ON GENERALIZED UNISERIAL ALGEBRAS OVER A PERFECT FIELD

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Let A be a ring with a unit element satisfying the minimum condition; let N be the radical of A. We call A a generalized uniserial ring if every indecomposable left [right] ideal of A possesses only one composition series. A generalized uniserial algebra over a field F is defined similarly. Recently H. Kupisch [3] discussed such rings and proved that a (two-sided) indecomposable generalized uniserial algebra over an algebraically closed field is completely determined up to isomorphism by a certain system of invariants. In the present note we shall generalize his method to the case of algebras over a perfect field, starting from the fact that the residue class algebra $\bar{A}=A/N$ of a (two-sided) indecomposable generalized uniserial algebra A over a field F (modulo the radical N) has the structure $B\times_F D$, where B is a split semisimple algebra over F and D is a division algebra over F.

NOTATIONS. Let

$$A=\sum_{\kappa=1}^{k}\sum_{i=1}^{f(\kappa)}Ae_{\kappa,i}=\sum_{\kappa=1}^{k}\sum_{i=1}^{f(\kappa)}e_{\kappa,i}A$$

be a decomposition of A into direct sum of indecomposable left [resp. right] ideals; $e_{\kappa,i}$ $(1\leqq\kappa\leqq k,\ 1\leqq i\leqq f(\kappa))$ are mutually orthogonal primitive idempotents; $Ae_{\kappa,i}\cong Ae_{\lambda,j}$ if and only if $\kappa=\lambda$; $e_\kappa=e_{\kappa,1}$, $E_\kappa=\sum_i e_{\kappa,i}$, and $E=\sum_\kappa E_\kappa$ is the unit element of A. $c_{\kappa,ij}$ $(1\leqq\kappa\leqq k,\ 1\leqq i,\ j\leqq f(\kappa))$ be a system of elements of A such that $c_{\kappa,ii}=e_{\kappa,i}$, $c_{\kappa,ij}c_{\kappa,kl}=\delta_{jk}c_{\kappa,il}$; $g(A)=k$ be the number of simple constituents of $\bar{A}=A/N$. $V=V^{(0)}\supset V^{(1)}\supset\cdots\supset V^{(d)}=0$ be the upper Loewy series of an A-left module V; here $V^{(m)}=N^mV$. $V=V_{(d)}\supset\cdots\supset V_{(1)}\supset V_{(0)}=0$ be the lower Loewy series of V; here $V_{(m)}=\{v\mid v\in V,\ N^mv=0\}$. $d(V)=d$ be the length of the upper and lower Loewy series of V; $d(A)=\rho$ is the index of N, i. e. $N^{\rho-1}\neq 0$, $N^\rho=0$.

1. A certain system of generators of composition factor modules of a two-sided composition series of a generalized uniserial ring.

Let A be a generalized uniserial ring and let N be its radical. We first consider an (A, A) composition series of A, which is a refinement of the series $A\supset N\supset N^2\supset\cdots\supset N^\rho=0$:

$$(1)\qquad A=\mathfrak{z}_0^0\supset\mathfrak{z}_1^0\supset\cdots\supset\mathfrak{z}_{r_0}^0=N=\mathfrak{z}_0^1\supset\cdots\supset\mathfrak{z}_{r_1}^1=N^2=\mathfrak{z}_0^2\supset\cdots\supset\mathfrak{z}_{r_{\rho-1}}^{\rho-1}=N^\rho=0.$$

