GENERAL CONNECTIONS $A\Gamma A$ AND THE PARALLELISM OF LEVI-CIVITA

BY TOMINOSUKE ŌTSUKI

In a previous paper [10], the author showed that for a normal general connection Γ[1] of an n-dimensional differentiable manifold $\mathfrak{X}$ we can define naturally two normal general connections $'\Gamma$ and $''\Gamma$ called the contravariant part and the covariant part of Γ respectively. In the present paper, the author will show that we can define products of a general connection and tensor fields of type (1, 1) on $\mathfrak{X}$ satisfying the associative law. According to this concept, $'\Gamma=Q\Gamma$ and $''\Gamma=\Gamma Q$, where Q is the inverse of P in the sense that $Q|P(T(\mathfrak{X}))=(P|P(T(\mathfrak{X})))^{-1}$ and $Q|P^{-1}(0)=P|P^{-1}(0)$ at each point of $\mathfrak{X}$. As an application, he will investigate a normal general connection $A\Gamma A$, where Γ is a metric regular general connection with respect to a metric tensor, A is a projection of $T(\mathfrak{X})$ and $A(T(\mathfrak{X}))$ and $A^{-1}(0)$ are invariant under $P=\lambda(\Gamma)$ respectively. Then, he will show that the well known parallelism of Levi-Civita in Riemannian geometry can be considered as a parallelism by means of a sort of general connections.

In this paper, the author will use the notations in [7], [8], [9], [10].

§1. Products of a general connection and tensor fields of type (1.1).

Let $\mathfrak{X}$ be a differentiable manifold of dimension n and Γ be a general connection of $\mathfrak{X}$ which is written in terms of local coordinates u^i as

(1.1) $$\Gamma=\partial u_j\otimes(P^j_i d^2u^i+\Gamma^j_{ih}du^i\otimes du^h)$$

or

(1.2) $$\Gamma=\partial u_j\otimes(d(P^j_i du^i)+\Lambda^j_{ih}du^i\otimes du^h),$$

where

(1.3) $$\Lambda^j_{ih}=\Gamma^j_{ih}-\frac{\partial P^j_i}{\partial u^h}.$$

For each coordinate neighborhood (U, u^i), we have two mappings

$$f_U\colon\ U\to\mathfrak{M}^2_n=\{(a^j_i,\ a^j_{ih})\}\ ^{2)}$$

by

(1.4) $$a^j_i\cdot f_U=P^j_i,\qquad a^j_{ih}\cdot f_U=\Gamma^j_{ih}$$

Received November 9, 1961.

1) See [8].

2) See [10], §2 or [7], §1.

and

$$\tilde{f}_U\colon\ U \to \tilde{\mathfrak{L}}_n^2 = \{(a_i^j,\ a_{ih}^j,\ p_i^j) \mid |a_i^j| \neq 0\}$$ [3]

by

(1.5) $$a_i^j\cdot\tilde{f}_U = \delta_i^j,\quad a_{ih}^j\cdot\tilde{f}_U = \Lambda_{ih}^j,\quad p_i^j\cdot\tilde{f}_U = -p_i^j,$$

and the systems $\{f_U\}$ and $\{\tilde{f}_U\}$ satisfy the equations:

(1.6) $$(\sigma\cdot g_{VU})f_U = f_V g_{VU}$$

and

(1.7) $$g_{VU}\tilde{f}_U = \tilde{f}_V(\sigma\cdot g_{VU}),$$

where

$$g_{VU}\colon\quad U\frown V \to \mathfrak{L}_n^2 = \{(a_i^j,\ a_{ih}^j) \mid |a_i^j| \neq 0\}$$

is given by

(1.8) $$a_i^j\cdot g_{VU} = \frac{\partial v^j}{\partial u^i},\qquad a_{ih}^j\cdot g_{VU} = \frac{\partial^2 v^j}{\partial u^h\partial u^i},$$

σ: $\mathfrak{M}_n^2 \to M_n^1 = \{(a_i^j)\}$ and $\tilde{\mathfrak{L}}_n^2 \to L_n^1 = \{(a_i^j) \mid |a_i^j| \neq 0\}$ is the homomorphism

$$\sigma((a_i^j,\ a_{ih}^j)) = (a_i^j),\qquad \sigma((a_i^j,\ a_{ih}^j,\ p_i^j)) = (a_i^j)$$

and $M_n^1 \subset \mathfrak{M}_n^2$, $\mathfrak{L}_n^2 \subset \tilde{\mathfrak{L}}_n^2$, putting

$$(a_i^j) = (a_i^j,\ 0),\qquad (a_i^j,\ a_{ih}^j) = (a_i^j,\ a_{ih}^j,\ a_i^j).$$

The two systems of mappings $\{f_U\}$ and $\{\tilde{f}_U\}$ satisfying (1.6) and (1.7) characterize the general connection Γ respectively.

