ON THE THEORY OF LINEAR DIFFERENCE-DIFFERENTIAL EQUATIONS

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Introduction.

Many authors have discussed the problems of linear difference-differential equations such that

$$\frac{dx(t)}{dt} = ax(t) + bx(t-1) + f(t) \tag{1}$$

for $0 \leqq t < \infty$. (Cf. [1]—[11], [15]—[22].)

However, few have been investigated concerning the problems of neutral systems. Hence, it may not be useless for us to consider a neutral linear equation such that

$$\frac{dx(t)}{dt} = ax(t) + bx(t-1) + c\frac{dx(t-1)}{dt} + f(t) \tag{2}$$

for $0 \leqq t < \infty$, where a, b, c are constant. Although neutral systems are essentially different from equations with $c = 0$, the similar method is applicable for it.

In § 1, we are going to discuss the location of zeros of the characteristic equation corresponding to (2). In § 2, the existence of solutions of homogeneous equations will be studied. Non-homogeneous equations together with the fundamental solution will be discussed in § 3. In § 4, stability problem will be discussed as the application of the results in § 3. In § 5, perturbation method will be described for equations having a parameter.

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§ 1. Location of zeros of the characteristic equations.

In this section, we first consider the linear difference-differential equation

$$\frac{dx(t+1)}{dt} = ax(t+1) + bx(t) + c\frac{dx(t)}{dt} \tag{1.1}$$

with the constant coefficients a, b, and c, where we suppose that $c \neq 0$.

In the equation (1.1), if we put $x(t) = e^{st}$, we have

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$$\frac{dx(t+1)}{dt}-\left(ax(t+1)+bx(t)+c\frac{dx(t)}{dt}\right)=(e^s(s-a)-(b+cs))e^{st}.$$

Hence, if we denote by s_0 a zero of the transcendental equation $e^s(s-a)-(b+cs)=0$, e^{s_0t} is a particular solution of (1.1). Then, the equation

$$f(s)\equiv e^s(s-a)-(b+cs)=0 \tag{1.2}$$

is called the characteristic equation of the linear difference-differential equation (1.1). In connection with the investigation of the linear difference-differential equations of higher order, Langer [13, 14] has considered the location of zeros of the equation $e^sP(s)+Q(s)=0$, where $P(s)$ and $Q(s)$ are polynomials of s. By Picard's theorem, there exists an infinite number of zeros of (1.2) in the neighborhood of the essential singularity at $s=\infty$. Then, it follows from (1.2) that $e^s=(b+cs)/(s-a)$, by which we obtain

$$s=\log c+O(1/s) \tag{1.3}$$

for any large $|s|$ with exception of an integral multiple of $2\pi i$. If we put $s=u+iv$, where u and v are real, (1.3) gives us that

$$u=\log|c|+O(1/n),\quad v=\arg c+2n\pi+O(1/n),$$

where n is an arbitrary integer, which implies that all the zeros of (1.2) lie in a strip parallel to the imaginary axis. In general, however, it is well known that if the degree of $P(s)$ is different from that of $Q(s)$, a finite number of zeros of $e^sP(s)+Q(s)=0$ lies in a strip parallel to the imaginary axis. Furthermore, Hayes [12] has investigated in detail the case where the degree of $P(s)$ and $Q(s)$ are equal to 1 and 0 respectively.

In the following sections, it is convenient to suppose that the real parts of zeros of (1.2) are all negative. It is, however, not sufficient to assume that $|c|$ is less than 1, for the equation

$$e^s(s-1)-\left(-\frac{1}{2}+\frac{1}{2}s\right)=0$$

has the real zeros 1 and $-\log 2$.

It follows from (1.2) that

$$f'(s)=e^s(s-a+1),\quad f^{(n)}(s)=e^s(s-a+n)\qquad (n\geqq 2),$$

which shows that the number of multiple roots of (1.2) and their multiplicity are finite.

§ 2. Linear homogeneous difference-differential equations with constant coefficients.

