\circ}{M}{}^{G}_{G-R},$$

we have

$$\underset{\circ}{\Lambda}{}^{*G}_{R} \geqq \Lambda_{*}{}^{G}_{R}.$$

Evidently we have

$$\Lambda_{*}{}^{G}_{R} = \Lambda_{*}{}^{G}_{\overset{\circ}{R}}.$$

THEOREM 41. $\Lambda^{*G}_{R} = \Lambda^{*G}_{\bar{R}} = \overset{\circ}{\Lambda}{}^{*G}_{R} \geqq \underset{\circ}{\Lambda}{}^{*G}_{R} \geqq \Lambda_{*}{}^{G}_{R} = \Lambda_{*}{}^{G}_{\overset{\circ}{R}}$.

Similarly we have the following theorem.

THEOREM 42. $\bar{M}^G_R = \bar{M}^G_{\bar{R}} \geqq \mathring{M}^G_R \geqq \underset{\circ}{M}^G_R \geqq \underline{M}^G_R = \underline{M}^G_{\mathring{R}}$.

The above four theorems clarify the situations of the former preparatory theorems.

§2. **The geometric, topological and potential-theoretic meanings.** By two theorems 37 and 38 it is reasonable to define that $\mathring{R}$ is *almost equal* to X_R in G or $\bar{R}$ is *almost equal* to $\bar{X}_R$ in G according as all the open kernel operators coincide with the inner operators or with the outer operators, respectively. Then we have the following fact: If $\mathring{R}$ is almost equal to X_R in G and simultaneously $\bar{R}$ is almost equal to $\bar{X}_R$ in G, then R is almost two-sided in G.

All the operators introduced above have their own geometric, topological or potential-theoretic meanings.

THEOREM 43. *If $X_R = \phi$, that is, $\bar{R}$ has no inner point, then all the inner operators vanish identically and further $M^{*G}_R = 0$. If $\mathring{R} = \phi$, that is, R has no inner point, then all the inner operators vanish identically.*

Proof. The assertions for all inner operators are evident by their definitions. Since $T^G_{\bar{R}} = 0$, we can say that $T^G_R = 0$ for a suitable open set with two-sidedness condition containing $\bar{R}$ by the modification theorem. This implies that $0 = \mu^{*G}_{\bar{U}} \geqq M^{*G}_{\bar{R}} = 0$.

THEOREM 44. *If $\mathring{R}$ is dispersive in G, that is, any connected component of $\mathring{R}$ is compact, then $M_{*}{}^G_{\bar{R}} = 0$. If $\bar{R}$ is dispersive in G, then $M^{*G}_R = 0$. If X_R is dispersive in G, then $M_{*}{}^G_R = 0$.*

Proof. If $\mathring{R}$ is dispersive in G, then any open set U with two-sidedness condition contained in $\mathring{R}$ is also dispersive in G. Then $T^G_{\bar{U}} = 0$ and hence $\mu^{*G}_{\bar{U}} = 0$. If $\bar{R}$ is dispersive in G, then there is an open set U with two-sidedness condition containing $\bar{R}$ which is also dispersive in G. Then $T^G_{\bar{U}} = 0$ and hence $\mu^{*G}_{\bar{U}} = 0$. Similarly we have the last assertion.

THEOREM 45. *If $\bar{R}$ clusters irregularly at Δ^G_1, then $\Lambda^{*G}_{\bar{R}} = 0$. If $\mathring{R}$ clusters irregularly at Δ^G_1, then $\Lambda_{*}{}^G_{\bar{R}} = 0$. If R irregularly clusters at Δ^G_1, then $\Lambda_{*}{}^G_R = 0$. If X_R irregularly clusters at Δ^G_1, then $\mathring{\Lambda}^{*G}_R = 0$.*

Proof. By the irregularity of $\bar{R}$ we have $T^G_{G-\bar{R}} u_q > 0$ for some $q \in \Delta^G_1$ and the corresponding minimal u_q. Then by the modification theorem we have $T^G_{G-\bar{U}} u_q > 0$ for some open set U with two-sidedness condition containing $\bar{R}$. This implies that $\mu^{*G}_{G-\bar{U}} u_q = u_q$ and hence $\lambda^{*G}_{\bar{U}} u_q = 0$. By the definition of $\Lambda^{*G}_{\bar{R}}$ we have $\Lambda^{*G}_{\bar{R}} = 0$.

If R can cluster only at $\Delta^G - \Delta^G_1$, then we say R a *non-effective set* in G.

THEOREM 46. *If $R(X_R$ or $\mathring{R})$ is non-effective in G, then $\bar{\Lambda}^G_R(\mathring{\Lambda}^G_R$ or $\underset{\circ}{\Lambda}^G_R)$ vanishes identically and vice versa.*

Further these results can be localized around a neighborhood of a single point

$q \in \Delta_1^G$. Let N_ε be a set defined by the condition $d_G(p, q) < \varepsilon$, $p \in G$. If we replace R by $N_\varepsilon \frown R$ we arrive at the similar localized results.

We can introduce several other notions concerning the geometric or potential-theoretic situations and further many other P-niceness of R in G and many other P-equivalency of R_1 and R_2 in G. We shall postpone the discussions and applications of almost all their parts.

Let γ be a curve tending to the ideal boundary of G. Let $\bar{\Lambda}_\gamma^G$ and Λ^{*G}_γ be two operators defined by

$$\bar{\Lambda}_\gamma^G = \inf_R \bar{\Lambda}_R^G \text{ and } \Lambda^{*G}_\gamma = \inf_R \lambda^{*G}_R,$$

respectively, where R is an arbitrary open set containing γ and satisfying the two-sidedness condition and the infimum is taken over all such domains.

If γ is a tangential path to a minimal point $q \in \Delta_1^G$, then $\Lambda^{*G}_\gamma =$ and $\bar{\Lambda}_\gamma^G \neq 0$. This is shown by the modification theorem and the definition of tangential path.

THEOREM 47. *If $\Lambda^{*G}_\gamma = 0$ and $\bar{\Lambda}_\gamma^G \neq 0$, then γ is tangential to any point in $\Delta_{1,\gamma}^{\;G}$.*

Proof. By the modification theorem we can construct an open set U with two-sidedness condition for which $\Delta_1^G(U) = \Delta_1^G(\gamma) = \phi$, $\Delta_1^G(G - \bar{U}) = \Delta_1^G(G - \gamma)$, $\Delta_{1,U}^{\;G} = \Delta_{1,\gamma}^{\;G}$ and $\Delta_{1,G-\bar{U}}^{\;G} = \Delta_{1,G-\gamma}^{\;G}$. Then by the assumption $\Lambda^{*G}_\gamma = 0$ we can say that $\Delta_1^G(G - \bar{U}) = \Delta_1^G(G - \gamma) = \Delta_1^G$. If q belongs to $\Delta_{1,\gamma}^{\;G}$, then $T_{G-\bar{U}}^G K_G(p, q) > 0$. Thus γ is tangential to q.

It should be here noted that γ is tangential to two minimal points simultaneously and hence it is not a tangential path, that is, γ does not determine a minimal point.

THEOREM 48. *There is no curve being tangential and approaching to any neighboring minimal point around a minimal point q if such a minimal point always exists in any neighborhood of q.*

Proof. If γ is tangential to a minimal point q and to any minimal point p, $p \neq q$, lying in $N_\varepsilon(q)$: $\{q \in \Delta_1^G, d_G(p, q) < \varepsilon\}$, then γ oscillates between p and q. Thus γ intersects almost all level curves $K_G(r, s)$ corresponding to a minimal points lying between p and q. This shows that γ is not tangential to s, which is a contradiction.

Chapter III. Linear mappings in *PHB*(*G*).

In this chapter we shall apply our results in Chapter I to the class $PHB(G)$ and then we shall obtain several sharpened results. In a general case T_R^G mapping is an into positively linear mapping from $PH(G)$ to $PH(R)$, however its restriction to the class $PHB(G)$ is an onto mapping from $PHB(G)$ to $PHB(R)$. Thus S_R^G mapping is an into univalent mapping from $PHB(R)$ to $PHB(G)$. This difference brings us some essential effects to our results. Further in PHB(G) there is a sort of canonical functions such as the harmonic measure.

Evidently we have

$$\lambda_R^G u(p) \equiv \lambda^*{}_R^G u(p) = \int_{\Delta_1^G - \Delta_1^G(G-\bar{R})} K_G(p, q)\ d\sigma_u(q)$$

and

$$\mu_R^G u(p) \equiv \mu^*{}_R^G u(p) = \int_{\Delta_1^G(R)} K_G(p, q)\ d\sigma_u(q)$$

for any $u \in PHB(G)$, when $u(p)$ is represented canonically by

$$u(p) = \int_{\Delta_1^G} K_G(p, q)\ d\sigma_u(q)\,.$$

We shall make use of these notations λ and μ instead of λ^* and μ^* through this chapter and we do not discuss the $^-$ mappings in this chapter, though there are many problems to be treated. Through this chapter we make use of a notation ω_G as the usally availed standard harmonic measure in G. This may be the constant 1, if $G \in O_{HB}$.

§1. ***B*-niceness and *B*-equivalency.** We can classify the relative situation of R in G in reference to the harmonic measure ω_G as in the following figures.