From (1.6), we get

$$(\sigma\cdot g_{VU})(\sigma\cdot f_U) = (\sigma\cdot f_V)(\sigma\cdot g_{VU}),$$

hence $\{\sigma\cdot f_U\}$ defines a tensor field of type (1.1) with local components P_i^j denoted by

(1.9) $$P = \partial u_j \otimes P_i^j du^i = \lambda(\Gamma).$$

Now, $Q = \partial u_j \otimes Q_i^j du^i$ be a tensor field on $\mathfrak{X}$. For each coordinate neighborhood $(U,\ u^i)$, we define two mappings

$$q_U\colon\ U \to \mathfrak{M}_n^2 \qquad\text{and}\qquad \tilde{q}_U\colon\ U \to \tilde{\mathfrak{L}}_n^2$$

by

(1.10) $$a_i^j\cdot q_U = Q_i^j,\qquad a_{ih}^j\cdot q_U = 0$$

and

(1.11) $$a_i^j\cdot\tilde{q}_U = \delta_i^j,\quad a_{ih}^j\cdot\tilde{q}_U = 0,\quad p_i^j\cdot\tilde{q}_U = Q_i^j.$$

They satisfy the equations:

(1.12) $$(\sigma\cdot g_{VU})q_U = q_V(\sigma\cdot g_{VU}) \quad\text{in}\quad \mathfrak{M}_n^2$$

---

3) See [10], §2 or [7], §8.

and

$$(\sigma\cdot g_{VU})\tilde{q}_U=\tilde{q}_V(\sigma\cdot g_{VU}) \quad \text{in} \quad \tilde{\mathfrak{L}}^2_n. \tag{1.13}$$

By virtue of (1.6) and (1.12), the system $\{q_U f_U\}$ defines a general connection which we denote by $Q\Gamma$. Analogously, by virtue of (1.7) and (1.13), the system $\{\tilde{f}_U\tilde{q}_U\}$ defines a general connection which we denote by ΓQ. They can be written in terms of local coordinates u^i as

$$Q\Gamma=\partial u_k Q^k_j\otimes(P^j_i d^2u^i+\Gamma^j_{ih}du^i\otimes du^h) \tag{1.14}$$

and

$$\begin{aligned}\Gamma Q&=\partial u_j\otimes(d(P^j_kQ^k_i du^i)+\Lambda^j_{kh}Q^k_i du^i\otimes du^h)\\&=\partial u_j\otimes(P^j_k d(Q^k_i du^i)+\Gamma^j_{kh}(Q^k_i du^i)\otimes du^h).\end{aligned} \tag{1.15}$$

Since we have

$$(Q^j_i,\ 0)(P^j_i,\ \Gamma^j_{ih})=(Q^j_kP^k_i,\ Q^j_k\Gamma^k_{ih})$$

and

$$(\delta^j_i,\ \Lambda^j_{ih},\ -P^j_i)(\delta^j_i,\ 0,\ Q^j_i)=(\delta^j_i,\ \Lambda^j_{kh}Q^k_i,\ -P^j_kQ^k_i).$$[4]

PROPOSITION 1.1. *The multiplication of general connections and tensor fields of type* (1.1) *satisfies the associative law.*

This is easily verified from (1.14) and (1.15). According to Proposition 1.1, we may write the products of a general connection Γ and tensor fields Q, R of type (1.1) as

$$R(Q\Gamma)=RQ\Gamma,\quad (\Gamma Q)R=\Gamma QR,\quad (Q\Gamma)R=Q(\Gamma R)=Q\Gamma R,\quad \text{etc.}$$

EXAMPLE 1. Let Γ be a normal general connection[5] of $\mathfrak{X}$ and put $P=\lambda(\Gamma)$. Let Q be the tensor field of type (1.1) on $\mathfrak{X}$ such that

$$Q\,|P(T(\mathfrak{X}))=(P\,|P(T(\mathfrak{X})))^{-1}\quad\text{and}\quad Q(P^{-1}(0))=0$$

at each point of $\mathfrak{X}$. $PQ=QP=A$ is the projection of $T(\mathfrak{X})$ onto $P(T(\mathfrak{X}))$ according to the direct sum decomposition

$$T_x(\mathfrak{X})\cong P(T_x(\mathfrak{X}))+P^{-1}(0),\qquad x\in\mathfrak{X}.$$

Then, the normal general connections $'\Gamma$ and $''\Gamma$ called the contravariant part[6] and the covariant part of Γ can be written as

$$'\Gamma=Q\Gamma\quad\text{and}\quad ''\Gamma=\Gamma Q.$$

Since the tensor field for A analogous to Q for P is A itself, we have

$$\begin{aligned}'('\Gamma)&=AQ\Gamma=Q\Gamma={}'\Gamma,\\ ''('\Gamma)&=\Gamma^{\cdot}=Q\Gamma A,\end{aligned}$$

4) See [10], § 2.
5) See [8], § 2 and § 3.
6) See [10], § 3.

$$'(''\Gamma) = \Gamma^{\cdot\cdot} = A\Gamma Q,$$
$$''(''\Gamma) = \Gamma QA = \Gamma Q = ''\Gamma.$$

And so, we have

$$'(\Gamma^{\cdot}) = A\Gamma^{\cdot} = AQ\Gamma A = Q\Gamma A = \Gamma^{\cdot},$$
$$''(\Gamma^{\cdot}) = \Gamma^{\cdot} A = Q\Gamma A^2 = Q\Gamma A = \Gamma^{\cdot}$$

and analogously, $'(\Gamma^{\cdot\cdot}) = ''(\Gamma^{\cdot\cdot}) = \Gamma^{\cdot\cdot}$. Thus we see that *the operations "$'$" and "$''$" are closed and the general connections $\Gamma^{\cdot}$ and $\Gamma^{\cdot\cdot}$ are stationary with respect to these operations.*

EXAMPLE 2. Let Γ be any general connection. $P = \lambda(\Gamma)$ has an integer $r(x)$ at each $x \in \mathfrak{X}$ such that

$$\operatorname{rank} P > \operatorname{rank} P^2 > \cdots > \operatorname{rank} P^{r(x)} = \operatorname{rank} P^{r(x)+1} = \cdots.$$

We have $\max r(x) \leqq n$. We assume that $\operatorname{rank} P^r = m$ is constant. Then the connections $P^q \Gamma P^{r-q-1}$, $q = 0, 1, \cdots, r$, are normal general connections.

§ 2. The general connection $A\Gamma A$.