We shall consider, in this section, the linear homogeneous difference-differential equation

$$\frac{dx(t+1)}{dt}=ax(t+1)+bx(t)+c\frac{dx(t)}{dt},\quad 1\leqq t<\infty, \tag{2.1}$$

where a, b, and c are constant and $c\neq 0$, with the conditions

$$x(t)=\phi(t)\ \ (0\leqq t<1),\quad x(1)=x_0, \tag{2.2}$$

where $\phi(t)$ is a given function continuous for $0 \leqq t < 1$ and $\phi(1-0)$ exists. Bellman [1] has already considered the case where $c = 0$. In order to obtain a solution of (2.1) with (2.2), we formally apply the Laplace transformation for (2.1) obtaining

$$\int_0^\infty \frac{dx(t+1)}{dt} e^{-st}dt = a\int_0^\infty x(t+1)e^{-st}dt + b\int_0^\infty x(t)e^{-st}dt + c\int_0^\infty \frac{dx(t)}{dt} e^{-st}dt.$$

Using the initial conditions (2.2), we have

$$(e^s(s-a) - (b+cs))\int_0^\infty x(t)e^{-st}dt = x_0 - c\phi(0) + e^s(s-a)\int_0^1 \phi(t_1)e^{-st_1}dt,$$

where we have assumed that $\lim_{t\to+\infty} e^{-st}x(t) = 0$. This condition will be affirmed later. Then, the Laplace inverse formula leads us to

$$x(t) = \frac{1}{2\pi i}\int_C \frac{x_0 - c\phi(0) + e^s(s-a)\int_0^1\phi(t_1)e^{-st_1}dt_1}{e^s(s-a) - (b+cs)} e^{st}ds, \tag{2.3}$$

which is considered as a positive solution of (2.1) with (2.2). C is a suitably chosen contour of integration parallel to the imaginary axis.

It is convenient to make use of the following assumption:

Every root of $e^s(s-a) - (b+cs) = 0$ lies to the left of the straight line $\Re s = -\delta < 0$.

Then, since the function

$$\frac{x_0 - c\phi(0) + e^s(s-a)\int_0^1\phi(t_1)e^{-st_1}dt_1}{e^s(s-a) - (b+cs)}$$

is regular in the half-plane $\Re s > -\delta$, we may take as the contour C the straight line $-\delta/2 + i\tau$, $-\infty < \tau < \infty$, parallel to the imaginary axis. It follows by (2.3) that

$$\begin{aligned} x(t) = &\frac{1}{2\pi i}\int_{-\delta/2-i\infty}^{-\delta/2+i\infty} \frac{x_0 - c\phi(0) + (b+cs)\int_0^1\phi(t_1)e^{-st_1}dt_1}{e^s(s-a) - (b+cs)} e^{st}ds \\ &+ \frac{1}{2\pi i}\int_{-\delta/2-i\infty}^{-\delta/2+i\infty}\left[\int_0^1 \phi(t_1)e^{-st_1}dt_1\right]e^{-st}ds. \end{aligned} \tag{2.4}$$

We denote by u_1 and u_2 the first and second integrals respectively, that is,

$$\begin{aligned} u_1 &= \frac{1}{2\pi i}\int_{-\delta/2-i\infty}^{-\delta/2+i\infty} \frac{x_0 - c\phi(0) + (b+cs)\int_0^1\phi(t_1)e^{-st_1}dt_1}{e^s(s-a) - (b+cs)} e^{st}ds, \\ u_2 &= \frac{1}{2\pi i}\int_{-\delta/2-i\infty}^{-\delta/2-i\infty}\left[\int_0^1 \phi(t_1)e^{-st_1}dt_1\right]e^{st}ds. \end{aligned} \tag{2.5}$$

Bellman [1] stated in (2.5) that u_1 vanishes for $0 \leqq t < 1$ if $c = 0$, u_2 attains the value $\phi(t)$ for $0 \leqq t \leqq 1$ and it vanishes for $1 < t < \infty$.

As for the case $c \neq 0$, since we can also apply the same method of contour integration as in [1], we obtain $u_1 = 0$ for $0 \leqq t < 1$. Furthermore, by the same method, it follows that

$$|u_1(t)| \leqq c_1 e^{-\delta t/2}$$

for $1 < t < \infty$, where c_1 is a constant (cf. [1]).