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In the above composition series every factor module $\mathfrak{M}_j^i=\mathfrak{z}_{j-1}^i/\mathfrak{z}_j^i$ $(0\leqq i\leqq \rho-1, 1\leqq j\leqq r_i)$ is a simple (A, A) double module and there exists a unique pair of positive integers (κ, λ) $(\kappa, \lambda\leqq k)$ such that $E_\kappa\mathfrak{M}_j^iE_\lambda=\mathfrak{M}_j^i$, i.e. $\mathfrak{M}_j^i$ is of type (κ, λ). When that is so, we see that $\mathfrak{M}_j^ie_\lambda$ is a simple left submodule of $\mathfrak{M}_j^i$. In fact, by the definition of generalized uniserial rings we have $\mathfrak{M}_j^ie_\lambda\cong\mathfrak{z}_{j-1}^ie_\lambda/\mathfrak{z}_j^ie_\lambda=N^ie_\lambda/N^{i+1}e_\lambda$ and $N^ie_\lambda/N^{i+1}e_\lambda$ is a simple left A-module. Similarly, $e_\kappa\mathfrak{M}_j^i$ is a simple right submodule of $\mathfrak{M}_j^i$.[1] Therefore $e_\kappa\mathfrak{M}_j^ie_\lambda$ is simple as left $e_\kappa Ae_\kappa$-module and, at the same time, as right $e_\lambda Ae_\lambda$-module. Let m be an arbitrary element of $e_\kappa\mathfrak{M}_j^ie_\lambda$. Then for any element x of $e_\kappa Ae_\kappa$ there exists an element y of $e_\lambda Ae_\lambda$ such that $xm=my$; the correspondence $\bar{x}\rightarrow\bar{y}$ gives an isomorphism between $\bar{e}_\kappa\bar{A}\bar{e}_\kappa$ and $\bar{e}_\lambda\bar{A}\bar{e}_\lambda$ (bars indicate the residue classes modulo N), which is determined uniquely up to inner automorphism of $\bar{e}_\lambda\bar{A}\bar{e}_\lambda$. From these arguments and from the Jordan-Hölder theorem we have the following

PROPOSITION 1. *Let A be a generalized uniserial ring. Let $\mathfrak{z}_1\supset\mathfrak{z}_2$ be two-sided ideals of A such that the factor module $\mathfrak{M}=\mathfrak{z}_1/\mathfrak{z}_2$ is a simple (A, A) module of type (κ, λ). Then $\mathfrak{M}e_\lambda$ and $e_\kappa\mathfrak{M}$ are simple left and right submodules of $\mathfrak{M}$, respectively. Moreover, by relation $xm=my$ $(x\in e_\kappa Ae_\kappa, y\in e_\lambda Ae_\lambda; m(\neq 0)\in\mathfrak{M})$ we have an isomorphism between $\bar{e}_\kappa\bar{A}\bar{e}_\kappa$ and $\bar{e}_\lambda\bar{A}\bar{e}_\lambda$: $\bar{x}\rightarrow\bar{y}$. The isomorphism is uniquely determined up to inner automorphism of $\bar{e}_\lambda\bar{A}\bar{e}_\lambda$.*

PROPOSITION 2. *Let A be a generalized uniserial ring; let N be the radical of A. Then for every i $(1\leqq i\leqq\rho)$ the factor module N^{i-1}/N^i is (two-sided) completely reducible (we set $N^0=A$). Moreover, the (two-sided) decomposition of N^{i-1}/N^i into direct sum of simple (A, A) modules is unique and is given by $N^{i-1}/N^i=\sum_\kappa E_\kappa(N^{i-1}/N^i)$ (κ runs through integers $1\leqq\kappa\leqq k$, for which $E_\kappa(N^{i-1}/N^i)\neq 0$).*

In fact, as $e_\kappa(N^{i-1}/N^i)\cong e_\kappa N^{i-1}/e_\kappa N^i$ is either 0 or a simple right module, $E_\kappa(N^{i-1}/N^i)$ is either 0 or a simple two-sided module $(1\leqq\kappa\leqq k)$; $N^{i-1}/N^i=\sum_\kappa E_\kappa(N^{i-1}/N^i)$ is therefore (two-sided) completely reducible. Our last assertion is now trivial.