Let Γ be a general connection and A be a tensor field of type (1.1). We denote the covariant differential operators for Γ and $\tilde{\Gamma} = A\Gamma A$ by D and $\tilde{D}$ respectively. Putting

$$\Gamma = \partial u_j \otimes (P^j_i d^2 u^i + \Gamma^j_{ih} du^i \otimes du^h) \quad \text{and} \quad \tilde{\Gamma} = \partial u_j \otimes (\tilde{P}^j_i d^2 u^i + \tilde{\Gamma}^j_{ih} du^i \otimes du^h),$$

we have by (1.14) and (1.15) the equations

$$\tilde{P}^j_i = A^j_k P^k_l A^l_i, \qquad \tilde{\Gamma}^j_{ih} = A^j_k \Gamma^k_{lh} A^l_i + A^j_k P^k_l \frac{\partial A^l_i}{\partial u^h}. \tag{2.1}$$

For any contravariant tensor field $V = V^i \partial u_i$, we have[7]

$$\begin{aligned} \tilde{D} V^j &= \tilde{P}^j_i dV^i + \tilde{\Gamma}^j_{ih} V^i du^h \\ &= A^j_k P^k_l A^l_i dV^i + A^j_k \left(\Gamma^k_{lh} A^l_i + P^k_l \frac{\partial A^l_i}{\partial u^h} \right) V^i du^h \\ &= A^j_k \{ P^k_l d(A^l_i V^i) + \Gamma^k_{lh} A^l_i V^i du^h \} \\ &= A^j_k D \tilde{V}^k, \end{aligned}$$

where

$$\tilde{V}^k = A^k_i V^i.$$

Analogously, for any covariant tensor field $W = W_i du^i$ we have

$$\begin{aligned} \tilde{D} W_i &= d(W_j \tilde{P}^j_i) - W_j \tilde{\Gamma}^j_{ih} du^h \\ &= d(W_j A^j_k P^k_l A^l_i) - W_i A^i_k \left(\Gamma^k_{lh} A^l_i + P^k_l \frac{\partial A^l_i}{\partial u^h} \right) du^h \end{aligned}$$

7) See [7], (2.15).

$$= \{d((W_j A^j_k)P^k_l) - (W_i A^i_k)\Gamma^k_{lh} du^h\} A^l_i$$
$$= A^j_i D\tilde{W}_j,$$

where

$$\tilde{W}_i = W_j A^j_i.$$

Making use of the homomorphisms A and ι_A of the tangent tensor bundles of $\mathfrak{X}$ naturally defined from A in [10],[8] the above equations give the formulas

$$\tilde{D}V = \iota_A DA(V) \quad \text{and} \quad \tilde{D}W = \iota_A DA(W), \tag{2.2}$$

where

$$V \in \Psi(T((\mathfrak{X})) \quad \text{and} \quad W \in \Psi(T^*(\mathfrak{X})).$$[9]

THEOREM 2.1. *The covariant differential operator $\tilde{D}$ for $A\Gamma A$ can be written as*

$$\tilde{D} = \iota_A DA.$$

Proof. It is sufficient to show that if for any two tensor fields V and W we have

$$\tilde{D}V = \iota_A DA(V) \quad \text{and} \quad \tilde{D}W = \iota_A DA(W),$$

then we get $\tilde{D}(V \otimes W) = \iota_A DA(V \otimes W)$. In fact, by means of the formula (2.19) in [7] we have

$$\begin{aligned}\tilde{D}(V \otimes W) &= \varepsilon(\tilde{D}V \otimes \tilde{P}(W)) + \tilde{P}(V) \otimes \tilde{D}W \\ &= \varepsilon(\iota_A DA(V) \otimes APA(W)) + APA(V) \otimes \iota_A DA(W) \\ &= \iota_A\{\varepsilon(DA(V) \otimes PA(W)) + PA(V) \otimes DA(W)\} \\ &= \iota_A D(A(V) \otimes A(W)) = \iota_A DA(V \otimes W). \qquad \text{q.e.d.}\end{aligned}$$

THEOREM 2.2. *If Γ is regular, A is a projection of $T(\mathfrak{X})$ and $A(T(\mathfrak{X}))$ is invariant under $P = \lambda(\Gamma)$, then the general connection $A\Gamma A$ is normal and proper.*[10]

Proof. When $A = 1$, the theorem is evident. When $A \neq 1$, $N = 1 - A$ is a projection of $T(\mathfrak{X})$. At each point x of $\mathfrak{X}$, we put $A_x = A(T_x(\mathfrak{X}))$ and $N_x = N(T_x(\mathfrak{X}))$, then

$$A_x \frown N_x = 0.$$

Since $P|A_x$ is an isomorphism and $A|A_x = 1$, we have

$$APA(T_x(\mathfrak{X})) = AP(A_x) = A(A_x)) = A_x$$

and $APA|A_x$ is an isomorphism. Since $T_x(\mathfrak{X}) \cong A_x \otimes N_x$ and $APA(N_x) = 0$, APA

8) See [10], (3.8).

9) For any vector bundle $\mathfrak{F} = (\mathfrak{Z}, \mathfrak{X}, \pi)$ over $\mathfrak{X}$, we denote the vector space consisting of all cross-sections of $\mathfrak{F}$ by $\Psi(\mathfrak{F})$.

10) See [8], §5 or [9], 1.

$=\lambda(A\Gamma A)$ is normal. Hence, the general connection $A\Gamma A$ is normal.

In the next place, the projection of $T(\mathfrak{X})$ onto $APA(T(\mathfrak{X}))$ corresponding to the normal tensor field APA is A itself. By virtue of Proposition 1.1, we have

$$N(A\Gamma A)=(NA)\Gamma A=O\Gamma A=0.$$

Hence, the general connection $A\Gamma A$ is proper.