Then, $x=u_1(t)$ seems to be a solution of (2.1) with (2.2) for $1<t<\infty$. However, we shall prove it indirectly. We denote the integral

$$\int_t^\infty u_1(t_1)\,dt_1$$

by $x(t)$. We shall prove that the function $x(t)$ satisfies the difference-differential equation

$$\frac{dx(t+1)}{dt}=ax(t+1)+bx(t)+c\frac{dx(t)}{dt} \tag{2.6}$$

for $t>1$. Then, we have

$$\begin{aligned}&\frac{dx(t+1)}{dt}-\left(ax(t+1)+bx(t)+c\frac{dx(t)}{dt}\right)\\ &=\frac{1}{2\pi i}\int_{-\delta/2-i\infty}^{-\delta/2+i\infty}\left[(x_0+c\phi(0)+(b+cs)\int_0^1\phi(t_1)e^{-st_1}dt_1\right]e^{st}\frac{e^{s(t-1)}}{s}\,ds.\end{aligned} \tag{2.7}$$

Since the integration by parts gives us that the member in the bracket of (2.7) is bounded in the left half-plane, and furthermore, since Cauchy's theorem is applicable for the integrand of (2.7) by its analyticity, we may shift the contour to the left. Then, it is easily established that the right hand side of (2.7) vanishes for $t>1$, which implies that $x(t)$ is a solution of (2.6).

We can prove that the uniqueness of solution of (2.6) is affirmed. To this end, we suppose that there exists another solution $y(t)$ with the initial conditions (2.2). If we put $z(t)=x(t)-y(t)$, it follows by the linearity of (2.1) that $z(t)$ is a solution of (2.1) with the initial conditions $z(t)=0$ for $0\leqq t\leqq 1$ and $z(0)=0$. Then, it is apparent that $z(t)$ is identically equal to zero.

§ 3. Nonhomogeneous difference-differential equations.

We shall consider the nonhomogenous difference-differential equation

$$\frac{dx(t+1)}{dt}=ax(t+1)+bx(t)+c\frac{dx(t)}{dt}+f(t) \tag{3.1}$$

with the initial conditions that $x(t)=\phi(t)$ for $0\leqq t<1$, $\phi(1-0)$ exists, and $x(1)=x_0$.

Formally multiplying e^{-st} and using the analogous method as in the preceding section, we obtain an equation

$$x(t)=x^0(t)+\int_0^\infty f(t_1)K(t-t_1)\,dt_1, \tag{3.2}$$

where $K(t)$ has the following integral representation

$$K(t)=\frac{1}{2\pi i}\int_{-\delta/2-i\infty}^{-\delta/2+i\infty}\frac{e^{st}}{e^s(s-a)-(b+cs)}\,ds, \tag{3.3}$$

and $x^0(t)$ is the solution of the homogenous equation (2.1) with the conditions (2.2).

We first summarize the results concerning the properties of $K(t)$. It is easily proved by the analogous method as in the preceding section that

$$|K(t)| \leqq c_2 e^{-\delta t/2} \tag{3.4}$$

and the continuity of $K(t)$ are guaranteed for $t > 1$, where c_2 is a constant.

Since (3.4) is rewritten as

$$K(t) = \frac{1}{2\pi i}\int_{-\delta/2-i\infty}^{-\delta/2+i\infty} \frac{e^{s(t-1)}}{s-a-e^{-s}(b+cs)}\,ds,$$

it follows that by the same method as above $K(t)$ vanishes for $t < 1$.

In order to establish the differentiability of $K(t)$, it is sufficient to prove the relations

$$K(t+1) - a\int_0^t K(t_1+1)dt_1 - b\int_0^t K(t_1)dt_1 - cK(t) = 1 \qquad (t>1) \tag{3.5}$$

and

$$K(t+1) - a\int_0^t K(t_1+1)dt_1 = 1 \qquad (0 < t < 1). \tag{3.6}$$

It follows by (3.3) that the right hand side of (3.5) is reduced to

$$\frac{1}{2\pi i}\int_{-\delta/2-i\infty}^{-\delta/2+i\infty} \frac{e^{st}}{s}ds + \frac{1}{2\pi i}\int_{-\delta/2-i\infty}^{-\delta/2+i\infty} \frac{ae^s+b}{e^s(s-a)-(b+cs)}\,\frac{ds}{s}. \tag{3.7}$$

Since the integrand of the first integral of (3.7) is regular in the left half-plane, by a simple calculation, it is equal to the integral

$$\frac{1}{2\pi i}\int_{-\alpha-i\infty}^{-\alpha+i\infty} \frac{e^{st}}{s}ds,$$

where α is an arbitrary constant greater than $\delta/2$. By the arbitrariness of α, if we shift the contour of integration far to the left, that is, if $\alpha \to +\infty$, the integral tends to zero for any finite $t > 0$.

