ECKMANN-FRÖLICHER CONNEXIONS ON ALMOST ANALYTIC SUBMANIFOLDS

By Kentaro Yano

1. Almost analytic submanifolds.

We consider a $2m$-dimensional differentiable manifold M_{2m} of class C^∞ and with an almost complex structure $F_j{}^h$:

$$F_j{}^i F_i{}^h = -A_j^h, \tag{1.1}$$

where A_j^h denotes a unit tensor and the indices $h, i, j, \cdots$ run over the range $1, 2, \cdots, 2m$. We call such a manifold an almost complex space.

It is well known that the condition for an almost complex structure to be induced from a complex structure is the vanishing of the Nijenhuis tensor [4]:

$$N_{ji}{}^h = F_j{}^l \partial_l F_i{}^h - F_i{}^l \partial_l F_j{}^h - (\partial_j F_i{}^l - \partial_i F_j{}^l) F_l{}^h, \tag{1.2}$$

where ∂_l denotes the partial differentiation with respect to the coordinates ξ^l. We call a complex space an almost complex space with vanishing Nijenhuis tensor.

We now consider a $2n$-dimensional submanifold M_{2n} $(2m > 2n)$:

$$\xi^h = \xi^h(\eta^a) \tag{1.3}$$

of class C^∞ where the indices $a, b, c, \cdots$ run over the range $1, 2, \cdots, 2n$. If the transform by $F_i{}^h$ of any vector tangent to M_{2n} is still tangent to M_{2n}, we call M_{2n} an almost analytic submanifold. A necessary and sufficient condition for M_{2n} to be almost analytic is

$$F_i{}^h B_b{}^i = 'F_b{}^a B_a{}^h, \tag{1.4}$$

where $'F_b{}^a$ is a certain tensor of M_{2n} and

$$B_a{}^h = \partial_a \xi^h; \qquad \partial_a = \partial/\partial\eta^a.$$

From (1.1) and (1.4), we find

$$'F_c{}^b\, 'F_b{}^a = -A_c^a. \tag{1.5}$$

Thus, we have [3]

THEOREM 1.1. *An almost analytic submanifold in an almost complex space is an almost complex space.*

On the other hand, by a straightforward computation, we have

$$B_c{}^j B_b{}^i N_{ji}{}^h = 'N_{cb}{}^a B_a{}^h, \tag{1.6}$$

Received December 25, 1961.

where

$$'N_{cb}{}^{a} = 'F_{c}{}^{d}\partial_{d}'F_{b}{}^{a} - 'F_{b}{}^{d}\partial_{d}'F_{c}{}^{a} - (\partial_{c}'F_{b}{}^{d} - \partial_{b}'F_{c}{}^{d})'F_{d}{}^{a} \tag{1.7}$$

is the Nijenhuis tensor of the almost analytic submanifold M_{2n}.

From (1.6) we have [3]

THEOREM 1.2. *An almost analytic submanifold in a complex space is a complex space.*

We now choose, at each point of the almost analytic submanifold M_{2n}, a vector C_{2n+1}^{h} which is not tangent to M_{2n} and put

$$C_{2n+2}^{h} = F_{i}{}^{h}C_{2n+1}^{i}.$$

It is clear that C_{2n+2}^{h} is linearly independent of C_{2n+1}^{h} and is not tangent to M_{2n}. We next choose a third vector C_{2n+3}^{h} which is linearly independent of C_{2n+1}^{h} and C_{2n+2}^{h} and is not tangent to M_{2n} and put

$$C_{2n+4}^{h} = F_{i}{}^{h}C_{2n+3}^{i}.$$

It is clear that C_{2n+4}^{h} is linearly independent of C_{2n+1}^{h}, C_{2n+2}^{h}, C_{2n+3}^{h} and is not tangent to M_{2n}.

Continuing in this way we can choose $2m-2n$ vectors $C_{x}{}^{h}$ which are linearly independent of each other and are not tangent to M_{2n}, where $x, y, z = 2n+1, 2n+2, \cdots, 2m$.

