A DISTORTION THEOREM OF UNIVALENT FUNCTIONS RELATED TO SYMMETRIC THREE POINTS

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1. Let Σ be a family of functions $g(z)$ meromorphic and univalent for $|z|>1$ with Laurent expansion for $|z|>1$ given by

$$g(z)=z+c_0+\frac{c_1}{z}+\cdots.$$

The distortion inequality

$$\frac{(1-r^{-2})^2}{4r^2(1+r^{-2})^2}\leqq\frac{|g'(z)g'(-z)|}{|g(z)-g(-z)|^2}\leqq\frac{(1+r^{-2})^2}{4r^2(1-r^{-2})^2}\qquad(z=re^{i\theta})$$

for $g(z)$ belonging to Σ is easily obtained by combining the classical results. It can be also shown that the left and right equalities are attained by the functions $z+e^{i2\theta}z^{-1}$ and $z-e^{i2\theta}z^{-1}$ respectively.

We are concerned in the present paper with an analogous problem relating to symmetric three points z, $ze^{i2\pi/3}$ and $ze^{i4\pi/3}$. Analogous bounds will be obtained and the extremal functions will be closely connected with the above two functions. We remark that a known coefficient inequality $|c_2|\leqq 2/3$ can be proved from our theorem with respect to Σ ([2], [5], [6]) and that a distortion theorem of this type relating to four points cannot be obtained by using elementary functions as extremal functions. We use Jenkins' general coefficient theorem ([3], [4]) to prove our theorem and make a slight discussion to verify the extremal functions.

2. We now state the theorem.

THEOREM. *For all functions $g(z)$ belonging to Σ the inequalities*

$$\frac{(1-r^{-3})}{3\sqrt{3}\,r^3(1+r^{-3})^3}\leqq\frac{|g'(z)g'(z\omega)g'(z\omega^2)|}{|g(z)-g(z\omega)|\,|g(z\omega)-g(z\omega^2)|\,|g(z\omega^2)-g(z)|}$$
$$\leqq\frac{(1+r^{-3})^3}{3\sqrt{3}\,r^3(1-r^{-3})^3}$$

hold where $z=re^{i\theta}$, $r>1$ and $\omega=e^{i2\pi/3}$. The left equality occurs only for the function $g(z)=z(1+e^{i3\theta}z^{-3})^{2/3}+k$ and the right only for the function $g(z)=z(1-e^{i3\theta}z^{-3})^{2/3}+k$ with k as an arbitrary constant.

Proof. We first prove the left inequality. We set $R_j=r(1+r^{-3})^{2/3}\omega^j$, $j=0, 1, 2$, and consider a quadratic differential

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$$Q(w)\,dw^2 = \frac{w\,dw^2}{(w-R_0)^2(w-R_1)^2(w-R_2)^2}$$

on the w-plane. We denote by $\varDelta$ the complementary domain of the union of three segments $[0, 2^{2/3}\omega^j]$, $j=0, 1, 2$, which is evidently an admissible domain with respect to $Q(w)\,dw^2$ ([4]). The function $g_0(z) = z(1+z^{-3})^{2/3}$ maps the exterior of the unit circle onto $\varDelta$. Let $\Phi(w)$ be the inverse of $g_0(z)$ and put

$$u(z) = g(\Phi(w))$$

for any $g(z)$ belonging to Σ. We define $v(w)$ by the equation

$$(1) \qquad \frac{v-R_j}{v-R_{j+1}}\,\frac{R_{j+2}-R_{j+1}}{R_{j+2}-R_j} = \frac{u-u_j}{u-u_{j+1}}\,\frac{u_{j+2}-u_{j+1}}{u_{j+2}-u_j} \qquad (\text{mod } 3)$$

where $u_j = u(R_j)$. Then $v(w)$ becomes an admissible function associated with $\varDelta$ ([4]). The quadratic differential $Q(w)\,dw^2$ has only three double poles at the points R_j for $j=0, 1, 2$ and its local expansion in a neighborhood of each R_j with a local parameter $W=(w-R_j)^{-1}$ is of the form