On the other hand, since the integrand of the second integral of (3.7) has a simple pole at $s = 0$ in the half-plane $\Re s > -\delta/2$, it follows by Cauchy's integral formula that the following equality is satisfied:

$$\frac{1}{2\pi i}\int_{-\delta/2-iR}^{-\delta/2+iR} \frac{ae^s+b}{e^s(s-a)-(b+cs)}\,\frac{ds}{s} = 1 + \frac{1}{2\pi i}\int_{C_R} \frac{ae^s+b}{e^s(s-a)-(b+cs)}\,\frac{ds}{s},$$

where C_R represents a right-hand semi-circle of the radius R with the center at $-\delta/2$. Since the integral of the right-hand side tends to zero as $R \to +\infty$ by the analogous way as used above, it follows that (3.5) is equal to 1.

As to the relation (3.6), we obtain

$$\begin{aligned} K(t+1) - a\int_0^t K(t_1+1)dt_1 &= \frac{1}{2\pi i}\int_{-\delta/2-i\infty}^{-\delta/2+i\infty} \frac{e^{st}(e^s(s-a))+ae^s}{e^s(s-a)-(b+cs)}\,\frac{ds}{s} \\ &= \frac{1}{2\pi i}\int_{-\delta/2-i\infty}^{-\delta/2+i\infty} \frac{e^{st}}{s}ds + \frac{1}{2\pi i}\int_{-\delta/2-i\infty}^{-\delta/2+i\infty} \frac{e^{st}(b+cs)+ae^s}{e^s(s-a)-(b+cs)}\,\frac{ds}{s}. \end{aligned}$$

By the same reason as before, the first integral of the right hand side vanishes for any $t > 0$.

On the other hand, since the integrand of the second integral of the right hand side has a simple pole at $s = 0$ in the half-plane $\Re s > -\delta/2$, we obtain

$$\frac{1}{2\pi i}\int_{-\delta/2-iR}^{-\delta/2+iR}\frac{e^{st}(b+cs)+ae^{s}}{e^{s}(s-a)-(b+cs)}\frac{ds}{s}=1+\frac{1}{2\pi i}\int_{C_R}\frac{e^{st}(b+cs)+ae^{s}}{e^{s}(s-a)-(b+cs)}\frac{ds}{s}.$$

By the same reason as before, the integral of the right hand side tends to zero as $R\to+\infty$ for $t<1$, which proves the relation (3.6), by which we obtain $K(1+0)=1$ if $t\to+0$.

As to the differentiability, it follows by (3.6) that

$$K'(t+1)=aK(t+1)\qquad(0<t<1),$$

that is, $K(t)$ is differentiable for $1<t<2$.

On the other hand, differentiating (3.5), we have

$$K'(t+1)=aK(t+1)+bK(t)+cK'(t)\qquad(1<t<2).$$

Hence, we obtain the differentiability of $K(t)$ for $n<t<n+1$ $(n=1,2\cdots)$. Summarizing the results obtained above, we have the following

THEOREM 1. (i) *$K(t)$ is continuous for $t>1$.*

(ii) *$|K(t)|\leqq c_2e^{-\delta t/2}$ for $t>1$.*

(iii) *$K(t)=0$ for $t<1$, if it is represented by the integral.*

(iv) *$K(t+1)-a\int_0^t K(t_1+1)\,dt-b\int_0^t K(t_1)\,dt_1-cK(t)=1$ for $1<t<\infty$.*

(v) *$K(t+1)-a\int_0^t K(t_1+1)\,dt_1=1$ for $0<t<1$.*

(vi) *$K(t)$ is differentiable for $n<t<n+1$ $(n=1,2,\cdots)$.*

Thus, if we refer to the properties of $K(t)$, it follows that the existence and uniqueness of solutions of (3.1) will be established.

THEOREM 2. *Suppose that all the roots of $e^{s}(s-a)-(b+cs)=0$ lie to the left of the straight line $\Re s=-\delta<0$ and if $f(t)$ is a continuous function of bounded variation over any finite interval such that*

$$\int_0^\infty|f(t_1)|e^{\delta t_1/2}dt_1<\infty. \tag{3.8}$$

Then, the solution $x(t)$ of (3.2) with $x(t)=\phi(t)$ for $0\leqq t<1$ and $x(1)=x_0$ is represented by

$$x(t)=x^0(t)+\int_0^{t-1}f(t_1)K(t-t_1)\,dt_1 \tag{3.9}$$

for $t>1$, where $x^0(t)$ is the solution of the homogeneous equation corresponding to (2.3), $K(t)$ is represented by