In the followings we assume that A is generalized uniserial and (two-sided) indecomposable. (The latter restriction is not essential.) We know then, owing to Kupisch [3], that *for a suitable reordering of κ's* (a) *$d(Ae_\kappa)\geqq 2$ for $\kappa<k$;* (b) *$Ne_\kappa/N^2e_\kappa\cong Ae_{\kappa+1}/Ne_{\kappa+1}$ for $\kappa<k$ and $Ne_k/N^2e_k\cong Ae_1/Ne_1$ if $Ne_k\neq 0$,* (c) *$d(Ae_{\kappa+1})\geqq d(Ae_\kappa)-1$, where the κ's are to be taken* mod k. By (b) we take for every κ $(<k)$ an element ${}_{\kappa+1}b_\kappa^1$ of $e_{\kappa+1}Ne_\kappa$ which does not lie in N^2; and, if $Ne_k\neq 0$, we take an element ${}_1b_k^1$ of e_1Ne_k which does not lie in N^2. Put ${}_{\kappa+p}b_{\kappa+p-1}^1\cdot{}_{\kappa+p-1}b_{\kappa+p-2}^1\cdots{}_{\kappa+1}b_\kappa^1={}_{\kappa+p}b_\kappa^p$, if the product of the left-hand side is not zero; here the subscripts are to be taken mod k, if necessary. Further, we put $e_\kappa={}_\kappa b_\kappa^0$ $(1\leqq\kappa\leqq k)$.

1) Cf. Asano [2], § 1.

LEMMA. *Every* ${}_{\kappa+p}b^p_\kappa$ *belongs to* N^p.[2)]

Proof. We have only to consider the case when $p>1$. Suppose that ${}_{\kappa+p}b^p_\kappa$ is an element of N^{p+1}. ${}_{\kappa+p}b^p_\kappa$ is then expressible as ${}_{\kappa+p}b^p_\kappa = x_1x_2\cdots x_q + y$ ($y \in N^{p+1}$), where $q \geqq p+1$ and every x_i ($1 \leqq i \leqq q$) belongs to N^1 and satisfies $e_{\mu(i)}x_ie_{\nu(i)} = x_i$ for some $\mu(i)$ ($\mu(1) = \kappa+p$) and $\nu(i)$; moreover, we may assume that $x_1x_2\cdots x_j$ belongs to N^j for every j ($\leqq q$). But, prop. 1 and prop. 2 show that for a suitable regular element c_1 of $e_{\kappa+p-1}Ae_{\kappa+p-1}$ we have $x_1 \equiv {}_{\kappa+p}b^1_{\kappa+p-1}c_1 \pmod{N^2}$, hence that $x_1x_2\cdots x_p \equiv {}_{\kappa+p}b^1_{\kappa+p-1}c_1x_2\cdots x_p \pmod{N^{p+1}}$; similarly proceeding, we get finally $x_1x_2\cdots x_p \equiv {}_{\kappa+p}b^p_\kappa c_p \equiv 0 \pmod{N^{p+1}}$, where c_p is an regular element of $e_\kappa Ae_\kappa$. This contradicts our assumption that $x_1x_2\cdots x_p$ belongs to N^p.

THEOREM 1. *Every* ${}_{\kappa+p}b^p_\kappa$ *is a (two-sided) generator of one and only one composition factor module* $\mathfrak{z}^p_{j-1}/\mathfrak{z}^p_j$ ($1 \leqq j \leqq r_p$) *of the (two-sided) composition series* (1) *of A. Conversely, every composition factor module* $\mathfrak{z}^q_{l-1}/\mathfrak{z}^q_l$ ($0 \leqq q \leqq \rho-1$, $1 \leqq l \leqq r_q$) *is generated by one and only one element* ${}_{\lambda+q}b^q_\lambda$.

Our first assertion follows immediately from prop. 2 and from lemma; our second assertion can be seen straightforwardly by a similar method as in the proof of lemma.