We choose these vectors $C_{x}{}^{h}$ as "affine normals" to the almost analytic submanifold M_{2n}. The affine normals satisfy

$$F_{i}{}^{h}C_{x}{}^{i} = ''F_{x}{}^{y}C_{y}{}^{h}, \tag{1.8}$$

from which

$$''F_{z}{}^{y}''F_{y}{}^{x} = -\delta_{z}^{x}. \tag{1.9}$$

Thus, we have

THEOREM 1.3. *The affine normal space of an almost analytic submanifold admits an almost complex structure.*

The vectors $B_{b}{}^{h}$ and $C_{y}{}^{h}$ being linearly independent of each other, we can form a matrix $(B^{a}{}_{i}, C^{x}{}_{i})$ inverse to $(B_{b}{}^{h}, C_{y}{}^{h})$.

If we have a vector v^{h} at a point of the almost analytic submanifold M_{2n}, we can project v^{h} on the submanifold and get

$$'v^{h} = B_{a}{}^{h}B^{a}{}_{i}v^{i} \tag{1.10}$$

or

$$'v^{h} = B_{a}{}^{h}\,'v^{a}, \tag{1.11}$$

where

$$'v^{a} = B^{a}{}_{i}v^{i}. \tag{1.12}$$

Now we assume that there is given an affine connexion Γ_{ji}^{h} in the almost complex space M_{2m} and denote by ∇_{j} the covariant differentiation with respect to the connexion Γ_{ji}^{h}; for example

$$\nabla_j v^h = \partial_j v^h + \Gamma_{ji}^{h} v^i \tag{1.13}$$

for a contravariant vector v^h.

Take a vector field v^h of M_{2m} which is tangent to M_{2n}:

$$v^h = B_a{}^h{}'v^a$$

and consider its covariant differential

$$\delta v^h = d(B_a{}^h{}'v^a) + \Gamma_{ji}^{h} B_c{}^j d\eta^c B_b{}^i{}'v^b$$

along M_{2n}. The covariant differential δv^h is not necessarily tangent to M_{2n} and so we consider the projection $B_a{}^h B^a{}_i \delta v^i = B_a{}^h \delta' v^a$ of δv^h on the tangent plane to M_{2n}, then we have

$$\delta' v^a = B^a{}_h \delta v^h = d'v^a + '\Gamma_{cb}^{a} d\eta^c{}'v^b, \tag{1.14}$$

where

$$'\Gamma_{cb}^{a} = (B_c{}^j B_b{}^i \Gamma_{ji}^{h} + \partial_c B_b{}^h) B^a{}_h. \tag{1.15}$$

The $'\Gamma_{cb}^{a}$ define an affine connexion on the almost analytic submanifold M_{2n} which we call an induced affine connexion on M_{2n}.

The covariant derivatives of $B_b{}^h$ and $C_x{}^h$ along the almost analytic submanifold M_{2n} are respectively given by

$$\begin{aligned} '\nabla_c B_b{}^h &= \partial_c B_b{}^h + B_c{}^j B_b{}^i \Gamma_{ji}^{h} - \Gamma_{cb}^{a} B_a{}^h, \\ '\nabla_c C_y{}^h &= \partial_c C_y{}^h + B_c{}^j C_y{}^i \Gamma_{ji}^{h}. \end{aligned} \tag{1.16}$$

Equation (1.15) and the first of (1.16) show that $'\nabla_c B_b{}^h$ are, as vectors of M_{2m}, normal to M_{2n} and consequently we can put

$$'\nabla_c B_b{}^h = H_{cb}{}^x C_x{}^h, \tag{1.17}$$

where $H_{cb}{}^x$ are so-called second fundamental tensors of M_{2n} with respect to affine normals $C_x{}^h$ and (1.17) is the equation of Gauss.