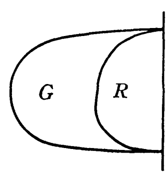


Fig. 13 $\omega = \lambda\omega = \mu\omega > 0$

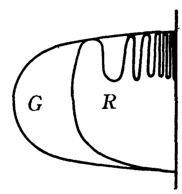


Fig. 14 $\omega = \lambda\omega > \mu\omega > 0$

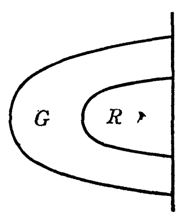


Fig. 15 $\omega > \lambda\omega = \mu\omega > 0$

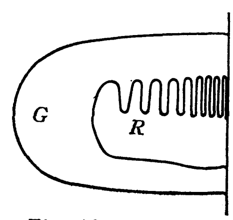


Fig. 16 $\omega > \lambda\omega > \mu\omega > 0$

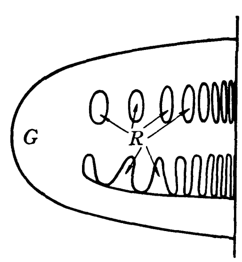


Fig. 17 $\omega > \lambda\omega > \mu\omega = 0$

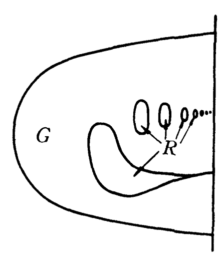


Fig. 18 $\omega > \lambda\omega = \mu\omega = 0$

If $\lambda_R^G u = \mu_R^G u$ for any $u \in PHB(G)$, then we say R a *B-nice subset* in G. If this holds for any $G \supset R$, then we say R an *extremally B-nice subset.*

If $\lambda_{R_1}^G u = \lambda_{R_2}^G u$ and $\mu_{R_1}^G u = \mu_{R_2}^G u$ for any $u \in PHB(G)$ and for given two subsets R_1 and R_2 of G, then we say that R_1 and R_2 are *B-equivalent* in G. If this equivalency holds for any $G \supset R_1 \cup R_2$, then we say that R_1 and R_2 are *perfectly B-equivalent.* We denote this $R_1 \overset{B}{\sim} R_2$. These notions of equivalency of two subsets R_1 and R_2 are different from the following fact: $PHB(R_1)$ is equivalent to $PHB(R_2)$ as linear spaces.

We should mention here two examples by figures. The first one shows that R is a B-nice subset in G but it is not extremally B-nice. The second one shows that R_1 and R_2 are B-equivalent in G but they are not perfectly B-equivalent.

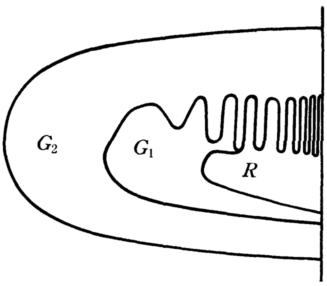


Fig. 19

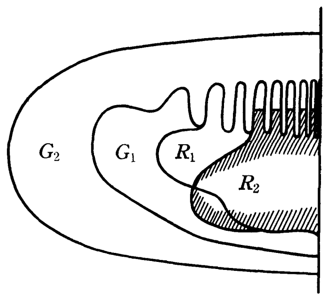


Fig. 20

$\lambda_R^{G_1} u = \mu_R^{G_1} u$ for any $u \in PHB(G_1)$ but there is a member u of $PHB(G_2)$ for which $\lambda_R^{G_2} u > \mu_R^{G_2} u$, for example the harmonic measure ω_{G_2}.

$\lambda_{R_1}^{G_1} u = \lambda_{R_2}^{G_1} u$, $\mu_{R_1}^{G_1} u = \mu_{R_2}^{G_1} u$ for any $u \in PHB(G_1)$ but there is an element u in $PHB(G_2)$ such that $\lambda_{R_1}^{G_2} u > \lambda_{R_2}^{G_2} u$, $\mu_{R_1}^{G_2} u = \mu_{R_2}^{G_2} u$, for example the harmonic measure ω_{G_2}.

If R is P-nice in G, then R is B-nice in G. However the inverse is not true in general. This is shown by Fig. 14. Further we can see by an example which is easy to construct that R_1 and R_2 are not P-equivalent in G even if they are B-equivalent in G.

§2. **Algebraic calculations.** In this section we shall list several algebraic properties of two mappings λ and μ.

(1) $\lambda_R^G u + \mu_{G-\bar{R}}^G u = u$, $\lambda_R^G u \wedge \mu_{G-\bar{R}}^G u \equiv$ G.H.M. $\min(\lambda_R^G u, \mu_{G-\bar{R}}^G u) = 0$ for any $u \in PHB(G)$.

(2) $T_{G-\bar{R}}^G \lambda_R^G u = T_{G-\bar{R}}^G \mu_R^G u = 0$ for any $u \in PHB(G)$.

(3) $\mu_R^G \lambda_R^G u = \lambda_R^G \mu_R^G u = \mu_R^G \mu_R^G u = \mu_R^G u$ and $\lambda_R^G \lambda_R^G u = \lambda_R^G u$ for each $u \in PHB(G)$.

(4) $\mu_{G-\bar{R}}^G \mu_R^G u = \mu_{G-\bar{R}}^G \lambda_R^G u = \lambda_R^G \mu_{G-\bar{R}}^G u = \lambda_{G-\bar{R}}^G \mu_R^G u = \mu_R^G \lambda_{G-\bar{R}}^G u = 0$ and $\lambda_{G-\bar{R}}^G \lambda_R^G u = \lambda_R^G \lambda_{G-\bar{R}}^G u$

$= \lambda^{G}_{R} u - \mu^{G}_{R} u = \lambda^{G}_{G-\bar{R}} u - \mu^{G}_{G-\bar{R}} u$ for each $u \in PHB(G)$.

(5) Let G_1 and $G_2(\supset G_1)$ be two subregions of W containing R, then for each $u \in PHB(G_1)$ we have

$$T^{G_2}_{G_1} \mu^{G_2}_{R} S^{G_2}_{G_1} u = \mu^{G_1}_{R} u \text{ and } T^{G_2}_{G_1} \lambda^{G_2}_{R} S^{G_2}_{G_1} u = \lambda^{G_1}_{R} u .$$

And for each $v \in PHB(G_2)$ we have

$$\mu^{G_2}_{R} v = S^{G_2}_{G_1} \mu^{G_1}_{R} T^{G_2}_{G_1} v \text{ and } \lambda^{G_2}_{R} v \geqq S^{G_2}_{G_1} \lambda^{G_1}_{R} T^{G_2}_{G_1} v .$$

Proof.

$$T^{G_2}_{G_1} \mu^{G_2}_{R} S^{G_2}_{G_1} = T^{G_2}_{G_1} S^{G_2}_{R} T^{G_1}_{R} T^{G_2}_{G_1} S^{G_2}_{G_1}$$

$$= T^{G_2}_{G_1} S^{G_2}_{G_1} S^{G_1}_{R} T^{G_1}_{R} = S^{G_1}_{R} T^{G_1}_{R} = \mu^{G_1}_{R} .$$

$$S^{G_2}_{G_1} \mu^{G_1}_{R} T^{G_2}_{G_1} = S^{G_2}_{G_1} S^{G_1}_{R} T^{G_1}_{R} T^{G_2}_{G_1} = S^{G_2}_{R} T^{G_2}_{R} = \mu^{G_2}_{R} .$$

By the definition of $S^{G_2}_{G_1}$ we have $S^{G_2}_{G_1} u \geqq u$ in G_1 and hence $\lambda^{G_2}_{R} S^{G_2}_{G_1} u \geqq \lambda^{G_1}_{R} u$ in G_1. This implies that $T^{G_2}_{G_1} \lambda^{G_2}_{R} S^{G_2}_{G_1} u \geqq \lambda^{G_1}_{R} u$ in G_1. On the other hand we can say the following relations by the first equality and by the monotoneity of μ

$$\lambda^{G_1}_{R} u = u - \mu^{G_1}_{G_1-\bar{R}} u = T^{G_2}_{G_1} S^{G_2}_{G_1} u - T^{G_2}_{G_1} \mu^{G_1}_{G_1-\bar{R}} S^{G_2}_{G_1} u$$

$$\geqq T^{G_2}_{G_1} S^{G_2}_{G_1} u - T^{G_2}_{G_1} \mu^{G_2}_{G_2-\bar{R}} S^{G_2}_{G_1} u = T^{G_2}_{G_1} \lambda^{G_2}_{R} S^{G_2}_{G_1} u$$

for any $u \in PHB(G)$. Thus we have the equality.

For any $v \in PHB(G_2)$ we have $v \geqq T^{G_2}_{G_1} v$ in G_1 and hence $\lambda^{G_2}_{R} v \geqq \lambda^{G_1}_{R} T^{G_2}_{G_1} v$ in G_1. Therefore we have $\lambda^{G_2}_{R} v \geqq S^{G_2}_{G_1} \lambda^{G_1}_{R} T^{G_2}_{G_1} v$ in G_2.

Evidently this (5) corresponds to (7) in Chap. I, §4. In that formulation the second relation is an inequality in general, however in this (5) the second relation is an equality. This difference occurs from a different point stated already at the top of this chapter.