§ 3. Some properties of $A\Gamma A$ when Γ is a metric general connection.

Let be given a non-singular symmetric tensor field $G=g_{ij}du^i\otimes du^j$. We say that a general connection Γ is *metric* with respect to G, if $DG=0$.

THEOREM 3.1. *If a regular general connection Γ is metric with respect to G and satisfies the conditions:*

$$S^j_{ih}=\frac{1}{2}(\Gamma^j_{ih}-\Gamma^j_{hi})=\frac{1}{2}(P^j_{i;h}-P^j_{h;i}),\text{[11]} \tag{3.1}$$

where the semi-colon "$;$" denotes the covariant derivative with respect to the Levi-Civita's connection made by G, then the covariant part $''\Gamma$ of Γ is the Levi-Civita's connection.

Proof. Since Γ is regular, we put $Q=P^{-1}$, $P=\lambda(\Gamma)$. Then we have

$$\Gamma^j_{ih}={}''\Gamma^j_{kh}P^k_i+\frac{\partial P^j_i}{\partial u^h}, \tag{3.2}$$

putting $''\Gamma=\Gamma Q=\partial u_j\otimes(d^2u^j+{}''\Gamma^j_{ih}du^i\otimes du^h)$. Putting these into (3.1), we have

$$\left({}''\Gamma^j_{kh}P^k_i+\frac{\partial P^j_i}{\partial u^h}\right)-\left({}''\Gamma^j_{ki}P^k_h+\frac{\partial P^j_h}{\partial u^i}\right)$$
$$=\left(\frac{\partial P^j_i}{\partial u^h}+\begin{Bmatrix} j\\ l\,h\end{Bmatrix}P^l_i-\begin{Bmatrix} l\\ ih\end{Bmatrix}P^j_l\right)-\left(\frac{\partial P^j_h}{\partial u^i}+\begin{Bmatrix} j\\ li\end{Bmatrix}P^l_h-\begin{Bmatrix} l\\ hi\end{Bmatrix}P^j_l\right),$$

hence

$$\left({}''\Gamma^j_{kh}-\begin{Bmatrix} j\\ kh\end{Bmatrix}\right)P^k_i=\left({}''\Gamma^j_{ki}-\begin{Bmatrix} j\\ ki\end{Bmatrix}\right)P^k_h,$$

where $\begin{Bmatrix} j\\ ih\end{Bmatrix}$ are the Christoffel symbols of the second kind made by g_{ij}. Let us put

$$X^j_{ih}={}''\Gamma^j_{ih}-\begin{Bmatrix} j\\ ih\end{Bmatrix},$$

then the above equations can be written as

$$X^j_{kh}P^k_i=X^j_{ki}P^k_h.$$

11) See [9], 2. The condition (iv) of Theorem 2 is written as this.

In the next place, we have

$$g_{ij,h} = P_i^l P_j^k g_{lk|h} = 0,$$

hence

$$g_{ij|h} = \frac{\partial g_{ij}}{\partial u^h} - g_{lj}{}''\Gamma_{ih}^l - g_{il}{}''\Gamma_{jh}^l = 0,$$

where the symbol "|" denotes the basic covariant derivative with respect to Γ.[12] As is well known, we have

$$g_{ij;h} = \frac{\partial g_{ij}}{\partial u^h} - g_{lj}\left\{{l \atop ih}\right\} - g_{il}\left\{{l \atop jh}\right\} = 0.$$

We get immediately from these two equations

$$g_{lj}X_i{}^l{}_h + g_{il}X_j{}^l{}_h = 0.$$

Putting $X_{ijh} = g_{jk}X_i{}^k{}_h$, we obtain

$$X_{ijh} + X_{jih} = 0 \quad \text{and} \quad X_{kjh}P_i^k = X_{kji}P_h^k.$$

Now, if we put $Y_{ijh} = X_{lkh}P_i^l P_j^k$, we get from the above equations

$$Y_{ijh} + Y_{jih} = 0 \quad \text{and} \quad Y_{ijh} = Y_{hji},$$

hence it must be that $Y_{ijh} = 0$ and so $X_{ijh} = 0$. Accordingly we have

$$''\Gamma_{ih}^j = \left\{{j \atop ih}\right\}, \tag{3.3}$$

which shows that the covariant part of Γ is the Levi-Civita's connection made by G.

REMAKR. In [9], the author showed that the condition (iv) in Theorem 2:

$$S_{kh}^l A_i^k = \frac{1}{2} A_l^j (P_{k;h}^l - P_{h;k}^l) A_i^k$$

is a generalization of the symmetric condition in the classical case. Theorem 3.1 shows that the condition is very natural.

THEOREM 3.2. *Let Γ be a metric regular general connection with respect to a non-singular symmetric tensor $G = g_{ij}du^i \otimes du^j$ on $\mathfrak{X}$ and A be a projection of $T(\mathfrak{X})$ such that A_x and N_x are invariant under P and orthogonal with respect to G at each point x of $\mathfrak{X}$, where $N = 1 - A$, $A_x = A(T_x(\mathfrak{X}))$ and $N_x = N(T_x(\mathfrak{X}))$. If Γ satisfies the condition* (3.1) *in Theorem* 3.1, *then $\tilde{\Gamma} = A\Gamma A$ is a normal, proper general connection which is metric with respect to G and $\tilde{G} = A(G) = g_{hk}A_i^h A_j^k du^i \otimes du^j$ and satisfies the generalized symmetric condition*:

$$\tilde{S}_{k}{}^{j}{}_{h} A_i^k = \frac{1}{2} A_l^j (\tilde{P}_{k;h}^l - \tilde{P}_{h;k}^l) A_i^k, \tag{3.4}$$