$$Q(W) = \alpha_j W + \text{decreasing powers of } W$$

where $\alpha_j = 1/3R_0^3$. On the other hand $v(w)$ has the expansion, with the same parameter,

$$v(W) = a_j W + \text{decreasing powers of } W$$

where

$$a_j = \frac{(R_{j+1}-R_{j+2})(u_j-u_{j+1})(u_{j+2}-u_j)}{(R_j-R_{j+1})(R_{j+2}-R_j)(u_{j+1}-u_{j+2})}\,\frac{g_0'(r\omega^j)}{g'(r\omega^j)} \qquad (\text{mod } 3).$$

Then the general coefficient theorem ([4]) is available and we get

$$(2) \qquad \operatorname{Re}\sum_{j=0}^{2}\alpha_j \log a_j \leqq 0, \quad \text{i. e.} \quad \sum_{j=0}^{2}\log|a_j| \leqq 0$$

which implies that

$$\frac{|u_0-u_1||u_1-u_2||u_2-u_0|}{|R_0-R_1||R_1-R_2||R_2-R_0|}\left|\frac{g_0'(r)g_0'(r\omega)g_0'(r\omega^2)}{g'(r)g'(r\omega)g'(r\omega^2)}\right| \leqq 1.$$

We have the desired inequality for real r. In fact, by inserting $|R_j-R_{j+1}| = \sqrt{3}\,r(1+r^{-3})^{-1/3}$ and $g_0'(r\omega) = (1+r^{-3})^{-1/3}(1-r^{-3})$, we get

$$(3) \qquad \frac{(1-r^{-3})^3}{3\sqrt{3}\,r^3(1+r^{-3})^3} \leqq \prod_{j=0}^{2}\left|\frac{g'(r\omega^j)}{g(r\omega^j)-g(r\omega^{j+1})}\right|.$$

For general z, $z=re^{i\theta}$, it is only necessary to insert $G(z)=e^{i\theta}g(e^{-i\theta}z)$ in (3) instead of $g(z)$.

We can only conclude that $|a_j|=1$, $j=0, 1, 2$, from the equality assertion of the general coefficient theorem ([4]). Hence we make a slight discussion to show that equality occurs in (3) only for the function $g_0(z) = z(1+z^{-3})^{2/3}+k$

which implies the equality assertion in our theorem. We consider a function defined by

$$\zeta(w)=\int^{w}(Q(w))^{1/2}\,dw$$

in the complementary domain D of the union of the positive real axis and two segments $[0, R_1]$ and $[0, R_2]$. A suitable branch of $\zeta(w)$ maps the domain D onto a covering surface $\mathfrak{D}$ of a horizontal strip $-2\pi(1/3R_0)^{1/2}\leqq \operatorname{Im}\zeta\leqq 0$. If the equality holds in (2), it is easily shown, in the same way as in the equality proof of the general coefficient theorem ([4]), that the induced mapping $\eta(\zeta)$ by the function $v(w)$ maps any horizontal line in $\mathfrak{D}$ onto a horizontal line in the η-plane and $\eta(\zeta)$ must be of the form $\pm\zeta+b$ with the projection ζ of $\mathfrak{D}$ as a local parameter. Since $v(w)$ fixes each R_j we deduce, by using its conformality in a neighborhood of the point at infinity, that $\eta(\zeta)$ must be the identity mapping, i. e. $v(w)=w$.

Thus we see from (1) that $u(w)$ must be a linear function of w. Since $u(\infty)=\infty$ and $u'(\infty)=1$, we have

$$u(w)=w+k,$$

k being a constant. This implies the equality assertion for real r.