From the above theorem it follows that there exists in A a system $S = \{{}_\kappa b^p_\lambda\}$ of generators of (two-sided) factor modules of (1) with the properties: (i) $\kappa \equiv p+\lambda \pmod{k}$; (ii) ${}_\kappa b^p_\lambda$ belongs to N^p, ${}_\kappa b^0_\kappa = e_\kappa$ and $e_{\kappa\kappa}b^p_\lambda e_\lambda = {}_\kappa b^p_\lambda$; (iii) S is closed under multiplication. We shall call such S a *(*)-generator system of A.*

REMARK 1. For an arbitrarily fixed pair (κ, λ) the number of elements in a (*)-generator system of type (κ, λ) is denoted by $c_{\kappa\lambda}$; the numbers $c_{\kappa\lambda}$ are the left (and at the same time the right) Cartan invariants of A. We shall write in the followings the elements of type (κ, λ) in a (*)-generator system as ${}_\kappa b^{(1)}_\lambda$, ${}_\kappa b^{(2)}_\lambda$, $\cdots$, ${}_\kappa b^{(c_{\kappa\lambda})}_\lambda$, if necessary.

REMARK 2. It is easy to see that a (*)-generator system consitutes a system of (two-sided) generators of composition factor modules of an arbitrary (two-sided) composition series of A.

2. (Two-sided) indecomposable generalized uniserial algebras over a perfect field.

Let A be a (two-sided) indecomposable generalized uniserial algebra over a field F. We now take, after a suitable reordering of $\kappa = 1, 2, \cdots, k$ as above, a (*)-generator system $S = \{{}_\kappa b^p_\lambda\}$. Since S is closed under multiplication, the subset $A^0_{(*)} = \sum_{\kappa,p} F\,{}_{\kappa+p}b^p_\kappa$ of A is a subalgebra of A (over F); similarly, the subset $A = \sum_{\kappa,i}\sum_{\lambda,j} c_{\kappa,i1}A^0_{(*)}c_{\lambda,1j}$ of A is also a subalgebra of A. $A^0_{(*)}$ and $A_{(*)}$ are themselves both split generalized uniserial algebras and $A^0_{(*)}$ is a basic algebra of $A_{(*)}$. These subalgebras $A^0_{(*)}$ and $A_{(*)}$ will be called a (*)-*basic*

2) An element of A is said to belong to N^p if $a \in N^p$ and $a \notin N^{p+1}$.

algebra and a $(*)$-*algebra* of A (related to the $(*)$-generator system S), respectively. The next proposition follows immediately from *Satz* 6 of Kupisch [3] and from the definitions.

PROPOSITION 3. *The $(*)$-algebra [the$(*)$-basic algebra] of a (two-sided) indecomposable generalized uniserial algebra A is uniquely determined by A up to isomorphism.*

It is obvious that the radical of $A_{(*)}$ is $N \frown A_{(*)}$ and that the radical of $A^0_{(*)}$ is $N \frown A^0_{(*)}$. We denote these by $N_{(*)}$ and by $N^0_{(*)}$, respectively. Furthermore we have

THEOREM 2. *Let A be a (two-sided) indecomposable generalized uniserial algebra over a field F with a radical N; let $A_{(*)}$ be a $(*)$-algebra of A. Then: 1) between two-sided ideals of A and those of $A_{(*)}$ there exists a 1-1 lattice-isomorphic correspondence, which is given by $\mathfrak{z} \to \mathfrak{z} \frown A_{(*)}$ ($\mathfrak{z}_{(*)} \to A\mathfrak{z}_{(*)}A$) where $\mathfrak{z}$ [$\mathfrak{z}_{(*)}$] is a two-sided ideal of A [$A_{(*)}$]; 2) each indecomposable left ideal $A_{(*)}e_\kappa$ of $A_{(*)}$ has the corresponding composition series to that of the indecomposable left ideal Ae_κ of A, i.e., $N^p_{(*)}e_\kappa/N^{p+1}_{(*)}e_\kappa \cong A_{(*)}e_\lambda/N_{(*)}e_\lambda$ if and only if $N^pe_\kappa/N^{p+1}e_\kappa \cong Ae_\lambda/Ne_\lambda$, and the same for right ideals. (The notations be the same as before.) Similar assertions are also true for a $(*)$-basic algebra $A^0_{(*)}$ of A.*