12) See [7], (3.7).

where

$$\tilde{S}_{k}{}^{j}{}_{h}=\frac{1}{2}(\tilde{\Gamma}^{j}_{kh}-\tilde{\Gamma}^{j}_{hk}),\qquad \tilde{P}=P^{j}_{i}\partial u_{j}\otimes du^{i}=\lambda(\tilde{\Gamma}).$$

Proof. From the assumption and Theorem 2.2, it is clear that $\tilde{\Gamma}=A\Gamma A$ is normal and proper. At first, we prove that $\tilde{\Gamma}=A\Gamma A$ is metric with respect to G. By means of Theorem 2.1, it is sufficient to prove that

$$\bar{g}_{lk,h}A^{l}_{i}A^{k}_{j}=0. \tag{3.5}$$

Since $\delta^{j}_{i}=A^{j}_{i}+N^{j}_{i}$ and Γ is metric with respect to G,

$$\bar{g}_{ij}=g_{lk}A^{l}_{i}A^{k}_{j}=g_{ij}-g_{lj}N^{l}_{i}-g_{ik}N^{k}_{j}+g_{lk}N^{l}_{i}N^{k}_{j}$$

and

$$g_{ij,h}=\frac{\partial(g_{lk}P^{l}_{i}P^{k}_{j})}{\partial u^{h}}-g_{lk}\Gamma^{l}_{ih}P^{k}_{j}-g_{lk}P^{l}_{i}\Gamma^{k}_{jh}=0.$$

Now, making use of these equations, we have

$$\begin{aligned}\bar{g}_{lk,h}A^{l}_{i}A^{k}_{j}&=\left\{\frac{\partial(\bar{g}_{st}P^{s}_{l}P^{t}_{k})}{\partial u^{h}}-\bar{g}_{st}\Gamma^{s}_{lh}P^{t}_{k}-\bar{g}_{st}P^{s}_{l}\Gamma^{t}_{kh}\right\}A^{l}_{i}A^{k}_{j}\\&=\left\{\frac{\partial}{\partial u^{h}}((g_{st}-g_{pt}N^{p}_{s}-g_{sq}N^{q}_{t}+g_{pq}N^{p}_{s}N^{q}_{t})P^{s}_{l}P^{t}_{k})\right.\\&\quad-(g_{st}-g_{pt}N^{p}_{s}-g_{sq}N^{q}_{t}+g_{pq}N^{p}_{s}N^{q}_{t})\Gamma^{s}_{lh}P^{t}_{k}\\&\quad\left.-(g_{st}-g_{pt}N^{p}_{s}-g_{sq}N^{q}_{t}+g_{pq}N^{p}_{s}N^{q}_{t})P^{s}_{l}\Gamma^{t}_{kh}\right\}A^{l}_{i}A^{k}_{j}\\&=\left\{-\left(g_{pt}\frac{\partial N^{p}_{s}}{\partial u^{h}}+g_{sq}\frac{\partial N^{q}_{t}}{\partial u^{h}}\right)P^{s}_{l}P^{t}_{k}\right.\\&\quad\left.+g_{pt}N^{p}_{s}\Gamma^{s}_{lh}P^{t}_{k}+g_{sq}N^{q}_{t}P^{s}_{l}\Gamma^{t}_{kh}\right\}A^{l}_{i}A^{k}_{j},\end{aligned}$$

since we have $N^{j}_{k}P^{k}_{l}A^{l}_{i}=N^{j}_{k}(A^{k}_{h}P^{h}_{l}A^{l}_{i})=0$ from the assumption A_x invariant under P and $AN=NA=0$. Since A_x and N_x are orthogonal with respect to G, the above equation becomes

$$\begin{aligned}\bar{g}_{k,h}A^{l}_{i}A^{k}_{j}&=-\left(g_{pt}\frac{\partial N^{p}_{s}}{\partial u^{h}}P^{s}_{l}A^{l}_{i}P^{t}_{k}A^{k}_{j}+g_{sq}\frac{\partial N^{q}_{t}}{\partial u^{h}}P^{t}_{k}A^{k}_{j}P^{s}_{l}A^{l}_{i}\right)\\&=g_{pt}N^{p}_{s}P^{t}_{k}A^{k}_{j}\frac{\partial(P^{s}_{l}A^{l}_{i})}{\partial u^{h}}+g_{sq}N^{q}_{t}P^{s}_{l}A^{l}_{i}\frac{\partial(P^{t}_{k}A^{k}_{j})}{\partial u^{h}}\\&=0.\end{aligned}$$

Thus, (3.5) is proved. Hence we have

$$\tilde{D}G=0.$$

By means of Theorem 2.1 and $A^{2}=A$, we get

$$\tilde{D}\bar{G}=\iota_{A}DA(A(G))=\iota_{A}DA(G)=\tilde{D}G=0.$$

Now, we shall prove (3.4). By means of (2.1), we have

$$\tilde{S}_{k}{}^{j}{}_{h}A_{i}^{k}=\frac{1}{2}\left[\left(A_{l}^{j}\Gamma_{mh}^{l}A_{k}^{m}+A_{l}^{j}P_{m}^{l}\frac{\partial A_{k}^{m}}{\partial u^{h}}\right)-\left(A_{l}^{j}\Gamma_{mk}^{l}A_{h}^{m}+A_{l}^{j}P_{m}^{l}\frac{\partial A_{h}^{m}}{\partial u^{k}}\right)\right]A_{i}^{k}$$
$$=\frac{1}{2}A_{l}^{j}\left[\Gamma_{kh}^{l}-\Gamma_{mk}^{l}A_{h}^{m}+P_{m}^{l}\left(\frac{\partial A_{k}^{m}}{\partial u^{h}}-\frac{\partial A_{h}^{m}}{\partial u^{k}}\right)\right]A_{i}^{k}.$$