(6) $\lambda^{G_2}_{R} \lambda^{G_2}_{G_1} u = \lambda^{G_2}_{R} u$, $\mu^{G_2}_{R} \lambda^{G_2}_{G_1} u = \mu^{G_2}_{R} \mu^{G_2}_{G_1} u = \mu^{G_2}_{R} u$ and $\mu^{G_2}_{R} u \leqq \lambda^{G_2}_{R} \mu^{G_2}_{G_1} u \leqq \lambda^{G_2}_{R} u$ for any $u \in PHB(G_2)$. If G_1 is B-nice in G_2 or more strongly G_1 is extremally B-nice and $G_1 \subset G_2$, then

$$\lambda^{G_2}_{R} \mu^{G_2}_{G_1} u = \lambda^{G_2}_{R} u .$$

(7) For any $u \in PHB(G_2)$ $\lambda^{G_2}_{G_1} \lambda^{G_2}_{R} u = \lambda^{G_2}_{R} u$, $\lambda^{G_2}_{G_1} \mu^{G_2}_{R} u = \mu^{G_2}_{G_1} \mu^{G_2}_{R} u = \mu^{G_2}_{R} u$ and $\lambda^{G_2}_{R} u \geqq \mu^{G_2}_{G_1} \lambda^{G_2}_{R} u \geqq \mu^{G_2}_{R} u$. If G_1 is B-nice in G_2, then $\lambda^{G_2}_{R} u = \mu^{G_2}_{G_1} \lambda^{G_2}_{R} u$ and $\lambda^{G_2}_{G_2-\bar{G}_1} \lambda^{G_2}_{R} u = 0$.

(8) If G_1 is B-nice in G_2, then for any $u \in PHB(G_2)$

$$\lambda^{G_2}_{R} u = S^{G_2}_{G_1} \lambda^{G_1}_{R} T^{G_2}_{G_1} u .$$

Proof. By (5) we have for any $u \in PHB(G_2)$

$$\lambda^{G_2}_{R} u \geqq S^{G_2}_{G_1} \lambda^{G_1}_{R} T^{G_2}_{G_1} u .$$

On the other hand by (6) and (7) we have

$$\lambda^{G_2}_{R} u = \lambda^{G_2}_{G_1} \lambda^{G_2}_{R} \lambda^{G_2}_{G_1} u = \mu^{G_2}_{G_1} \lambda^{G_2}_{R} \mu^{G_2}_{G_1} u = S^{G_2}_{G_1} T^{G_2}_{G_1} \lambda^{G_2}_{R} S^{G_2}_{G_1} T^{G_2}_{G_1} u$$

$$= S^{G_2}_{G_1} T^{G_2}_{G_1} S^{G_2}_{G_1} T^{G_2}_{G_1} u - S^{G_2}_{G_1} T^{G_2}_{G_1} \mu^{G_2}_{G_2 - \bar{R}} S^{G_2}_{G_1} T^{G_2}_{G_1} u$$
$$= S^{G_2}_{G_1} T^{G_2}_{G_1} u - S^{G_2}_{G_1} T^{G_2}_{G_1} \mu^{G_2}_{G_2 - \bar{R}} S^{G_2}_{G_1} T^{G_2}_{G_1} u .$$

Further since $\mu^{G_2}_{G_2 - \bar{R}} x \geqq \mu^{G_2}_{G_1 - \bar{R}} x$ for any $x \in PHB(G_2)$, we have

$$S^{G_2}_{G_1} T^{G_2}_{G_1} u^{G_2}_{G_2 - \bar{R}} S^{G_2}_{G_1} T^{G_2}_{G_1} u \geqq S^{G_2}_{G_1} T^{G_2}_{G_1} \mu^{G_2}_{G_1 - \bar{R}} S^{G_2}_{G_1} T^{G_2}_{G_1} u$$
$$= S^{G_2}_{G_1} \mu^{G_1}_{G_1 - \bar{R}} T^{G_2}_{G_1} u$$

by (5). Thus we have an inequality

$$\lambda^{G_2}_{R} u \leqq S^{G_2}_{G_1} T^{G_2}_{G_1} u - S^{G_2}_{G_1} \mu^{G_1}_{G_1 - \bar{R}} T^{G_2}_{G_1} u$$
$$= S^{G_2}_{G_1} \lambda^{G_1}_{R} T^{G_2}_{G_1} u .$$

Therefore we can say that the desired result $\lambda^{G_2}_{R} u = S^{G_2}_{G_1} \lambda^{G_1}_{R} T^{G_2}_{G_1} u$ holds for any $u \in PHB(G_2)$, if G_1 is B-nice in G_2.

It should be here remarked that in the inequality $\lambda^{G_2}_{R} u \geqq S^{G_2}_{G_1} \lambda^{G_1}_{R} T^{G_2}_{G_1} u$ in (5) there is an example by which the equality sign must be excluded if any other assumption such as in (8) is not made of. In the following Fig. 21 we can say that $\lambda^{G_2}_{R} u_{\overline{pq}} = u_{\overline{pq}} > 0$ and $S^{G_2}_{G_1} \lambda^{G_1}_{R} T^{G_2}_{G_1} u_{\overline{pq}} = 0$, where $u_{\overline{pq}}$ is the harmonic measure $\omega(z, \overline{pq}, G_2)$. On the other hand $\lambda^{G_2}_{G_1} \gneqq \mu^{G_2}_{G_1}$, that is, G_1 is not B-nice in G_2.

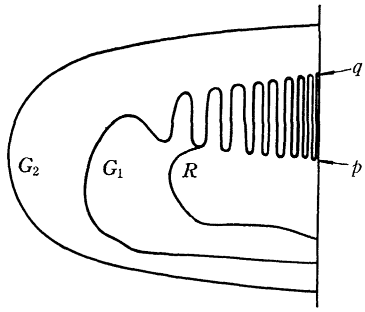


Fig. 21

(9) Let R_1 and R_2 be two open sets in G, then $\mu^{G}_{R_1} \mu^{G}_{R_2} = \mu^{G}_{R_2} \mu^{G}_{R_1} = \mu^{G}_{R_1 \cap R_2}$.

(10) $\lambda^{G}_{R_1} \lambda^{G}_{R_2} = \lambda^{G}_{R_2} \lambda^{G}_{R_1} = \lambda^{G}_{R_1} + \lambda^{G}_{R_2} - \lambda^{G}_{R_1 \cup R_2}$.

§ 3. Applications.

THEOREM 49. *Let G be an extremally B-nice subregion. If R is B-nice in G, then R is also extremally B-nice.*

Proof. Let G_1 be any subregion of G and $G_1 \supset R$. By the assumption $\lambda^{G}_{R} = \mu^{G}_{R}$ and by (5) we have

$$\mu^{G_1}_{R} u = T^{G_2}_{G_1} \mu^{G}_{R} S^{G}_{G_1} u = T^{G}_{G_1} \lambda^{G}_{R} S^{G}_{G_1} u = \lambda^{G_1}_{R} u \geqq \mu^{G_1}_{R} u ,$$

which shows that R is B-nice in G_1. By (8) we have $\lambda^{G'}_{G} = \mu^{G'}_{G}$ for any $G' \supset G$ by the extremal B-niceness of G and

$$\lambda^{G'}_{R} v = S^{G'}_{G} \lambda^{G}_{R} T^{G'}_{G} v$$

for any $v \in PHB(G')$. Therefore

$$\lambda^{G'}_{R} v = S^{G'}_{G} \lambda^{G}_{R} T^{G'}_{G} v = \mu^{G'}_{R} v ,$$

which shows that R is also B-nice in G'. This and the first part of our proof imply that R is B-nice in any subregion of W containing R. Therefore R is extremally B-nice.

Any collection of ideal boundary components of W with compact relative boundary is extremally B-nice. This is easy to prove.

THEOREM 50. *Let G be an extremally B-nice subregion and R_1 and R_2 be B-equivalent in G, then R_1 and R_2 are B-equivalent in any B-nice subregion G_1 of W.*

Proof. Since G is extremally B-nice, we have $\lambda_G^W=\mu_G^W$, and hence we can apply (5) and (8). Therefore by the assumption $\lambda_{R_1}^G=\lambda_{R_2}^G$ and $\mu_{R_1}^G=\mu_{R_2}^G$ we have

$$\lambda_{R_1}^W u = S_G^W \lambda_{R_1}^G T_G^W u = S_G^W \lambda_{R_2}^G T_G^W u = \lambda_{R_2}^W u$$

and

$$\mu_{R_1}^W u = S_G^W \mu_{R_1}^G T_G^W u = S_G^W \mu_{R_2}^G T_G^W u = \mu_{R_2}^W u .$$

Let G_1 be any B-nice subregion in W, then we can apply (5) and see that

$$\lambda_{R_1}^{G_1} u = T_{G_1}^W \lambda_{R_1}^W S_{G_1}^W u = T_{G_1}^W \lambda_{R_2}^W S_{G_1}^W u = \lambda_{R_2}^{G_1} u$$

and

$$\mu_{R_1}^{G_1} u = T_{G_1}^W \mu_{R_1}^W S_{G_1}^W u = T_{G_1}^W \mu_{R_2}^W S_{G_1}^W u = \mu_{R_2}^{G_1} u .$$

Therefore R_1 and R_2 are B-equivalent in any B-nice subregion G_1 containing $R_1 \cup R_2$, whence follows the desired result.