Proof. 1) Let $A_{(*)}$ be the $(*)$-algebra of A related to a $(*)$-generator system of A, $S = \{_{\kappa+p}b^p_\kappa\}$, and let $\mathfrak{z}$ be a two-sided ideal of A. By what we have remarked (remark 2), $\mathfrak{z}$ is generated by a subset S' of S; so that $\mathfrak{z} \frown A_{(*)}$ contains S' and hence $A(\mathfrak{z} \frown A_{(*)})A = \mathfrak{z}$. Conversely, let $\mathfrak{z}_{(*)}$ be a two-sided ideal of $A_{(*)}$. Then $\mathfrak{z}_{(*)}$ is generated by a subset $S_{(*)}$ of S which satisfies $SS_{(*)}S = S_{(*)}$. However, the two-sided ideal $\mathfrak{z}'$ (of A) generated by $S_{(*)}$ can not contain the elements of S other than those of $S_{(*)}$. (This fact can be verified straightforwardly by a similar method as in the proof of lemma.) We must therefore have $\mathfrak{z}' \frown A_{(*)} = A\mathfrak{z}_{(*)}A \frown A_{(*)} = \mathfrak{z}_{(*)}$. 2) Since the element $_{\kappa+p}b^p_\kappa$ is a generator of the $A_{(*)}$-left module $N^p_{(*)}e_\kappa/N^{p+1}_{(*)}e_\kappa$ and since at the same time it is a generator of the A-left module $N^pe_\kappa/N^{p+1}e_\kappa$, $A_{(*)}e_\kappa$ and Ae_κ must have the corresponding composition series.

Hereafter we shall assume that the underlying field F is a perfect field. A is then expressible as a direct sum of the radical N and a semisimple subalgebra $A^*(\cong \bar{A} = A/N)$, and we may assume that the elements $c_{\kappa,ij}$ $(1 \leqq \kappa \leqq k,$ $1 \leqq i,\ j \leqq f(\kappa))$ are in A^*. Prop. 1 shows that the division algebras $e_\kappa A^* e_\kappa$ $(1 \leqq \kappa \leqq k)$ are all isomorphic over F. By the well-known structure theorems of semisimple algebras we have the following

PROPOSITION 4. *A^* is expressible as $A^*_{(*)} \times_F D$, where $A^*_{(*)}$ is a split semisimple algebra over F and D is a division algebra over F.*

The division subalgebra D of A in this proposition may be taken such that every element x of D satisfies $x_{\kappa+1}b_{\kappa}^{1} \equiv {}_{\kappa+1}b_{\kappa}^{1}x \pmod{N^2}$ for $\kappa < k$. If $Ne_k = 0$, then clearly $x_{\kappa+p}b_{\kappa}^{p} = {}_{\kappa+p}b_{\kappa}^{p}x$ $(x \in D)$. If, on the other hand, $Ne_k \neq 0$, then ${}_1b_k^1 \neq 0$ and we have an automorphism σ: $x \to x'$ of D by the relation $x'{}_1b_k^1 \equiv {}_1b_k^1x$ $\pmod{N^2}$; by prop. 1 it follows that σ is uniquely determined up to inner automorphism. And, in this case, we have for every ${}_{\kappa}b_{\lambda}^{(i)} = {}_{\kappa}b_{\lambda}^{p}$ $(1 \leqq i \leqq c_{\kappa\lambda})$ and for every x in D that $x^{\sigma^{i-1}}{}_{\kappa}b_{\lambda}^{(i)} \equiv {}_{\kappa}b_{\lambda}^{(i)}x \pmod{N^{p+1}}$ when $\kappa \geqq \lambda$ and that $x^{\sigma^{i}}{}_{\kappa}b_{\lambda}^{(i)} \equiv {}_{\kappa}b_{\lambda}^{(i)}x$ $\pmod{N^{p+1}}$ when $\kappa < \lambda$