Making use of (3.2) and (3.3), the above equations can be written as

$$\tilde{S}_{k}{}^{j}{}_{h}A_{i}^{k}=\frac{1}{2}A_{l}^{j}\left[\left\{{l \atop th}\right\}P_{k}^{t}+\frac{\partial P_{k}^{l}}{\partial u^{h}}-\left\{{l \atop tk}\right\}P_{m}^{t}A_{h}^{m}-\frac{\partial P_{m}^{l}}{\partial u^{k}}A_{h}^{m}+P_{m}^{l}\left(\frac{\partial A_{k}^{m}}{\partial u^{h}}-\frac{\partial A_{h}^{m}}{\partial u^{k}}\right)\right]A_{i}^{k}$$
$$=\frac{1}{2}A_{l}^{j}\left[\left\{{l \atop th}\right\}P_{m}^{t}A_{k}^{m}+\frac{\partial(P_{m}^{l}A_{k}^{m})}{\partial u^{h}}-\left\{{l \atop tk}\right\}P_{m}^{t}A_{h}^{m}-\frac{\partial(P_{m}^{l}A_{h}^{m})}{\partial u^{k}}\right]A_{i}^{k}$$
$$=\frac{1}{2}A_{l}^{j}\left[(P_{m}^{l}A_{k}^{m})_{;h}-(P_{m}^{l}A_{h}^{m})_{;k}\right]A_{i}^{k}$$
$$=\frac{1}{2}A_{l}^{j}\left[A_{t}^{l}P_{m}^{t}A_{k}^{m})_{;h}-(A_{t}^{l}P_{m}^{t}A_{h}^{m})_{;k}-A_{t;h}^{l}P_{m}^{t}A_{k}^{m}+A_{t;k}^{l}P_{m}^{t}A_{h}^{m}\right]A_{i}^{k}$$
$$=\frac{1}{2}A_{l}^{j}(\tilde{P}_{k;h}^{l}-\tilde{P}_{h;k}^{l})A_{i}^{k}+\frac{1}{2}A_{l}^{j}(N_{t;h}^{l}P_{m}^{t}A_{i}^{m}-N_{t;k}^{l}P_{m}^{t}A_{h}^{m}A_{i}^{k}).$$

Since we have

$$A_{l}^{j}N_{t;h}^{l}P_{m}^{t}A_{i}^{m}=-A_{l;h}^{j}(N_{t}^{l}P_{m}^{t}A_{i}^{m})$$
$$=0,$$

finally we obtain the equations:

$$\tilde{S}_{k}{}^{j}{}_{h}A_{i}^{k}=\frac{1}{2}A_{l}^{j}(\tilde{P}_{k;h}^{l}-\tilde{P}_{h;k}^{l})A_{i}^{k}. \qquad \text{q.e.d.}$$

§ 4. A geometrical meaning of $A\Gamma A$ when Γ is the Levi-Civita's connection.

Let us consider an n-dimensional Riemann space $\mathfrak{X}$ with a metric tensor $G=g_{ij}du^{i}\otimes du^{j}$ and Γ be the Levi-Civita's connection made by G. Let $\mathfrak{Y}$:

$$u^{j}=u^{j}(v^{1}, v^{2}, \cdots, v^{m})$$

be an m-dimensional subspace of $\mathfrak{X}$. Putting

$$(4.1)\qquad \begin{cases} B_{\alpha}^{i}=\dfrac{\partial u^{j}}{\partial v^{\alpha}}, & g_{\alpha\beta}=g_{ij}B_{\alpha}^{i}B_{\beta}^{j}, \\ B_{i}^{\alpha}=g^{\alpha\beta}g_{ij}B_{\beta}^{j}, & \alpha, \beta=1, 2, \cdots, m, \end{cases}$$

the formulas:

$$(4.2)\qquad \left\{{\beta \atop \alpha\gamma}\right\}_{v}=B_{j}^{\beta}\frac{\partial B_{\alpha}^{j}}{\partial v^{\gamma}}+B_{j}^{\beta}B_{\alpha}^{i}B_{\gamma}^{h}\left\{{j \atop ih}\right\}_{u}$$

are well known, where $\left\{{j \atop ih}\right\}_{u}$ are the Christoffel symbols of the second kind made by g_{ij} and $\left\{{\beta \atop \alpha\gamma}\right\}_{v}$ are the ones made by $g_{\alpha\beta}$ which are the local components of the metric tensor of $\mathfrak{Y}$ induced from G.

Now, we may consider $\mathfrak{Y}$ as one of a family of m-dimensional subspaces of $\mathfrak{X}$:

$$u^j = u^j(v^1, v^2, \cdots, v^m, c^{m+1}, \cdots, c^n) \tag{4.3}$$

such that

$$\left|\frac{\partial u^j}{\partial v^i}\right| \neq 0, \qquad \text{putting} \quad c^{m+1} = v^{m+1}, \cdots, c^n = v^n.$$