In order to prove the right inequality, we consider a function $g_1(z) = z(1-z^{-3})^{2/3}$ and put $g_1(r\omega^j)=R_j^*$, $j=0, 1, 2$. Taking a quadratic differential

$$Q^*(w)\,dw^2=\frac{-w\,dw^2}{(w-R_0^*)^2(w-R_1^*)^2(w-R_2^*)^2},$$

we can prove the inequality in the same way as above. For the argument on the equality we consider

$$\zeta^*(w)=\int^{w}(Q^*(w))^{1/2}\,dw$$

in the w-plane slit along positive real axis and two segments $[0, R_1^*]$ and $[0, R_2^*]$ which are portions of the closure of orthogonal trajectories of $Q^*(w)\,dw^2$. The proof proceeds then on the same lines as before.

3. Let Σ_0 be a subfamily of Σ consisting of functions $h(z)$ which do not take the value zero in $|z|>1$. Then if $h(z)$ belongs to Σ_0, $f(z)=(h(z^{-1}))^{-1}$ is regular, univalent for $|z|<1$ and normalized at the origin by $f(0)=0$ and $f'(0)=1$. It belongs to the so-called family S. We obtain the following corollary.

COROLLARY 1. *If a function $f(z)$ belongs to S we have*

$$\frac{(1-r^3)^3}{3\sqrt{3}\,r^3(1+r^3)^3}\leqq\frac{|f'(z)f'(z\omega)f'(z\omega^2)|}{\prod_{j=0}^{2}|f(z\omega^j)-f(z\omega^{j+1})|}\leqq\frac{(1+r^3)^3}{3\sqrt{3}\,(1-r^3)^3}\qquad (z=re^{i\theta}).$$

The left equality occurs only for the function $f(z)=z\{(1+e^{i3\theta}z^3)^{2/3}+\omega^j tz\}^{-1}$ and the right only for the function $f(z)=z\{(1-e^{i3\theta}z^3)^{2/3}-\omega^j tz\}^{-1}$ where $j=0,$

1, 2 *and* $0 \leqq t \leqq 2^{2/3}$.

Using our distortion theorem we can prove a known coefficient inequality $|c_2| \leqq 2/3$ with respect to the family Σ ([2], [3], [4]).

COROLLARY 2. *If* $g(z)$ *belongs to* Σ *and has Laurent expansion about the point at infinity*

$$g(z) = z + c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots,$$

then it holds that $|c_2| \leqq 2/3$.

Proof. We use the left inequality in the theorem for real r. It is easily shown that

$$\left|\prod_{j=0}^{2} g'(r\omega^j)\right| = \left|1 - \frac{6c_2}{r^3} + o\left(\frac{1}{r^3}\right)\right|$$

and

$$\prod_{j=0}^{2} |g(r\omega^j) - g(r\omega^{j+1})| = 3\sqrt{3}\, r \left|1 + \frac{3c_2}{r^3} + o\left(\frac{1}{r^3}\right)\right|$$

Hence we have

$$\frac{1}{r^3}(-3\,\mathrm{Re}\, c_2 + 2 + o(1)) \geqq 0.$$

By multiplying by r^3 and then letting r tending to infinity, we have $\mathrm{Re}\, c_2 \leqq 2/3$. Since $e^{-i\theta} g(e^{i\theta} z)$ belongs to Σ for any real θ and we can choose θ such that $\mathrm{Re}\, c_2 e^{-i\theta} = |c_2|$, and we have

$$|c_2| \leqq \frac{2}{3}.$$

Finally we remark that the extremal functions for the distortion problem relating to symmetric four points are not given by the functions $z(1 \pm e^{i4\theta} z^{-4})^{1/2}$, i. e. the functions obtained by symmetrizing the functions $z \pm 2 + e^{i2\theta}/z$, contrary to the case of three points. Indeed, if it were valid, we would deduce an inequality $|c_3| \leqq 1/2$. However it contradicts the result of Garabedian and Schiffer $|c_3| \leqq 1/2 + e^{-6}$ ([1], [5]).

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