$$K(t)=\frac{1}{2\pi i}\int_{-\delta/2-i\infty}^{-\delta/2+i\infty}\frac{e^{st}}{e^{s}(s-a)-(b+cs)}ds \tag{3.10}$$

for $t>1$, and

$$\int_0^\infty|K(t_1)|dt_1<\infty. \tag{3.11}$$

Next, we consider the nonlinear equation

$$\frac{dx(t+1)}{dt}=ax(t+1)+bx(t)+c\frac{dx(t)}{dt}+f(t,\ x(t+1),\ x(t)) \tag{3.12}$$

for $t>1$, where we suppose that $f(t, x, y)$ is continuous and satisfies the Lipschitz condition

$$|f(t,\ x_1,\ y_1)-f(t,\ x_2,\ y_2)|\leqq a(t)|x_1-x_2|+b(t)|y_1-y_2|. \tag{3.13}$$

It is convenient to assume that $a(t)$ and $b(t)$ is continuous and to consider the nonlinear integral equation of Volterra type

$$x(t)=x^0(t)+\int_0^{t-1}f(t_1,\ x(t_1),\ x(t_1-1))K(t-t_1)\,dt_1 \tag{3.14}$$

for $t>1$, where $K(t)$ is represented by (3.10) and $x^0(t)$ is the solution of the homogeneous equation corresponding to (3.12). We define a sequence $\{x_n(t)\}_{n=0}^{\infty}$ as follows:

$$x_0(t)=x^0(t),$$
$$x_{n+1}(t)=x^0(t)+\int_0^{t-1}f(t_1,\ x_n(t_1),\ x_n(t_1-1))K(t-t_1)dt_1 \qquad (n=0,\ 1,\ 2,\ \cdots).$$

Then, by the explicit representation of $x^0(t)$ and the properties of $K(t)$, it follows that each function $x_n(t)$ is bounded. Furthermore, since the Lipschitz condition (3.13) is fulfilled, it follows that the sequence $\{x_n(t)\}_{n=0}^{\infty}$ converges uniformly to $x(t)$, which is the unique solution of (3.14).

THEOREM 3. *Suppose that in* (3.12) $f(t, x, y)$ *is continuous, bounded, and satisfies the Lipschitz condition* (3.13). *Then, there exists the unique solution of* (3.13). *Furthermore, if we suppose that the integral*

$$\int_0^{\infty}|f(t_1,\ x,\ y)|e^{\delta t_1/2}dt_1$$

is bounded, the solution tends to 0 *as* $t\to+\infty$.

In fact, the explicit representation $x^0(t)$ and the properties of $K(t)$, it follows that

$$|x_n(t)|\leqq Ke^{-\delta t/2} \qquad (n=1,\ 2,\ \cdots),$$

where K is a constant, which proves the second part of the theorem.

§ 4. Stability of solutions.

In this section, we shall propose to consider the behavior of the solutions of the difference-differential equation

$$\frac{dy(t+1)}{dt}=ay(t+1)+by(t)+f(t,\ y(t+1),\ y(t)), \tag{4.1}$$

where a, b, are constant and f is small as $t\to+\infty$ in some sense, by using the behavior of the solution of

$$\frac{dx(t+1)}{dt} = ax(t+1) + bx(t), \tag{4.2}$$

which has been investigated in [1]. If we consider a special type of (4.1), that is, the equation

$$\frac{dy(t+1)}{dt} = (a + a(t))y(t+1) + (b + b(t))y(t), \tag{4.3}$$

where $a(t)$ and $b(t)$ are continuous and bounded for $0 \leqq t < \infty$, the existence and uniqueness of solutions are guaranteed with the conditions that $y(t) = \phi(t)$ $(0 \leqq t < 1)$, $\phi(1-0)$ exists, and $y(1) = y_0$.

However, instead of the equation (4.3), it is convenient to consider the integral equation

$$y(t+1) = x(t+1) + \int_0^t (a(t_1)y(t_1+1) + b(t_1)y(t_1))K(t-t_1)\,dt_1, \tag{4.4}$$

where $x(t+1)$ is the solution of (4.2) with the conditions $x(t) = \phi(t)$ $(0 \leqq t < 1)$ and $x(1) = x_0$, and $K(t)$ is represented by (3.10) with $c = 0$.