Hence, $v^1, v^2, \cdots, v^n$ may be considered as local coordinates of $\mathfrak{X}$. Then,

$$A_i^j = B_\alpha^j B_i^\alpha \tag{4.4}$$

are clearly the local components of the orthogonal projection of $T_x(\mathfrak{X})$ onto the tangent space of the subspace of (4.3) through x with respect to local coordinates u^i which are independent of v^α. Denoting this projection of $T(\mathfrak{X})$ by A, let us consider the general connection $\tilde{\Gamma} = A\Gamma A$. Since Γ is a classical affine connection, that is $\lambda(\Gamma) = 1$, we have

$$\lambda(\tilde{\Gamma}) = A\lambda(\Gamma)A = A^2 = A \tag{4.5}$$

and

$$\tilde{\Gamma} = \partial u_j \otimes (A_i^j d^2u^i + \tilde{\Gamma}^j_{ih} du^i \otimes du^h),$$

where

$$\tilde{\Gamma}^j_{ih} = A_k^j\left(\begin{Bmatrix} k \\ l\,h \end{Bmatrix}_u A_i^l + \frac{\partial A_i^k}{\partial u^h}\right) \tag{4.6}$$

by virtue of (2.1). Denoting the components in local coordinates $v^1, \cdots, v^n$ by the notations with Greek indices λ, μ, ν, we have

$$\underset{v}{\tilde{\Gamma}}{}^\mu_{\lambda\nu} = \frac{\partial v^\mu}{\partial u^j}\left(A_l^j \frac{\partial^2 u^l}{\partial v^\nu \partial v^\lambda} + \tilde{\Gamma}^j_{ih} \frac{\partial u^i}{\partial v^\lambda}\frac{\partial u^h}{\partial v^\nu}\right)^{13)} \tag{4.7}$$

by (4.5). Putting (4.4), (4.6) into (4.7), we have

$$\begin{aligned}\underset{v}{\tilde{\Gamma}}{}^\mu_{\lambda\nu} &= \frac{\partial v^\mu}{\partial u^j} B_\alpha^j B_l^\alpha\left[\frac{\partial^2 u^l}{\partial v^\nu \partial v^\lambda} + \left(\begin{Bmatrix} l \\ kh \end{Bmatrix}_u A_i^k + \frac{\partial A_i^l}{\partial u^h}\right)\frac{\partial u^i}{\partial v^\lambda}\frac{\partial u^h}{\partial v^\nu}\right] \\ &= \delta_\alpha^\mu B_j^\alpha\left[\frac{\partial^2 u^j}{\partial v^\nu \partial v^\lambda} + \left(\begin{Bmatrix} j \\ kh \end{Bmatrix}_u A_i^k + \frac{\partial A_i^j}{\partial u^h}\right)\frac{\partial u^i}{\partial v^\lambda}\frac{\partial u^h}{\partial v^\nu}\right].\end{aligned}$$

Hence, we have

$$\underset{v}{\tilde{\Gamma}}{}^E_{\lambda\nu} = 0, \qquad E = m+1, \cdots, n, \tag{4.8}$$

and

$$\underset{v}{\tilde{\Gamma}}{}^\beta_{\lambda\nu} = B_j^\beta\left[\frac{\partial^2 u^j}{\partial v^\nu \partial v^\lambda} + \left(\begin{Bmatrix} j \\ kh \end{Bmatrix}_u A_i^k + \frac{\partial A_i^j}{\partial u^h}\right)\frac{\partial u^i}{\partial u^\lambda}\frac{\partial u^h}{\partial v^\nu}\right]. \tag{4.9}$$

---

13) See [7], (2.27).

Since B^i_α and B^α_i are invariant under A as tangent vectors and cotangent vectors of $\mathfrak{X}$, we have

$$B^\beta_j \frac{\partial A^j_i}{\partial v^\gamma}\frac{\partial u^i}{\partial v^\alpha} = B^\beta_j\left(\frac{\partial(A^j_i A^i_\alpha)}{\partial v^\gamma} - A^j_i\frac{\partial B^i_\alpha}{\partial v^\gamma}\right)$$
$$= B^\beta_j\frac{\partial B^j_\alpha}{\partial v^\gamma} - B^\beta_i\frac{\partial B^i_\alpha}{\partial v^\gamma} = 0,$$

hence

(4.10) $$\underset{v}{\tilde{\Gamma}}{}^\beta_{\alpha\gamma} = B^\beta_j\frac{\partial B^j_\alpha}{\partial v^\gamma} + B^\beta_j B^i_\alpha B^h_\gamma \left\{ {j \atop ih} \right\}_u = \left\{ {\beta \atop \alpha\gamma} \right\}_v.$$

On the other hand, the components of $\bar{G} = A(G)$ and A with respect to local coordinates v^λ are

(4.11) $$\begin{cases} \underset{v}{\bar{g}}_{\lambda\mu} = \bar{g}_{ij}\dfrac{\partial u^i}{\partial v^\lambda}\dfrac{\partial u^j}{\partial v^\mu} = g_{hk}A^h_i A^k_j \dfrac{\partial u^i}{\partial v^\lambda}\dfrac{\partial u^j}{\partial v^\mu}, \\ \underset{v}{A}{}^\mu_\lambda = \dfrac{\partial v^\mu}{\partial u^j}A^j_i\dfrac{\partial u^i}{\partial v^\lambda} = \delta^\mu_\alpha B^\alpha_i\dfrac{\partial u^i}{\partial v^\lambda}, \end{cases}$$

especially

(4.12) $$\begin{cases} \underset{v}{\bar{g}}_{\alpha\beta} = g_{hk}A^h_i B^i_\alpha A^k_j B^j_\beta = g_{hk}B^h_\alpha B^k_\beta = g_{\alpha\beta}, \\ \underset{v}{A}{}^\mu_\alpha = \delta^\mu_\alpha. \end{cases}$$