Evidently $R_1 \overset{B}{\sim} R_2$ if $\lambda_{R_1}^G \omega_G = \lambda_{R_2}^G \omega_G = 0$ for any G and this is the case if $\lambda_{R_1}^W \omega_W = \lambda_{R_2}^W \omega_W = 0$, where ω_G and ω_W are the harmonic measures in G and in W, respectively.

Now we shall define *the breadth of oscillation* of ∂R in G by the function $\lambda_{G-\bar{R}}^G \lambda_R^G \omega_G = \lambda_R^G \omega_G - \mu_R^G \omega_G$.

Let γ be a curve tending to the ideal boundary of G or a collection of curves or point-sets. Let R be any collection of domains being compact or non-compact and containing γ. *The oscillation measure $OM_G(\gamma)$* of γ in G is defined by $\inf_R (\lambda_R^G - \mu_R^G)\omega_G$.

We say that an element u in $PHB(G)$ is a *generalized harmonic measure* if $u \frown (\omega_G - u) = 0$. Then two mappings λ and μ preserve the generalized harmonic measure. Thus the breadth of oscillation of ∂R in G $(\lambda_R^G - \mu_R^G)\omega_G = \lambda_R^G \lambda_{G-\bar{R}}^G \omega_G$ is a generalized harmonic measure.

THEOREM 51. *If $G_1 \subset G_2$, then $OM_{G_1}(\gamma) \leqq OM_{G_2}(\gamma)$. If γ be a collection of curves tending to the ideal boundary of G, then $OM_G(\gamma) = \inf_R \lambda_R^G \omega_G$.*

Proof. Let R be an open set containing γ for which

$$OM_{G_2}(\gamma) + \varepsilon > \lambda_R^{G_2} \omega_{G_2} - \mu_R^{G_2} \omega_{G_2} \geqq OM_{G_2}(\gamma).$$

By (5) we have

$$\lambda_R^{G_2}\omega_{G_2} - \mu_R^{G_2}\omega_{G_2} \geqq S_{G_1}^{G_2}\lambda_R^{G_1}T_{G_1}^{G_2}\omega_{G_2} - S_{G_1}^{G_2}\mu_R^{G_1}T_{G_1}^{G_2}\omega_{G_2}$$
$$= S_{G_1}^{G_2}(\lambda_R^{G_1}-\mu_R^{G_1})\omega_{G_1} \geqq (\lambda_R^{G_1}-\mu_R^{G_1})\omega_{G_1} \geqq OM_{G_1}(\gamma).$$

This implies

$$OM_{G_2}(\gamma) + \varepsilon \geqq OM_{G_1}(\gamma).$$

Since ε is arbitrary, we have the first desired result.

In order to prove the second half of this theorem we need a modified modification theorem stated in Chap. I, §7. By the modification theorem we can select R so near to γ that a relation

$$(\lambda_R^G - \mu_R^G)\omega_G(p) = 0$$

holds at any ideal boundary point which cannot be approached by any point-sequence on γ. Since γ is a collection of curves, $T_R^G\omega_G = 0$ for these R. Thus we can select R so that $\mu_R^G\omega_G = 0$. This implies that $OM_G(\gamma) = \inf_R \lambda_R^G\omega_G$.

If γ is of zero oscillation measure $OM_G(\gamma)$ in G, then we say γ *a non-oscillatory set* in G. If γ is non-oscillatory in any G, then we say γ an *absolutely non-oscillatory set.*

THEOREM 52. *If G is an extremally B-nice subregion and if γ is non-oscillatory in G, then γ is absolutely non-oscillatory.*

Proof. Since G is extremally B-nice, we have $\lambda_G^W = \mu_G^W$. Hence we can apply (5) and (8). Then we can select an open set R containing γ such that for any $\varepsilon > 0$

$$0 \leqq \lambda_R^G\omega_G - \mu_R^G\omega_G < \varepsilon$$

holds in G. Since we have

$$\lambda_R^W\omega_W = S_G^W\lambda_R^G T_G^W\omega_W \text{ and } \mu_R^W\omega_W = S_G^W\mu_R^G T_G^W\omega_W$$

and further $T_G^W\omega_W = \omega_G$, we can say that

$$0 \leqq \lambda_R^W\omega_W - \mu_R^W\omega_W = S_G^W(\lambda_R^G\omega_G - \mu_R^G\omega_G) \leqq \varepsilon$$

in W by the maximum principle. This implies that γ is non-oscilatory in W. Let G_1 be any subregion of W containing R, then

$$0 \leqq OM_{G_1}(\gamma) \leqq OM_W(\gamma) \leqq \varepsilon$$

by theorem 51. Therefore γ is absolutely non-oscillatory. Of course $\omega_W \equiv 1$ by the definition of $PHB(W)$ in the above proof.

THEOREM 53. *A necessary and sufficient condition for a set R in G to be B-nice in G is that $OM_G(R) = 0$ or equivalently $(\lambda_R^G - \mu_R^G)\omega_G = 0$.*

Proof. We can construct a set R' so near to $R(R' \supset R)$ that $(\lambda_{R'}^G - \mu_{R'}^G)\omega_G = (\lambda_R^G - \mu_R^G)\omega_G$. This is a simple consequence of the modification theorem. Since ω_G

$\in PHB(G)$, the B-niceness of R in G implies $(\lambda_R^G - \mu_R^G)\omega_G = 0$. Thus we can say that

$$0 \leqq OM_G(R) = \inf_{R'' \supset R} (\lambda_{R''}^G - \mu_{R''}^G)\omega_G \leqq (\lambda_{R'}^G - \mu_{R'}^G)\omega_G = 0 .$$

This implies the necessity part of the theorem.

If $u \leqq v$ in G and $u, v \in PHB(G)$, then $(\lambda_R^G - \mu_R^G)u \leqq (\lambda_R^G - \mu_R^G)v$ in G. In fact this is a simple consequence of the positivity of $\lambda_R^G - \mu_R^G$ or $\lambda_{G-\bar{R}}^G \lambda_R^G$. For any $u \in PHB(G)$ there holds $u \leqq M\omega_G$ for some positive number M. By the assumption we have

$$0 \leqq (\lambda_R^G - \mu_R^G)u \leqq (\lambda_R^G - \mu_R^G)M\omega_G$$
$$= M(\lambda_R^G - \mu_R^G)\omega_G = 0 ,$$

whence follows the niceness of R in G. Again we can construct a set R' so near that $(\lambda_{R'}^G - \mu_{R'}^G)\omega_G = (\lambda_R^G - \mu_R^G)\omega_G$. Further we can assume that $OM_G(R) + \varepsilon > (\lambda_{R'}^G - \mu_{R'}^G)\omega_G$ for any positive number ε. By the assumption $OM_G(R) = 0$ we can say that $(\lambda_R^G - \mu_R^G)\omega_G = 0$, since ε is arbitrary. This is the desired sufficiency part of the theorem.

Let R_1 and R_2 be two open subsets in G. If $R_1 \frown R_2 = \phi$ and $\lambda_{R_1}^G u + \lambda_{R_2}^G u = \lambda_{R_1+R_2}^G u$ for any $u \in PHB(G)$, then we say R_1 and R_2 *relatively disjoint* in G. If this is the case for any B-nice subregion G containing R_1 and R_2, then we say R_1 and R_2 *perfectly disjoint.*

$I(R_1, R_2 ; G)\omega_G \equiv (\lambda_{R_1}^G + \lambda_{R_2}^G - \lambda_{R_1 \smile R_2}^G)\omega_G$ is called *the ideal intersection measure* of R_1 and R_2 in G. It is the restriction of the first ideal intersection operator $I_1(R_1, R_2 ; G)$ to ω_G. By (10) we see that

$$I(R_1, R_2 ; G)\omega_G = \lambda_{R_1}^G \lambda_{R_2}^G \omega_G .$$

Thus this is also a generalized harmonic measure.

For any open sets R_1 and R_2 in G we have

$$\lambda_{R_1}^G u + \lambda_{R_2}^G u \geqq \lambda_{R_1 \cup R_2}^G u \text{ and } \mu_{R_1}^G u + \mu_{R_2}^G u \geqq \mu_{R_1 \cup R_2}^G u .$$

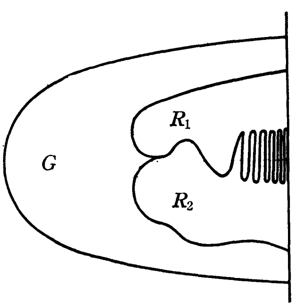


Fig. 22

$R_1 \frown R_2 = \phi$ implies $\mu_{R_1}^G u + \mu_{R_2}^G u = \mu_{R_1+R_2}^G u$, however it does not imply $\lambda_{R_1}^G u + \lambda_{R_2}^G u = \lambda_{R_1+R_2}^G u$ in general as Fig. 22 shows.