By the hypotheses on $a(t)$ and $b(t)$, the existence and uniqueness of solutions are established. Then, we obtain the following

THEOREM 4. *We suppose that in the equation* (4.4) *the following conditions are satisfied*:

(i) *every root of* $e^s(s-a) - b = 0$ *lies to the left of the straight line* $\Re s = -\delta < 0$;

(ii) $a(t)$ *and* $b(t)$ *are continuous for* $0 \leqq t < \infty$ *and bounded, that is*,

$$|a(t)| \leqq A, \quad |b(t)| \leqq B;$$

(iii) $A + B < \delta/2$.

Then, the solution of (4.3) *approaches zero as* $t \to +\infty$.

Proof. By the hypothesis on $a(t)$, $b(t)$ and the property (ii) in Theorem 1, it follows that

$$|y(t+1)| \leqq |x(t+1)| + \int_0^t (A|y(t_1+1)| + B|y(t_1)|)e^{-\delta(t-t_1)/2}dt_1. \tag{4.5}$$

Since we may take as t a large number, (4.5) is reduced to

$$\begin{aligned} |y(t+1)|e^{\delta(t+1)/2} \leqq |x(t+1)|e^{\delta(t+1)/2} + B\int_0^1 |\phi(t_1)|e^{\delta(t_1+1)/2}dt_1 \\ + \int_0^t (A+B)|y(t_1+1)|e^{\delta(t_1+1)/2}dt_1. \end{aligned} \tag{4.6}$$

It is well known that there exists a constant c_1 such that $|x(t)| \leqq c_1 e^{-\delta t/2}$ for $t > 1$. Hence, we obtain

$$|y(t+1)|e^{\delta(t+1)/2} \leqq c_2 + (A+B)\int_0^t |y(t_1+1)|e^{\delta(t_1+1)/2}dt_1, \tag{4.7}$$

where

$$c_2 = c_1 + \int_0^1 B|\phi(t_1)|e^{\delta(t_1+1)/2}dt_1,$$

which is a constant. Now, we can apply for the inequality the following result.

LEMMA. *In the inequality*

$$u(x) \leqq f(x) + \int_0^x K(t)u(t)\,dt,$$

we suppose that $x \geqq 0$ *and* $K(t) \geqq 0$ *for* $0 \leqq t \leqq x$. *Then, we have*

$$u(x) \leqq f(x) + \int_0^x f(t)K(t)\exp\left(\int_t^x K(s)ds\right)dt.$$

Thus, it follows by the above result that

$$|y(t+1)|e^{\delta(t+1)/2} \leqq c_2 e^{(A+B)t},$$

i.e.,

$$|y(t+1)| \leqq c_2 e^{(A+B-\delta/2)t}$$

for $1 < t < \infty$. Hence, if $A + B < \delta/2$, $y(t+1)$ approaches zero as $t \to +\infty$.

Next we consider the equation

$$y(t+1) = x(t+1) + \int_0^t f(t_1,\ y(t_1+1),\ y(t_1))K(t-t_1)\,dt_1, \tag{4.8}$$

where $x(t+1)$ is the solution of (4.2) with the conditions $x(t) = \phi(t)$ $(0 \leqq t < 1)$, $x(1) = x_0$, and $K(t)$ is represented by (3.3) with $c = 0$. Then, we obtain the following

THEOREM 5. *Suppose that in the equation* (4.8) *the following conditions are satisfied*:

(i) *every root of the characteristic equation* $e^s(s-a) - b = 0$ *lies to the left of the straight line* $\Re s = -\delta < 0$;

(ii) $f(t, 0, 0) = 0$ *and*

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leqq h(t)(|x_1 - x_2| + |y_1 - y_2|)$$

for $0 \leqq t < \infty$, $|x_i| < \infty$, $|y_i| < \infty$;

(iii) $h(\tau)$ *is continuous in* $0 \leqq \tau \leqq t$ *for any finite* t, *and there exists a constant* $\varepsilon > 0$ *such that*

$$\frac{\delta}{2} - \frac{1}{t+1}\int_0^{t+1} h(\tau)\,d\tau > \varepsilon$$

for any finite t.

Then, the solution is uniquely determined and tends to zero as $t \to +\infty$.

Proof. The successive approximation method shows us that the existence and uniqueness of solutions of (4.8) are guaranteed under the hypotheses cited above.