On the other hand, the equation of Weingarten takes the form

$$'\nabla_c C_y{}^h = -K_c{}^a{}_y B_a{}^h + L_{cy}{}^x C_x{}^h. \tag{1.18}$$

2. Eckmann-Frölicher connexion.

If we can introduce an affine connexion Γ_{ji}^{h} in M_{2m} such that

$$\nabla_j F_i{}^h = 0 \tag{2.1}$$

and

$$S_{ji}{}^h = \frac{1}{2}(\Gamma_{ji}^{h} - \Gamma_{ij}^{h}) = 0, \tag{2.2}$$

then we have

$$N_{ji}{}^h = 0, \tag{2.3}$$

because $N_{ji}{}^h$ can be written as

$$N_{ji}{}^{h} = F_{j}{}^{l}\nabla_{l}F_{i}{}^{h} - F_{i}{}^{l}\nabla_{l}F_{j}{}^{h} - (\nabla_{j}F_{i}{}^{l} - \nabla_{i}F_{j}{}^{l})F_{l}{}^{h} \tag{2.4}$$

with respect to a symmetric affine connexion.

Conversely if the Nijenhuis tensor vanishes, we can introduce an affine connexion Γ^{h}_{ji} which satisfies (2.1) and (2.2). Indeed, we first introduce a Riemannian metric in M_{2m} and denote by $\mathring{\nabla}_{j}$ the covariant differentiation with respect to this Riemannian metric. If we put

$$\begin{aligned}\Gamma^{h}_{ji} = \mathring{\Gamma}^{h}_{ji} &- \frac{1}{4}(\mathring{\nabla}_{j}F_{i}{}^{l} + \mathring{\nabla}_{i}F_{j}{}^{l})F_{l}{}^{h} \\ &+ \frac{1}{4}(\mathring{\nabla}_{j}F_{l}{}^{h} - \mathring{\nabla}_{l}F_{j}{}^{h})F_{i}{}^{l}\end{aligned} \tag{2.5}$$

where $\mathring{\Gamma}^{h}_{ji}$ are Christoffel symbols, then we have

$$\nabla_{j}F_{i}{}^{h} = 0$$

and

$$S_{ji}{}^{h} = \frac{1}{2}(\Gamma^{h}_{ji} - \Gamma^{h}_{ij}) = \frac{1}{8}N_{ji}{}^{h}, \tag{2.6}$$

which show that if the Nijenhuis tensor vanishes, then the Γ^{h}_{ji} defined by (2.5) satisfies (2.1) and (2.2).

We call such an affine connexion Eckmann-Frölicher connexion [1], [2].

Now suppose that there is given an Eckmann-Frölicher connexion Γ^{h}_{ji} in M_{2m} and define the induced affine connexion $'\Gamma^{a}_{cb}$ on M_{2n} by (1.15). Then from (1.4) we find, by covariant differentiation along M_{2n},

$$F_{i}{}^{h}H_{cb}{}^{x}C_{x}{}^{i} = ('\nabla_{c}'F_{b}{}^{a})B_{a}{}^{h} + 'F_{b}{}^{d}H_{cd}{}^{x}C_{x}{}^{h},$$

from which

$$'\nabla_{c}'F_{b}{}^{a} = 0 \tag{2.7}$$

and

$$H_{cb}{}^{y}\,''F_{y}{}^{x} = H_{cd}{}^{x}\,'F_{b}{}^{d} \tag{2.8}$$

by virtue of (1.8).