THEOREM 54. *If R_1 and R_2 are relatively disjoint for an extremally B-nice subregion G, then R_1 and R_2 are perfectly disjoint.*

Proof. Since G is extremally B-nice, $\lambda_G^W = \mu_G^W$ holds and hence (8) is applicable. For any $u \in PHB(W)$

$$\lambda_{R_1}^W u + \lambda_{R_2}^W u = S_G^W \lambda_{R_1}^G T_G^W u + S_G^W \lambda_{R_2}^G T_G^W u$$
$$= S_G^W(\lambda_{R_1}^G + \lambda_{R_2}^G) T_G^W u = S_G^W \lambda_{R_1+R_2}^G T_G^W u = \lambda_{R_1+R_2}^W u .$$

For any B-nice subregion G_1 in W we can again apply (5) similarly and we have the desired result.

THEOREM 55. *If the ideal intersection measure $I(R_1, R_2\ G)\omega_G$ of two disjoint open subsets R_1 and R_2 in G is equal to zero, then R_1 and R_2 are relatively disjoint in G.*

Proof. If $u \leqq v$ in G and $u, v \in PHB(G)$, then $I(R_1, R_2; G)u \leqq I(R_1, R_2; G)v$. Now we can put $u \leqq M\omega_G$ and apply $I(R_1, R_2; G)\omega_G = 0$, then we have the relative disjointness of R_1 and R_2 in G.

THEOREM 56. *The ideal intersection measure $I(R_1, R_2; G)\omega_G$ is monotone increasing with an increasing sequence of subregions.*

Proof. If $G_1 \subset G_2$, then

$$I(R_1, R_2; G_2)\omega_{G_2} = \lambda_{R_1}^{G_2}\lambda_{R_2}^{G_2}\omega_{G_2} \geqq S_{G_1}^{G_2}\lambda_{R_1}^{G_1}T_{G_1}^{G_2}S_{G_1}^{G_2}\lambda_{R_2}^{G_1}T_{G_1}^{G_2}\omega_{G_2}$$
$$= S_{G_1}^{G_2}\lambda_{R_1}^{G_1}\lambda_{R_2}^{G_1}\omega_{G_1} \geqq \lambda_{R_1}^{G_1}\lambda_{R_2}^{G_1}\omega_{G_1} = I(R_1, R_2; G_1)\omega_{G_1}.$$

THEOREM 57. *Let R_1 and R_2 be two open subsets in G. Then there holds an inequality*

$$\mu_{R_1}^G u + \mu_{R_2}^G u \leqq \mu_{R_1\smile R_2}^G u + \mu_{R_1\frown R_2}^G u$$

for any $u \in PHB(G)$. Further this implies an inequality

$$\lambda_{R_1\smile R_2}^G u + \lambda_{R_1\frown R_2}^G u \leqq \lambda_{R_1}^G u + \lambda_{R_2}^G u .$$

THEOREM 58. *If R_1 and R_2 are B-nice in G, then $R_1\frown R_2$ and $R_1\smile R_2$ are B-nice in G. Especially the extremal B-niceness of R_1 and R_2 implies that of $R_1\frown R_2$ and $R_1\smile R_2$. Further if G is extremally B-nice and if R_1 and R_2 are B-nice in G, then $R_1\frown R_2$ and $R_1\smile R_2$ are also extremally B-nice*

THEOREM 59. *If G is an extremally B-nice subregion and if $\lambda_{R_1\frown R_2}^G\omega_G = 0$ or $\mu_{R_1\frown R_2}^G\omega_G = 0$, then for any $G_1 \supset R_1\frown R_2$ $\lambda_{R_1\frown R_2}^{G_1}\omega_{G_1} = 0$ or $\mu_{R_1\frown R_2}^{G_1}\omega_{G_1} = 0$ and for any $u \in PHB(G_1)$ $\lambda_{R_1\frown R_2}^{G_1} u = 0$ or $\mu_{R_1\frown R_2}^{G} u = 0$.*

If in theorem 57 the equality signs hold in two inequalities, then we say that R_1 and R_2 are *situated B-regularly* in G. This is the case if R_1 and R_2 are B-nice in G and $\mu_{R_1}^G u + \mu_{R_2}^G u = \mu_{R_1\frown R_2}^G u + \mu_{R_1\smile R_2}^G u$. Indeed by theorem 58 all the members R_1, R_2, $R_1\smile R_2$ and $R_1\frown R_2$ are B-nice in G and if R is B-nice in G, then $G-R$ is also B-nice in G, that is, $\lambda_R^G = \mu_R^G$ implies $\lambda_{G-\bar{R}}^G = \mu_{G-\bar{R}}^G$. however we can strengthen this fact as in the following theorem.

THEOREM 60. *If R_1 and R_2 are extremally B-nice, then R_1 and R_2 are regularly situated in any $G \supset R_1\smile R_2$ or if R_1 and R_2 are B-nice in G, then R_1 and R_2 are regularly situated in G. Further there holds*

$$\lambda_{R_1}^G u + \lambda_{R_2}^G u = \lambda_{R_1\smile R_2}^G u + \lambda_{R_1\frown R_2}^G u$$

$$= \mu^G_{R_1\smile R_2}u + \mu^G_{R_1\frown R_2}u = \mu^G_{R_1}u + \mu^G_{R_2}u$$

for any $u \in PHB(G)$.

THEOREM 61. *If G is an extremally B-nice subregion and if R_1 and R_2 are regularly situated in G, then R_1 and R_2 are regularly situated in any subregion G_1 of W.*

Proof. We have $\lambda^W_G = \mu^W_G$ by the B-nicenss of G in W. Thus we can apply (8) and (5). Therefore we have

$$I^W_{R_1,R_2}u \equiv (\lambda^W_{R_1} + \lambda^W_{R_2} - \lambda^W_{R_1\smile R_2} - \lambda^W_{R_1\frown R_2})u$$
$$= S^W_G\, I^G_{R_1,R_2}\, T^W_G u = 0$$

and

$$L_{R_1,R_2}^W u \equiv (\mu^W_{R_1} + \mu^W_{R_2} - \mu^W_{R_1\smile R_2} - \mu^W_{R_1\frown R_2})u$$
$$= S^W_G\, L_{R_1,R_2}^G\, T^W_G u = 0\,.$$

Again by (5) we have

$$I_{R_1,R_2}^{G_1}u = T^W_{G_1} I_{R_1,R_2}^W S^W_{G_1}u = 0$$

and

$$L_{R_1,R_2}^{G_1}u = T^W_{G_1} L_{R_1,R_2}^W S^W_{G_1}u = 0\,.$$

Let $M(R_1, R_2; G)u$ be $(I_{R_1,R_2}^G - L_{R_1,R_2}^G)u$ for any $u \in PHB(G)$. Then $M(R_1, R_2; G)u \in PHB(G)$. If G is a B-nice subregion in G_1, then

$$M(R_1, R_2; G_1)\omega_{G_1} = S^{G_1}_G M(R_1, R_2; G)\, T^{G_1}_G \omega_{G_1} \geqq M(R_1, R_2; G)\omega_G.$$

If $M(R_1, R_2; G)\omega_G = 0$, then $M(R_1, R_2; G)u = 0$ for any $u \in PHB(G)$ and vice versa. If G is extremally B-nice and if $M(R_1, R_2; G)\omega_G = 0$, then $M(R_1, R_2; G_1)\omega_{G_1} = 0$ for any subregion G_1 of W.

THEOREM 62. *If R_1 and R_2 are two open sets in G and $R_1 \subset R_2$, then*

$$\lambda^G_{R_2}u \leqq \lambda^G_{R_2-\bar{R}_1}u + \lambda^G_{R_1}u \text{ and } \mu^G_{R_2}u \geqq \mu^G_{R_2-\bar{R}_1}u + \mu^G_{R_1}u$$

for any $u \in PHB(G)$.

Let R_1 and R_2 be two open subsets in G. If $\lambda^G_{R_1-\overline{R_1\frown R_2}}u = \lambda^G_{R_2-\overline{R_1\frown R_2}}u = 0$ for any $u \in PHB(G)$, then we say R_1 and R_2 *truely B-equivalent* in G. Evidently if $\lambda^G_{R_1-\overline{R_2\frown R_1}}\omega_G = \lambda^G_{R_2-\overline{R_1\frown R_2}}\omega_G = 0$, then R_1 and R_2 are truely B-equivalent in G and vice versa.