It follows by (ii) that

$$|y(t+1)| \leqq |x(t+1)| + \int_0^t h(t_1)(|y(t_1+1)| + |y(t_1)|)|K(t-t_1)|dt_1.$$

By the same reason as in the proof of Theorem 4, we have

$$|y(t+1)|e^{\delta(t+1)/2} \leqq |x(t+1)|e^{\delta(t+1)/2} + \int_0^1 h(t_1)|\phi(t_1)|e^{\delta(t_1+1)/2}dt_1$$
$$+ \int_0^t (h(t_1) + h(t_1+1))|y(t_1+1)|e^{\delta(t_1+1)/2}dt_1.$$

It follows by Lemma that

$$|y(t+1)| \leqq c_2' \exp\left(-\frac{\delta}{2}(t+1) + \int_0^t (h(t_1) + h(t_1+1))\,dt_1\right),$$

where

$$c_2' = c_1 + \int_0^1 h(t_1)|\phi(t_1)|e^{\delta(t_1+1)/2}dt_1 \text{ and } |x(t+1)| \leqq c_1 e^{-\delta(t+1)/2}.$$

It follows from the hypothesis (iii) that $y(t)$ approaches zero as $t \to +\infty$, which proves the theorem.

§ 5. Perturbation of equations having periodic solutions.

In this section, we shall consider the equation

$$\frac{dx(t)}{dt} = f(t,\ x(t),\ x(t-1),\ \mu) \tag{5.1}$$

with the conditions $x(t-1) = \phi(t)$ $(0 \leqq t < 1)$ and $x(0) = x_0$, where $\phi(t)$ is a continuous function independent on μ, which is constant. In the sequel we suppose that $f(t, x, y, \mu)$ satisfies the following conditions:

(i) $f(t, x, y, \mu)$ is a periodic function of t with the period T;

(ii) the equation (5.1) corresponding to $\mu = 0$ has a periodic solution $x = p(t)$ of the period T with the conditions $p(t-1) = \phi(t)$ $(0 \leqq t < 1)$ and $p(0) = x_0$;

(iii) $f(t, x, y, \mu)$, $f_x(t, x, y, \mu)$, $f_y(t, x, y, \mu)$ are continuous for $0 \leqq t \leqq T$, $|x - x_0| \leqq K$, $|y - x_0| \leqq K$.

The problem to be investigated is whether or not the equation (5.1) has a periodic solution for μ which does not vanish but near zero.

1. First variation. In general, we consider the equation

$$\frac{dx(t)}{dt} = f(t,\ x(t),\ x(t-1)) \tag{5.2}$$

with the conditions that $x(t-1) = \phi(t)$ $(0 \leqq t < 1)$, $\phi(1-0)$ exists, and $x(0) = x_0$. We suppose that $f(t, x, y)$, $f_x(t, x, y)$, $f_y(t, x, y)$ are continuous for

$$0 \leqq t \leqq t_0,\ |x - x_0| \leqq K,\ |y - x_0| \leqq K.$$

Then, there exists a unique solution $x = \phi_0(t)$ with $\phi_0(t-1) = \phi(t)$ $(0 \leqq t < 1)$ and $\phi_0(0) = x_0$. If we put $x(t) = \phi_0(t) + y(t)$, (5.2) is reduced to the equation

$$\frac{dy(t)}{dt} = f(t,\ \phi_0(t) + y(t),\ \phi_0(t-1) + y(t-1)) - f(t,\ \phi_0(t),\ \phi_0(t-1)) \tag{5.3}$$

with the condition $y(t-1)=0$ for $0 \leqq t \leqq 1$. The mean value theorem leads us to the equation

$$\frac{dy(t)}{dt}=f_x(t,\ \phi_0(t),\ \phi_0(t-1))y(t)+f_y(t,\ \phi_0(t),\ \phi_0(t-1))y(t-1)+f_1, \tag{5.4}$$

where $f=o(y)$ as $y \to 0$. Then, we call the equation

$$\frac{dy(t)}{dt}=f_x(t,\ \phi_0(t),\ \phi_0(t-1))y(t)+f_y(t,\ \phi_0(t),\ \phi_0(t-1))y(t-1) \tag{5.5}$$

the first variation of (5.3) with the condition $y(t-1)=0$ $(0 \leqq t \leqq 1)$.

2. Existence of periodic solutions. It is noted that in the equation (5.1) there exist a solution $x=x(t,\ x_0,\ \mu)$ $(x_0=x(0,\ x_0,\ \mu))$ and its partial derivative $\partial x(t,\ x_0,\ \mu)/\partial x_0$ for $0 \leqq t \leqq T$, sufficiently small μ and $|x_0-p(0)|$.