On the other hand, we have from (1.15)

$$\begin{aligned}'S_{cb}{}^{a} &= \frac{1}{2}('\Gamma^{a}_{cb} - '\Gamma^{a}_{bc}) \\ &= B_{c}{}^{j}B_{b}{}^{i}B^{a}{}_{h}S_{ji}{}^{h} \\ &= \frac{1}{8}B_{c}{}^{j}B_{b}{}^{i}B^{a}{}_{h}N_{ji}{}^{h} \\ &= \frac{1}{8}{}'N_{cb}{}^{a}.\end{aligned} \tag{2.9}$$

Thus we have

THEOREM 2.1. *Affine connexion induced on an almost analytic submanifold from an Eckmann-Frölicher connexion with respect to affine normals is also an Eckmann-Frölicher connexion.*

From the equation of Gauss

$$'\nabla_c B_b{}^h = \partial_c B_b{}^h + B_c{}^j B_b{}^i \Gamma_{ji}^h - '\Gamma_{cb}^a B_a{}^h = H_{cb}{}^x C_x{}^h,$$

we find

$$B_c{}^i B_b{}^i S_{ji}{}^h - 'S_{cb}{}^a B_a{}^h = \frac{1}{2}(H_{cb}{}^x - H_{bc}{}^x) C_x{}^h$$

or

(2.10) $$H_{cb}{}^x = H_{bc}{}^x$$

by virtue of

$$S_{ji}{}^h = \frac{1}{8} N_{ji}{}^h, \qquad 'S_{cb}{}^a = \frac{1}{8} 'N_{cb}{}^a$$

and

$$B_c{}^j B_b{}^i N_{ji}{}^h = 'N_{cb}{}^a B_a{}^h.$$

Thus we have

THEOREM 2.2. *The second fundamental form of an almost analytic submanifold in an almost complex space with Eckmann-Frölicher connexion with respect to affine normals is symmetric*

Now from (2.8) and (2.10), we find

$$H_{cd}{}^x {'F_b{}^d} - H_{bd}{}^x {'F_c{}^d} = 0$$

or

(2.11) $$H_{cb}{}^x + 'F_c{}^e {'F_b{}^d} H_{ed}{}^x = 0,$$

from which

THEOREM 2.3. *For an almost analytic submanifold in an almost complex space with an Eckmann-Frölicher connexion, the second fundamental tensor with respect to affine normals is pure with respect to two lower indices.*

On the other hand, from (1.8) we have, by covariant differentiation along M_{2n},

$$F_i{}^h(-K_c{}^a{}_y B_a{}^i + L_{cy}{}^x C_x{}^i) = ('\nabla_c {''F_y{}^x}) C_x{}^h + ''F_y{}^z(-K_c{}^a{}_z B_a{}^h + L_{cz}{}^x C_x{}^h),$$

from which

(2.12) $$K_c{}^b{}_y {'F_b{}^a} = K_c{}^a{}_z {''F_y{}^z}$$

and

(2.13) $$L_{cy}{}^z {''F_z{}^x} = '\nabla_c {''F_y{}^x} + L_{cz}{}^x {''F_y{}^z}.$$

When a Hermitian metric g_{ji} is given in the almost complex space M_{2m}, we have

$$F_j{}^m F_i{}^l g_{ml} = g_{ji}$$

and consequently the induced Riemannian metric

$$'g_{cb} = B_c{}^j B_b{}^i g_{ji}$$

satisfies

$$'F_c{}^e\, 'F_b{}^d\, 'g_{ed} = 'g_{cb},$$

that is, $'g_{cb}$ is also Hermitian. Thus from Theorem 2.3, we have

$$'g^{cb} H_{cb}{}^x = 0. \tag{2.14}$$

Thus

THEOREM 2.4. *For an almost analytic submanifold in an almost complex space with a Hermitian metric and an Eckmann-Frölicher connexion, the second fundamental form* $H_{cb}{}^x$ *with respect to affine normals satisfies* (2.14).

BIBLIOGRAPHY

[1] ECKMANN, B., Structures complexes et transformations infinitésimales. Convegno di Geom. Diff. (1954), 1-9.

[2] FRÖLICHER, A., Zur Differentialgeometrie der komplexen Strukturen. Math. Ann. **129** (1955), 50-95.

[3] SCHOUTEN J. A., AND K. YANO, On invariant subspaces in the almost complex X_{2n}. Indag. Math. **17** (1955), 261-269.

[4] YANO, K., The theory of Lie derivatives and its applications. Amsterdam (1957).

DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.