THEOREM 63. *If R_1 and R_2 are truely B-equivalent in G, then R_1 and R_2 are B-equivalent in G and are regularly situated in G.*

Proof. By theorem 62 and by the assumption $\lambda^G_{R_1-\overline{R_1\frown R_2}}u = 0$ we have

$$\lambda^G_{R_1-\overline{R_1\frown R_2}}u + \lambda^G_{R_1\frown R_2}u \geqq \lambda^G_{R_1}u \geqq \lambda^G_{R_1\frown R_2}u$$

for any $u \in PHB(G)$, which implies $\lambda^G_{R_1} u = \lambda^G_{R_1 \frown R_2} u$. Similarly we have $\lambda^G_{R_2} u = \lambda^G_{R_1 \frown R_2} u$. In a way we have $\lambda^G_{R_1} u = \lambda^G_{R_1 \smile R_2} u$ by theorem 57. And further we have by a simple calculation

$$0 = \lambda^G_{R_1 - \overline{R_1 \frown R_2}} u = \lambda^G_{G - \bar{R}_2 - \overline{(G-\bar{R}_1) \frown (G-\bar{R}_2)}} u$$
$$= \lambda^G_{R_2 - \overline{R_1 \frown R_2}} u = \lambda^G_{G - \bar{R}_1 - \overline{(G-\bar{R}_1) \frown (G-\bar{R}_2)}} u .$$

Thus we have

$$\lambda^G_{G-\bar{R}_1} u = \lambda^G_{G-\bar{R}_2} u = \lambda^G_{(G-\bar{R}_1) \frown (G-\bar{R}_2)} u = \lambda^G_{(G-\bar{R}_1) \smile (G-\bar{R}_2)} u$$

for any $u \in PHB(G)$. This shows that

$$\mu^G_{R_1} u = \mu^G_{R_2} u = \mu^G_{R_1 \smile R_2} u = \mu^G_{R_1 \frown R_2} u$$

for any $u \in PHB(G)$. Thus our proof is now complete.

Inverse statement of theorem 63 does not remain true in general, which is shown by the following figure. Indeed, R_1 and R_2 are B-equivalent in G, however they are not truely B-equivalent in G.

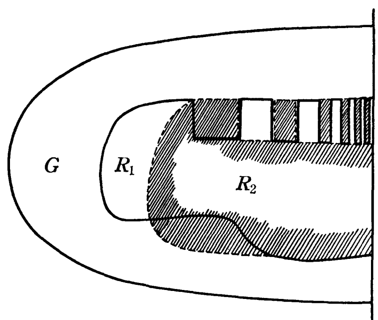


Fig. 23

THEOREM 64. *If R_1 and R_2 are B-nice and B-equivalent in G, then R_1 and R_2 are truely B-equivalent in G. If R_1 is B-nice in G and if R_1 and R_2 are truely B-equivalent in G, then R_2 is B-nice in G.*

Proof. To that end, we prove the following fact:

If A and B are disjoint open sets in G and $\mu^G_A u = \mu^R_B u$ for any $u \in PHB(G)$, then $\mu^G_A u = 0$.

If $A \in SO_{HB}$ or $B \in SO_{HB}$, then $\mu^G_A u = S^G_A T^G_A u = 0$ or $\mu^G_B u = S^G_B T^G_B u = 0$. If both A and B do not belong to SO_{HB}, we put $u = \mu^G_A \omega_G$. Then $\mu^G_A \mu^G_A \omega_G = \mu^G_A \omega_G \neq 0$ and $0 \leqq \mu^G_B \mu^G_A \omega_G \leqq \mu^G_{G-\bar{A}} \mu^G_A \omega_G = 0$. This is a contradiction.

The remaining part of the proof is quite similar to theorem 21. Thus we omit the remaining part of the proof.

THEOREM 65. *If G is extremally B-nice and two open sets R_1 and R_2 are truely B-equivalent in G, then these sets are truely B-equivalent in any $G_1 \supset R_1 \smile R_2$.*

By the proof of theorem 63 we can say that $\lambda^G_{R_1 - \overline{R_1 \frown R_2}} u = 0$ implies $\lambda^G_R u = \lambda^G_{R_1 \frown R_2} u$, $\lambda^G_{R_2} u = \lambda^G_{R_1 \smile R_2} u$ and $\mu^G_{R_1} u = \mu^G_{R_1 \frown R_2} u$, $\mu^G_{R_2} u = u^G_{R_1 \smile R_2} u$. Therefore we can say that R_2 almost contains R_1 in G if $\lambda^G_{R_1 - \overline{R_1 \frown R_2}} u = 0$. This is equivalent to $\lambda^G_{R_1 - \overline{R_1 \frown R_2}} \omega_G = 0$.

If R_2 almost contains R_1 in G, then R_1 and R_2 are regularly situated in G. If $R_1 \subset R_2$, then evidently R_2 almost contains R_1 in any $G \supset R_2$. If R_2 almost contains

R_1 in an extremally B-nice subregion G, then R_2 almost contains R_1 in any $G \supset R_1 \cup R_2$.

Let γ_1, γ_2 be two non-compact simple curves in G tending to the ideal boundary of G and being oscillatory in G, that is, $OM_G(\gamma_1)>0$ and $OM_G(\gamma_2)>0$. We can define two sides of a non-compact simple curve in G with respect to the defining parameter of the curve and the local parameter of the surface. Let R_1 and R_2 be any two subregions of G such that $\partial R_1 \supset \gamma_1$, $\partial R_2 \supset \gamma_2$ and R_1 and R_2 be situated simultaneously on the same side of γ_1 and γ_2 respectively. If for any such R_1 and R_2

$$\inf_{R_1, R_2} (\lambda^G_{R_1 - \overline{R_1 \frown R_2}} \omega_G + \lambda^G_{R_2 - \overline{R_1 \frown R_2}} \omega_G) = 0,$$

then we say that γ_1 and γ_2 are *near each other* in G. If this is the case for any $G \supset \gamma_1 \cup \gamma_2$, then we say that γ_1 and γ_2 are *absolutely near each other.*

THEOREM 66. *Let γ_1, γ_2 be two non-compact simple curves in G tending to the ideal boundary of G and being oscillatory in G. Further we assume that two curves are disjoint and are near each other in G. Then we cannot select two subregions R_1 and R_2 lying the same side of γ_1 aud γ_2 respectively and satisfying the disjointness $R_1 \frown R_2 = \phi$ and staisfying an inequality for any ε*

$$\lambda^G_{R_1 - \overline{R_1 \frown R_2}} \omega_G + \lambda^G_{R_2 - \overline{R_1 \frown R_2}} \omega_G < \varepsilon .$$

Proof. If this is not the case, then we have

$$\lambda^G_{R_1} \omega_G < \varepsilon \text{ and } \lambda^G_{R_2} \omega_G < \varepsilon .$$

This is now rejected by the oscillatory properties of γ_1 and γ_2.

If γ_1 and γ_2 are near each other in an extremally B-nice subregion G, then γ_1 and γ_2 are absolutely near each other. If γ_1 and γ_2 are near each other in G, then we have for any $u \in PHB(G)$

$$\inf(\lambda^G_{R_1 - \overline{R_1 \frown R_2}} u + \lambda^G_{R_2 - \overline{R_1 \frown R_2}} u) = 0 .$$

Chapter IV. Determination of the coordinate.

We shall here introduce a method which determines the coordinates of Martin minimal points of Δ^R_1 in reference to a point or a set of Δ^G_1. To that end we shall introduce several notions showing that a point $r \in \Delta^R_1$ lies over a point $q \in \Delta^G_1$ or spreads over a set $E \subset \Delta^G_1$.

DEFINITION 1. Let r be a point in Δ^R_1 and let q be a point in Δ^G_1. Let $N_l(q)$ be a level domain defined by an inequality $K_G(p, q) > l$ and $M_k(r)$ a level domain defined by an inequality $K_R(p, r) > k$. If for any given $l(< \sup_G K_G(p, q))$ there is a positive number k_0 for which $M_k(r) \subset N_l(q)$ for any $k \geqq k_0$ $(k \leqq \sup_R K_R(p, r))$, then we say that r *lies over* q.

Is there a point $r \in \Delta^R_1$ which does not lie over any point $q \in \Delta^G_1$, if $G \notin O_{HP}$?

The answer is positive. Let G be the unit disc and R be a set $N_l(1)$ plus a set S lying in a set $\{|z|<1\}-\overline{N_l(1)}$ and being tangential to $|z|=1$ at only one point 1 from only one direction. There are two points r and r' in $\Delta_1^R \equiv \Delta^R$. Let r be a point $T_R^G 1$, that is, a point of Δ_1^R corresponding to 1 by T_R^G. Then the remaining point r' does not lie over the point 1 by definition. However r' is attached to 1 in the following sense.

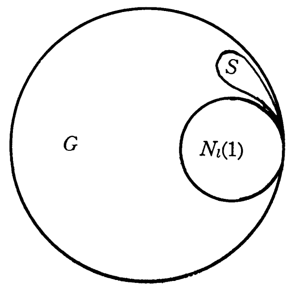


Fig. 24

DEFINITION 2. Let r and q be the same as in definition 1. Let $\overline{U_\varepsilon(q)}$ be an ε-neighborhood of a point $q \in \Delta_1^G$ defined by an inequality $d_G(q, p) < \varepsilon$, $p \in G \frown \Delta^G$ and $U_\varepsilon(q) = \overline{U_\varepsilon(q)} \frown G$, where $d_G(q, p)$ is the Martin distance in G. If for any given ε there is a positive number k_0 for which $M_k(r) \subset U_\varepsilon(q)$ for any $k \geqq k_0$, then we say that r is attached to q. If further any level domain $M_k(r)$ for $k \geqq k_0$ lies outside of some level domain $N_l(q)$, then we say that r is attached tangentially to q.

It is not necessary that any point in Δ_1^R being attached to q coincides with some point being attached tangentially to q.

Is there a point $r \in \Delta_1^R$ which is not attached to any point $q \in \Delta_1^G$, if $G \notin O_{HP}$?