THEOREM 6. *Suppose that $f(t,\ x,\ y,\ \mu)$ satisfies the conditions* (i), (ii), (iii) *stated above. Let $x=x(t,\ x_0,\ \mu)$ be a solution of* (5.1) *with the conditions $x(t-1,\ x_0,\ \mu)=\phi(t)$ $(0 \leqq t < 1)$, $x(0,\ x_0,\ \mu)=x_0$. Then, if*

$$\frac{\partial}{\partial x_0}x(T,\ p(0),\ 0) \neq 1, \tag{5.6}$$

there exists a unique solution $x=x(t,\ x_0(\mu),\ \mu)$ of (5.1) *with the period T for sufficiently small μ, where $x_0(\mu)$ is a continuous function of μ and $x_0(0)=x_0$.*

Proof. The necessary and sufficient condition that a solution $x=x(t,\ x_0,\ \mu)$ of (5.1) with the condition $x(t-1,\ x_0,\ \mu)=\phi(t)$ $(0 \leqq t < 1)$ and $x_0=x(0,\ x_0,\ \mu)$ has period T is that

$$x(T,\ x_0,\ \mu)=x_0. \tag{5.7}$$

Since (5.1) has a periodic solution $x=p(t)$ for $\mu=0$, we obtain $x(T,\ p(0),\ 0)=x_0$ $(x_0=p(0))$. It follows by a well known theorem of implicit function that if

$$\left(\frac{\partial}{\partial x_0}(x(T,\ x_0,\ \mu)-x_0)\right)_{\substack{\mu=0\\ x_0=p(0)}} \neq 0, \tag{5.8}$$

there exists a function $x_0=x_0(\mu)$ continuous and uniquely determined for sufficiently small μ, satisfying (5.7) and $x_0(0)=p(0)$. (5.8) is equivalent to (5.6), which proves the theorem.

Next, we consider the first variation of (5.1) with respect to $p(t)$ as $\mu=0$, that is, the equation

$$\frac{dy(t)}{dt}=f_x(t,\ p(t),\ p(t-1),\ 0)y(t)+f_y(t,\ p(t),\ p(t-1),\ 0)y(t-1) \tag{5.9}$$

with the condition $y(t-1)=0$ $(0 \leqq t \leqq 1)$. Since f_x, f_y, and $p(t)$ are the periodic functions of the period T, (5.9) is the equation with the periodic coefficients of the period T. Then, Theorem 6 is equivalent to the following

THEOREM 7. *If the equation* (5.9) *has no periodic solutions of the period T, the result in Theorem 6 remains valid.*

Proof. Let $x=x(t, x_0, \mu)$ be the same solution as in the proof of Theorem 6. Since $dx(t, x_0, \mu)/dt=f(t, x(t, x_0, \mu), x(t-1, x_0, \mu), \mu)$, we obtain an equation

$$\frac{d\psi(t)}{dt}=f_x(t, p(t), p(t-1), 0)\psi(t)+f_y(t, p(t), p(t-1), 0)\psi(t-1),$$

where

$$\psi(t)=\left(\frac{\partial}{\partial x_0}x(t, x_0, \mu)\right)_{\substack{\mu=0\\ x_0=p(0)}}, \text{ and } \psi(0)=1.$$

Since $y(t-1)=0$ $(0\leqq t\leqq 1)$ in (5.9), any solution $y(t)$ of (5.9) with the conditions $y(t-1)=0$ and $y(0)=c$ is represented by $y(t)=c\psi(t)$. In fact, the result is easily obtained by the linearity and uniqueness of solutions of (5.9).

If the equation (5.9) has a periodic solution $y_0(t)$ of the period T with the conditions $y_0(t-1)=0$ and $y_0(0)=a(\neq 0)$, it is represented by $y_0(t)=a\psi(t)$. Furthermore, we obtain

$$a\psi(T)=a\psi(0)=a, \text{ i.e., } \psi(T)=1.$$

Conversely, if $\psi(T)=1$, $a\psi(t)$ is a periodic solution of (5.9) for any constant $a\neq 0$.

On the other hand, it is easily obtained that

$$\psi(T)=\left(\frac{\partial}{\partial x_0}x(T, x_0, \mu)\right)_{\substack{\mu=0\\ x_0=p(0)}},$$

which completes the proof.

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