Let $(u_1, u_2, \cdots, u_n)$ be a basis of D over F. From $u_{i\ \kappa+1}b_{\kappa}^{(1)} \equiv {}_{\kappa+1}b_{\kappa}^{(1)}u_i$ $\pmod{N^2}$ $(\kappa < k)$ it follows

$$(2) \qquad {}_{\kappa+1}b_{\kappa}^{(1)}u_i = u_{i\ \kappa+1}b_{\kappa}^{(1)} + \sum_{j=1}^{n}\sum_{l=2}^{c_{\kappa+1,\kappa}} t_{ijl}^{\kappa}\, u_{j\ \kappa+1}b_{\kappa}^{(l)},$$

where t_{ijl}^{κ} $(1 \leqq i,\ j \leqq n,\ 1 \leqq \kappa \leqq k-1,\ 2 \leqq l \leqq c_{\kappa+1,\kappa})$ are elements of F; similarly, from $u_i^{\sigma}{}_1b_k^{(1)} \equiv {}_1b_k^{(1)}u_i \pmod{N^2}$ it follows

$$(3) \qquad {}_1b_k^{(1)}u_i = u_i^{\sigma}{}_1b_k^{(1)} + \sum_{j=1}^{n}\sum_{l=2}^{c_{1k}} t_{ijl}^{k}\, u_j\, {}_1b_k^{(l)},$$

where t_{ijl}^{k} $(1 \leqq i,\ j \leqq n,\ 2 \leqq l \leqq c_{1k})$ are elements of F. On the other hand, we have the following proposition, which is a direct consequence of the definitions and of prop. 1.

PROPOSITION 5. *Notations and assumptions being as above, we have* $A = DA_{(*)} = A_{(*)}D$.

It is now easy to see that the multiplication table of the basis elements of A over F is completely determined by the coefficients of (2) and (3). We have thus proved the following

THEOREM 3. *Let A be a (two-sided) indecomposable generalized uniserial algebra over a perfect field F; let the notations be as before. If $Ne_k \neq 0$, then A is expressible as $A_{(*)} \times_F D$. If, on the other hand, $Ne_k \neq 0$, then the structure of A is completely determined by $A_{(*)}$ and D, by the automorphism σ of D and by the coefficients of* (2) *and* (3).

REFERENCES

[1] ARTIN, E., C. J. NESBITT, AND R. M. THRALL, Rings with Minimum Condition. Univ. Michigan Publ. Math. No. 1 (1944).

[2] ASANO, S., On the radical of quasi-Frobenius algebras. Kōdai Math. Sem. Rep. 13 (1961), 135–151.

[3] KUPISCH, H., Beiträge zur Theorie nichthalbeinfacher Ringe mit Minimalbedingung. J. reine angew. Math. 201 (1959), 100–112.

[4] NAKAYAMA, T., Note on uni-serial and generalized uni-serial rings. Proc. Imp. Acad. Tokyo 16 (1941), 285–289.

[5] NAKAYAMA T., On Frobeniusean algebras II. Ann. Math. 42 (1941), 1–21.

[6] NESBITT, C., AND W. M. SCOTT, Some remarks on algebras over an algebraically closed field. Ann. Math 44 (1943), 534-553.
[7] OSIMA, M., Notes on basic rings. Math. J. Okayama Univ. 2 (1953), 103-110.
[8] OSIMA, M., Notes on basic rings II. Math. J. Okayama Univ. 3 (1954), 121-127.

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