Even if the problem is changed as above, the answer is still positive. In his recent lectures at Hiroshima M. Brelot gave the following example; see also [0]. Let G be a square defined by $0 < x < 1$, $0 < y < 1$. The line $x=1$ is considered as the ideal boundary of W: $x < 1$. Let R be a subregion of G with an infinite number of slits as shown in the figure. If two holes are sufficiently narrow, then

$$\lim K_R(p, z_n) = \lim K_R(p, z'_n)$$

remains true for any $p \in R$. This shows that there is only one point r in Δ_1^R. Let $K_R(p, r)$ be the corresponding Martin minimal function. Then the level domain $N_k(r)$ is a domain as shown in the Fig. 26 for any k. Therefore for any $q \in \Delta_1^G$ $M_k(r)$

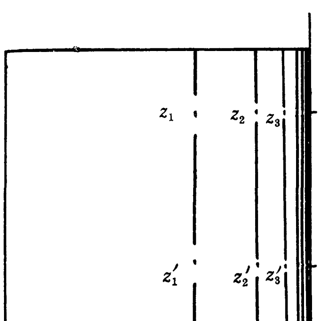


Fig. 25

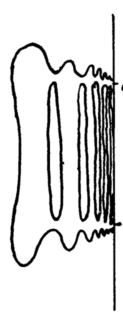


Fig. 26

$\subset U_\varepsilon(q)$ does not hold. This implies that r is attached to no point in Δ_1^G. However r spreads over the closed segment $\overline{q_2 q_1}$ in the following sense.

DEFINITION 3. Let r be a point in Δ_1^R. Let E be a set in Δ_1^G for which

$$M_k(r) \subset \bigcup_{q \in E} U_\varepsilon(q)$$

holds for any given ε with any $k \geqq k_0(\varepsilon)$. If E_r is the smallest closed set with the above property, then we say that r *spreads over* E_r.

If $r \in \Delta_1^R$ corresponds to $q \in \Delta_1^G$ by T_R^G, then r lies over q.

Proof. By the assumption

$$T_R^G K_G(p, q) = K_R(p, r) \leqq K_G(p, q)$$

in R. Let $N_l(q)$ be any level domain, then $K_R(p, r) \leqq l$ in $R - \overline{N_l(q)}$. Thus the level domain $M_l(r)$ belongs to $N_l(q)$. This is equivalent to the desired fact.

We have the following implications:

r corresponds to q by T_R^G. $\Rightarrow$ r lies over q. $\Rightarrow$ r is attached to q. $\Rightarrow$ r spreads over q.

LEMMA I. If there is a positive number m such that $N_m(q) \subset R$, then for any k there is an l_0 for which $M_k(r) \supset N_l(q)$ for any $l \geqq l_0$.

Proof. By the definition of $N_m(q)$,

$$T_{N_m(q)}^G K_G(p, q) = K_G(p, q) - m > 0 .$$

Therefore $T_R^G K_G(p, q) > 0$, since $S_{N_m(q)}^R (K_G(p, q) - m) \leqq K_G(p, q)$. This shows that $T_R^G K_G(p, q) \equiv K_R(p, r)$ is a Martin minimal function in R, that is, there is a minimal point $r \in \Delta_1^R$ which corresponds to q by T_R^G. Since $N_m(q) \subset R$, there exists a positive number M such that

$$\sup_{\partial R} K_G(p, q) \equiv M \leqq m < \infty .$$

Since $T_{M_k(r)}^R K_R(p, r) = K_R(p, r) - k$ by the definition,

$$T_{M_k(r)}^G K_G(p, q) = T_{M_k(r)}^R T_R^G K_G(p, q)$$
$$= T_{M_k(r)}^R (K_G(p, q) - u(p)) = K_G(p, q) - u(p) - k ,$$

where $u(p)$ is defined by $T_R^G K_G(p, q) = K_G(p, q) - u(p)$ in R. This function $u(p)$ is also defined by the following limiting process

$$\lim_{n \to \infty} u_n(p) ,$$

where $u_n(p)$ is a positive harmonic function in $W_n \frown R$ with the boundary value 0 on $\partial W_n \frown R$ and $K_G(p, q)$ on $\partial R \frown W_n$. $u_n(p)$ increases with n and is bounded by M. Thus $u(p) \leqq M$ in R. Therefore we have $K_G(p, q) \leqq K_R(p. r) + M$ in R and hence $K_G(p, q) \leqq k + M$ on $\partial M_k(r)$. Next we choose l_0 as $k + M$, then $N_{k+M}(q) \subset M_k(r)$ and hence $N_l(q) \subset M_k(r)$ for any $l \geqq l_0$.

LEMMA 2. *If $r_1 \in \varDelta_1^R$ lies over $q \in \varDelta_1^G$ and $r \in \varDelta_1^R$ corresponds to q by T_R^G, then r_1 coincides with r, unless there is a part of ∂R in any $N_l(q)$ for $l \geqq l_0$.*

Proof. By Lemma 1 for any k there is an l_0 for which $M_k(r) \supset N_l(q)$ for any $l \geqq l_0$. On the other hand we can choose k' for which $M_{k''}(r_1) \subset N_l(q)$ for any $k'' \geqq k'$. If $r_1 \neq r$, we can choose $M_{k''}(r_1)$ such that $M_{k''}(r_1) \frown M_k(r) = \phi$ for any given k. This is untenable, since $M_{k''}(r_1) \subset M_k(r)$. Therefore $r_1 = r$.

LEMMA 3. *If $r_1 \in \varDelta_1^R$, $r_2 \in \varDelta_1^R$ lie over a point $q \in \varDelta_1^G$, then $r_1 = r_2$, unless there is a part of ∂R in any $N_l(q)$ for $l \geqq l_0$.*

Proof. If there is no part of ∂R in some $N_l(q)$ for $l \geqq l_0$, then there is a point r of $\varDelta_1^R$ which corresponds to q by T_R^G. Then by Lemma 2 we have $r_1 = r = r_2$

THEOREM 67. *If any point $r \in \varDelta_1^R$ always lies over some point $q(r) \in \varDelta_1^G$ and if R is most P-nice in G, then R is onto P-nice in G.*

Proof. Let r be an arbitrary point of $\varDelta_1^R$, then r lies over a point $q(r) \in \varDelta_1^G$. By theorem 2 $q(r)$ is not accessible by two point-sequences one of which belongs to R and another of which belongs to $G-R$. Hence $q(r)$ is accessible by only a point-sequence belonging to R. By this fact there is no part of ∂R in some neighborhood $U_\varepsilon(q(r))$, that is, $U_\varepsilon(q(r)) \subset R$. Then we can say that $N_l(q(r)) \subset U_\varepsilon(q(r)) \subset R$ for a suitable l. This implies that

$$T_{N_l(q(r))}^G K_G(p, q(r)) = T_{N_l(q(r))}^R T_R^G K_G(p, q(r))$$

is positive. Thus we have

$$T_R^G K_G(p, q(r)) > 0 .$$

Let r' be a corresponding minimal point in $\varDelta_1^R$ by T_R^G. Then by Lemma 3 r coincides with this r'. This shows that $S_R^G K_R(p, r) = K_G(p, q(r))$. This is the desired result.

THEOREM 68. *If there is a point r of $\varDelta_1^R$ which spreads over a set $E_r (\subset \varDelta_1^G)$ containing at least two points, then R is neither onto nor most P-nice in G.*

Proof. By theorem 2 we can say that $\varDelta_1^R$ is imbedded in $\varDelta_1^G$ by T_R^G if R is onto P-nice in G. Let $q(r)$ be a point of $\varDelta_1^G$ such that r corresponds to $q(r)$ by T_R^G. Then for any ε there is a set $M_k(r)$ such that $M_k(r) \subset U_\varepsilon(q(r))$. On the other hand $M_k(r)$ oscillates on the whole E_r. Thus the diameter of $M_k(r)$ is not less than that of E_r which is also not less than the diameter or distance of two preassigned points in E_r being positive by the assumption. This is absurd, since the diameter of $M_k(r)$ is less than that of $U_\varepsilon(q(r))$ which is smaller than 2ε and ε is any positive number. If R is most P-nice in G, then by theorem 2 any point of E_r is not accessible by any point-sequence in $G-R$, since it is accessible by some point-sequence in R. Then two different points r_1 and r_2 in $\varDelta_1^R$ are determined by two preassigned points q_1 and q_2 in E_r in such a manner that r_i corresponds to q_i by $T_R^G (i=1, 2)$. On the other hand if two points of $\varDelta_1^R$ are two clustering points of any level curve L_l defined by a minimal function $K_R(p, r)$, then two points coincide with r and hence with each other. This shows that $r_1 = r = r_2$, which is absurd.