Here, *we assume that the tangent vectors*

$$\frac{\partial u^i}{\partial v^E}, \qquad E = m+1, \cdots, n,$$

are orthogonal to A_x *at each point* $x \in \mathfrak{X}$. Then, from (4.9), (4.11), we get

$$\underset{v}{\tilde{\Gamma}}{}^\beta_{E\nu} = B^\beta_j\left(\frac{\partial^2 u^j}{\partial v^\nu \partial v^E} + \frac{\partial A^j_i}{\partial v^\nu}\frac{\partial u^i}{\partial v^E}\right)$$
$$= B^\beta_j\frac{\partial^2 u^j}{\partial v^\nu\partial v^E} - B^\beta_j A^j_i\frac{\partial^2 u^i}{\partial v^\nu \partial v^E} = 0,$$

that is

(4.13) $$\underset{v}{\tilde{\Gamma}}{}^\beta_{E\nu} = 0$$

and

(4.14) $$\underset{v}{\bar{g}}_{E\mu} = 0, \qquad \underset{v}{A}{}^\mu_\lambda = \delta^\mu_\alpha\delta^\alpha_\lambda.$$

Accordingly, with respect to such local coordinates $v^1, \cdots, v^n$, the components with the indices $m+1, \cdots, n$ of a tensor invariant under A vanish. Hence, the covariant differential of such a tensor field, for example

$$V = V^\mu_\lambda \partial v_\mu \otimes dv^\lambda = V^\beta_\alpha \partial v_\beta \otimes dv^\alpha$$

is given by

$$\tilde{D}V = \partial v_\mu \otimes dv^\lambda \otimes \tilde{D}V^\mu_\lambda,$$

$$\tilde{D}V^\mu_\lambda = \underset{v}{A^\mu_\rho} d \underset{v}{V^\rho_\sigma} \underset{v}{A^\sigma_\lambda} + \underset{v}{\tilde{\Gamma}^\mu_{\rho\nu}} V^\rho_\sigma \underset{v}{A^\sigma_\lambda} dv^\nu - \underset{v}{A^\mu_\rho} V^\rho_\sigma \underset{v}{\tilde{\Gamma}^\sigma_{\lambda\nu}} dv^\nu.$$

Substituting the above equations into this, we have

$$\tilde{D}V^\mu_\lambda = \delta^\mu_\beta dV^\beta_\alpha \delta^\alpha_\lambda + \underset{v}{\tilde{\Gamma}^\mu_{\beta\nu}} V^\beta_\alpha \delta^\alpha_\lambda dv^\nu - \delta^\mu_\beta V^\beta_\alpha \underset{v}{\tilde{\Gamma}^\alpha_{\lambda\nu}} dv^\nu,$$

that is

$$(4.15) \qquad \begin{cases} \tilde{D}V^\beta_\alpha = dV^\beta_\alpha + \left\{ \begin{matrix} \beta \\ \delta\gamma \end{matrix} \right\}_v V^\delta_\alpha dv^\gamma - V^\beta_\delta \left\{ \begin{matrix} \delta \\ \alpha\gamma \end{matrix} \right\}_v dv^\gamma + \underset{v}{\tilde{\Gamma}^\beta_{\delta E}} V^\delta_\alpha dv^E - V^\beta_\delta \underset{v}{\tilde{\Gamma}^\delta_{\alpha E}} dv^E, \\ \tilde{D}V^E_\lambda = 0, \qquad \tilde{D}V^\lambda_E = 0. \end{cases}$$

When V is defined on the subspace (4.3), the first equation can be written as

$$\tilde{D}V^\beta_\alpha = dV^\beta_\alpha + \left\{ \begin{matrix} \beta \\ \delta\gamma \end{matrix} \right\}_v V^\delta_\alpha dv^\gamma - V^\beta_\delta \left\{ \begin{matrix} \delta \\ \alpha\gamma \end{matrix} \right\}_v dv^\gamma,$$

which shows that $\tilde{D}V^\beta_\alpha$ is the covariant differential of V^β_α with respect to the Levi-Civita's connection of the subspace. Thus, we obtain the following

THEOREM 4.1. *Let Γ be the Levi-Civita's connection of a Riemann space with a metric tensor G and $\mathfrak{Y}(c^{m+1}, \cdots, c^n)$ be a family of an m-dimensional subspaces simply covering $\mathfrak{X}$. Let A be the orthogonal projection of $T(\mathfrak{X})$ onto the tangent space of the family. The normal general connection $A\Gamma A$ is identical with the Levi-Civita's connections of the subspaces with the induced metric tensor from G, for the tensor fields defined on the subspaces which are invariant under A.*

This theorem shows us that the parallelism of Levi-Civita on a subspace in a Euclidean space is understood as a sort of parallelism by means of a metric general connection of the space.

REFERENCES

[1] CHERN, S. S., Lecture note on differential geometry. Chicago Univ. (1950).

[2] EHRESMANN, G., Les connexions infinitésimales dans un espace fibré différentiable. Colloque de Topologie (Espaces fibrés). (1950), 29–55.

[3] EHRESMANN, G., Les prolongements d'une variété différentiable, I. Calcul des jets, prolongement principal. C. R. Paris 233 (1951), 598–600.

[4] ŌTSUKI, T., Geometries of connections. Kyōritsu Shuppan Co. (1957). (in Japanese)

[5] ŌTSUKI, T., On tangent bundles of order 2 and affine connections. Proc. Japan Acad. 34 (1958), 325–330.

[6] ŌTSUKI, T., Tangent bundles of order 2 and general connections. Math. J. Okayama Univ. 8 (1958), 143–179.

[7] ŌTSUKI, T., On general connections, I. Math. J. Okayama Univ. **9** (1960), 99–164.

[8] ŌTSUKI, T., On general connections, II. Math. J. Okayama Univ. **10** (1961), 113–124.

[9] ŌTSUKI, T., On metric general connections. Proc. Japan Acad. **37** (1961), 183–188.

[10] ŌTSUKI, T., On normal general connections. Kōdai Math. Sem. Rep. **13** (1961), 152–166.

DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.