LEMMA 4. *If $r \in \Delta_1^R$ is attached to $q \in \Delta_1^G$ and $U_\varepsilon(q) \subset R$ for some ε, then r corresponds to q by T_R^G.*

Proof. If $r \in \Delta_1^R$ lies over $q \in \Delta_1^G$, then the assertion is already established by Lemma 2. If r does not lie over q but is attached to q, then we can choose a level domain $M_k(r)$ such that $M_k(r) \subset U_\varepsilon(q)$. If a part γ_s of any level line $K_R(p, r) = s$, $s \geqq s_0$ is not tangential to q, then this part γ_s intersects any level curves L_l: $K_G(p, q) = l$ for $l \geqq l_0$. On the other hand by Lemma 1 we can choose a level domain $N_m(q)$ in such a manner that $N_m(q) \subset M_k(r(q))$ for any k, where $r(q)$ is a point of Δ_1^R corresponding to q by T_R^G. This shows that any level curves L_n: $K_R(p, r(q)) = n$ intersect γ_s for any $n \geqq n_0$ and for any $s \geqq s_0$. On the other hand if $r \neq r(q)$, then we can choose that any level domain $M_n(r(q))$, $n \geqq n_0$ and $M_s(r)$, $s \geqq s_0$ are disjoint. This is untenable. Thus we have the desired result: $r = r(q)$.

THEOREM 69. *If any point $r \in \Delta_1^R$ is always attached to some point $q \in \Delta_1^G$, and if R is most P-nice in G, then R is onto P-nice in G.*

Proof. A quite similar method is available as in theorem 67. In this case we use Lemma 4 instead of Lemma 3.

Theorem 69 is a precision of theorem 67.

Next we shall prove a precision of Lemma 2.

THEOREM 70. *If $r' \in \Delta_1^R$ is attached to $q \in \Delta_1^G$ but is not attached tangentially to q and $r \in \Delta_1^R$ corresponds to $q \in \Delta_1^G$, then r' coincides with r.*

Proof. Since r' is not attached tangentially to q and is attached to q, there is a level curve L_k: $K_R(p, r') = k$, $k \geqq k_0$ intersecting any level curves Γ_l: $K_G(p, q) = l$ for $l \geqq l_0$. Thus L_k is a non-tangential path to q, if we choose a suitable component of L_k. On the other hand any level line γ_l: $K_R(p, r) = l$ is a tangential path to q, and hence any γ_l intersects L_k for any $l \geqq l_1 \geqq l_0$ and for any $k \geqq k_0$. Therefore r' must coincide with r.

Refering to the defining tail of a minimal point we can also define a sort of coordinates of $r \in \Delta_1^R$ over $q \in \Delta_1^G$.

DEFINITION 4. If any defining tail γ of $q \in \Delta_1^G$ on which $K_G(p, q)$ tends to its $\sup_G K_G(p, q)$, then we say that γ is a *principal defining tail.*

DEFINITION 5. If any defining tail of $r \in \Delta_1^R$ is a defining tail of $q \in \Delta_1^G$, then we say that *r is attached to q by the defining tail.*

DEFINITION 6. If r is attached to q by the defining tail and any principal defining tail of $r \in \Delta_1^R$ is a principal defining tail of $q \in \Delta_1^G$, then we say that *r lies over q by the defining tail.*

DEFINITION 7. If there exists a principal defining tail γ of $r \in \Delta_1^R$ satisfying $d_G(q, \gamma) = 0$ for any point q of $E \subset \Delta_1^G$, then we say that r spreads over E by the defining tail γ. Let E_r be the largest set among E with the above property, then we say that r *spreads over* E_r *by the defining tail.*

We can prove the following theorems:

THEOREM 71. *If* $r \in \Delta_1^R$ *is attached to* $q \in \Delta_1^G$ *by the defining tail, then* r *is attached to* q.

THEOREM 72. *If* $r \in \Delta_1^R$ *lies over* $q \in \Delta_1^G$, *then* r *lies over* q *by the defining tail and vice versa.*

THEOREM 73. *If* $r \in \Delta_1^R$ *spreads over* $E \subset \Delta_1^G$, *then* r *spreads over* E *by the defining tail and vice versa.*

Proof of theorem 71. If any η-neighborhood $U_\eta(r)$ of r satisfies $U_\eta(r) \not\subset U_\varepsilon(q)$, then there is a defining tail γ or r satisfying $\lim \sup_{p \in \gamma} d_G(p, q) > \varepsilon$. However by the assumption γ is also a defining tail of q in G, so $\lim_{p \in \gamma} d_G(p, q)$ exists and is equal to zero. This is absurd. Thus we can choose such an η-neighborhood $U_\eta(r)$ of r that $U_\eta(r) \subset U_\varepsilon(q)$. Then we can choose a positive number k_0 such that $M_k(r) \subset U_\eta(r)$ for any $k \geqq k_0$, which implies the desired result: $M_k(r) \subset U_\varepsilon(q)$, that is, r is attached to q.

Proof of theorem 72. Let $N_l(q)$ be any level domain and γ be any principal defining tail of r, then γ is a defining tail of q. Since by the assumption there is a positive number k_0 such that $M_k(r) \subset N_l(q)$ for any $k \geqq k_0$ and $\gamma(t) \subset M_k(r)$ for any $t \geqq t_0$, we can say that $\gamma(t) \subset N_l(q)$ for any $t \geqq t_0$. This holds for any l. Therefore γ is also a principal defining tail of q.

Inversely let $N_l(q)$ be any level domain lying inside of $U_\varepsilon(q)$. Let $M_k(r)$ be any level domain of r. If k is sufficiently near to $\sup_R K_R(p, r)$, then any principal defining tail is contained in $M_k(r)$. If any k $M_k(r)$ is not contained in $N_l(q)$, then there is a principal tail γ of r which is not contained in $N_l(q)$. Then $\liminf_{p \in \gamma} K_G(p, q) \leqq l < \sup_G K_G(p, q)$. This contradicts our assumption, that is, γ is also a principal defining tail of q. Therefore there is a level domain $M_k(r)$ contained in $N_l(q)$, which is the desired result.

Proof of theorem 73. Let r spread over E_r being closed in Δ_1^G. Any level domain $M_l(r)$ is connected in R and hence in G. Let $\{\varepsilon_n\}$ be a sequence of positive number tending to zero as n tends to ∞. Let l_n be a corresponding positive number for which $M_l(r) \subset \bigcup_{q \in E_r} U_\varepsilon(q)$ for any $l \geqq l_n$. We may choose l_n such that $l_n < l_m$ if $n < m$. Then $M_{l_n}(r) \supsetneqq M_{l_m}(r)$. Let γ_n be a curve lying in $M_{l_n}(r)$ and connecting two points p_n and p_{n+1}, $p_n \in \partial M_{l_n}(r)$, $p_{n+1} \in \partial M_{l_{n+1}}(r)$ and satisfying $d_G(p, q) < \varepsilon_n$ for any $q \in E_r$ and for some $p \in \gamma_n$. We introduce a parameter t, $n \leqq t \leqq n+1$ such that the trace of $\gamma(t)$ coincides with γ_n and $\gamma(n) = p_n$ and $\gamma(n+1) = p_{n+1}$. Connecting these curves $\{\gamma_n\}$, we have a curve $\gamma(t)$, $1 \leqq t < \infty$. For this curve we have $d_R(\gamma(t), r) \to 0$ for $t \to \infty$, since $\gamma(t)$, $n \leqq t$ belongs to $\overline{M_{l_n}(r)}$. Thus $\gamma(t)$ is a principal defining tail of r satisfying $\underline{\lim}_{t\to\infty} d_G(\gamma(t), q) = 0$ for any $q \in E_r$. This implies that E_T over which r spreads by the defining tail includes E_r. Next we shall prove $E_T = E_r$. If $r \in \Delta_1^R$ spreads over E_T by the defining tail, then there is a principal defining tail γ of r for which $d_G(\gamma, q) = 0$ for any $q \in E_T$. By the triangular inequality $d_G(\gamma, q) \leqq d_G(\gamma, q_n) + d_G(q_n, q)$ we can say that E_T is closed in Δ_1^G. If $E_T \supsetneqq E_r$, then there is a point q in $E_T - E_r$ for which $U_\varepsilon(q) \frown M_l(r) = \phi$ for some $\varepsilon > 0$ and for any l satisfying $l \geqq l_0$. On the other hand there is a principal tail γ of r for which $\gamma \frown U_\varepsilon(q) \neq \phi$ and $\gamma \frown M_l(r) \neq \phi$ for any $\varepsilon > 0$ and for any $l \geqq l_0$. γ oscillates between $U_\varepsilon(q)$ and $M_l(r)$. Thus there is a point-sequence $\{p_n(l)\} \in R$ on which $K_R(p_n(l), r) < l$ and $\{p_n(l)\}$ defines the r. Therefore γ is not a principal defining tail of r. This is a contradiction. Thus we have the desired fact: $E_T = E_r$.

Some unsolved problems. We shall here list our unsolved problems.

1) In connection with theorem 1 we ask for whether λ^{*G}_R (resp. μ^{*G}_R) coincides with λ^G_R (resp. μ^G_R) in $PH(G)$ or not.

2) Is there a point $r \in \Delta_1^R$ which spreads over some closed set E containing non-minimal point of Δ^G?

It is very plausible to conjecture that the problem is affirmatively answered.

3) Does the most P-niceness of R in G imply the onto P-niceness of R in G?

4) Is there a continuous non-compact curve γ for which $\bar{\Lambda}^G_\gamma = 0$, if $G \notin O_{HP}$?

5) To construct a tangential path to an isolated minimal point in Δ^G, when G is Heins' end of Heins' harmonic dimension one or G has more general nature such as $G \in O_{HP} - O_G$.

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