ON THE EQUATION $\Delta u + \lambda f(x, y)u = 0$ UNDER THE FIXED BOUNDARY CONDITION

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In the present paper, we shall consider some problems concerning the equation

$$\Delta u + \lambda f(x, y)u = 0$$

under the fixed boundary condition for an arbitrary bounded domain, where $f(x, y)$ is a bounded continuous function with continuous first derivatives and satisfies $M \geqq f(x, y) \geqq \mu > 0$, M and μ being constants.

For convenience of description, we shall treat the first eigenvalue and the first eigenfunction in theorem 1, the k-th eigenvalue and the k-th eigenfunction in theorem 2.[1]

§1. Theorem 1. *Let D be a bounded domain on the x, y-plane and C its boundary. Let $\{D_n\}$ be a sequence of domains exhausting D:*

$$D_1 \subset D_2 \subset \cdots \subset D_n \subset \cdots,$$

that is, a sequence such that $\lim_{n\to\infty} D_n = D$, where the boundary C_n of the domain D_n consists of a finite number of smooth curves. Let further $\lambda_{1,n}$ and $u_{1,n}$ be the first eigenvalue and the first eigenfunction, respectively, of the problem

$$\begin{cases} \Delta u + \lambda f(x, y)u = 0 & \text{in } D_n, \\ u = 0 & \text{on } C_n \end{cases}$$

and $u_{1,n}$ be normalized by

$$\iint_{D_n} f(x, y) u_{1,n}^2 d\sigma = 1.$$

Then

(1) $$\lim_{n\to\infty} \lambda_{1,n} = \lambda_1 \quad \text{and} \quad \lim_{n\to\infty} u_{1,n} = u_1$$

exist and are determined independently of a choice of exhausting sequence.

(2) *The normalization condition*

$$\iint_D f(x, y) u_1^2 d\sigma = 1$$

holds.

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1) In our previous paper [2] and [3], we have studied on the same problem for the equation with constant coefficient, i.e. the equation $\Delta u + \lambda u = 0$.

(3) *The limit function* u_1 *and the limit value* λ_1 *satisfy the equation*

$$\Delta u_1 + \lambda_1 f(x, y) u_1 = 0 \qquad in \quad D$$

together with the condition $u_1 = 0$ *on* C *except for a set of capacity zero. The exceptional points are identical with those of the ordinary Green's function for the same domain.*

(4) *In particular, if* C *consists of a finite number of closed smooth curves, the limit value* λ_1 *and the limit function* u_1 *coincide, respectively, with the first eigenvalue and the first eigenfunction of* D.

We would remark that our theorem 1 contains the continuity theorems of the first eigenvalue and the first eigenfunction with respect to the domain bounded by a finite number of closed smooth curves. It suggests further the limit value and the limit function might be regarded, respectively, as the definitions of the first eigenvalue and the first eigenfunction of our equation for general bounded domain.

The proof of theorem 1 will be given in several steps.

§**2.** Let D^* be another domain which contains D and its boundary consists of a finite number of closed smooth curves. Let further λ^* be the first eigenvalue of the same problem for D^*. Since our eigenvalue is a monotone decreasing domain function in the strict sense, we have

$$\lambda_{1,n} > \lambda_{1,n+1} \quad \text{and} \quad \lambda_{1,n} > \lambda^* \quad \text{for every} \quad n.$$

Hence there exists a limit value of $\lambda_{1,n}$ as $n \to \infty$. Put $\lim_{n\to\infty} \lambda_{1,n} = \lambda_1$ then $\lambda_1 \geqq \lambda^*$. From the monotoneity it is evident that our limit value λ_1 is determined independently of a choice of exhausting sequence.

§**3.** For our later reasoning, we shall give here two lemmas.

LEMMA 1. *Let* R *be a domain bounded by a finite number of smooth curves. Consider the boundary value problems*

(1) $$\Delta w + \lambda M w = 0$$

and

(2) $$\Delta u + \lambda f u = 0$$

in R *under the fixed boundary condition,* M *being a positive constant such that* $M \geqq f(x, y)$. *Let* μ^* *be the first eigenvalue of the problem* (1), *and* μ_1 *that of the problem of* (2). *Then* $\mu_1 \geqq \mu_1^*$ *holds.*

In fact, denoting by $G(p, q)$ the ordinary Green's function of the domain R, we get

$$\frac{1}{\mu_1} = \max_{\psi} \iint G(p, q) f(p) \psi(p) \psi(q) \, dp dq$$
$$\leqq \max_{\psi} \iint G(p, q) M \psi(p) M \psi(q) \, dp dq = \frac{1}{\mu_1^*}$$

i. e.

$$\mu_1 \geqq \mu_1^*$$

LEMMA 2. *Let w and u satisfy the equations*

$$\Delta w+Mw=0 \qquad in \quad R,$$

$$\Delta u+f(x, y)u=0 \qquad in \quad R,$$

where M is a positive constant and $f(x, y)$ a positive function such that $M\geqq f(x, y)$ in R. Here R is a domain for which there exists only trivial solution, i. e. identically zero, of the above equations under the fixed boundary condition.[2] *If $w=u>0$ on the boundary of R, then $w\geqq u$ in R.*

In fact, from the equations for u and w, we get

$$\Delta(u-w)+M(u-w)=(M-f)u.$$

Namely, putting $u-w\equiv v$, v satisfies

$$\Delta v+Mv=(M-f)u \qquad \text{in} \quad R,$$

$$v=0 \qquad \text{on the boundary of} \quad R.$$

Let $\Gamma(p, q)$ be the Green's function for the operator $\Delta+M$, with the fixed boundary condition. We get

$$v=-\iint \Gamma(p, q)(M-f(q))u(q)\,d\sigma(q)$$

where p and q denote the points in the domain and $d\sigma$ the areal element. As $u>0$, $M\geqq f(q)>0$, $\Gamma(p, q)>0$ in R, we obtain $v\leqq 0$ in R, i. e. $w\geqq u$ in R.

Now, let A be a subdomain of D with a positive distance 2ε from the boundary of D, and A' another subdomain of D with the positive distance ε from the boundary of D.

Fixing ε, there exists m such that $A'\subset D_n$ for $n\geqq m$.

(a) $\{u_{1,n}\}$ is uniformly bounded in A.

To show this, let p be an arbitrary point in A and K a circle of radius ε about p. Since K is contained in A' and also in D_n for n large enough. By taking ε sufficiently small, let K have the same property as R, that is, there exist no non-trivial solutions of the equations $\Delta w+Mw=0$ and $\Delta u_{1,n}+\lambda f(x, y)u_{1,n}=0$ in K under the fixed boundary condition where $M\geqq \lambda_{1,n}f(x, y)\geqq \mu>0$. By lemma 2, if $u_{1,n}=w$ on the boundary of K, then we obtain the

2) In fact, let the area of R be smaller than that of the circle whose first eigenvalue for the equation $\Delta w+\lambda w=0$ under the fixed boundary condition is M. Then, by a theorem on isoperimetric inequalities, the first eigenvalue of the same problem for any domain with the same area of the above circle, is larger than M. By monotone decreasing properties of the eigenvalue, all eigenvalues of the equation $\Delta w+\lambda w=0$ with the fixed boundary condition, are greater than M. Hence the equation $\Delta w+Mw=0$ under the fixed boundary condition has only trivial solution in R. By lemma 1, the same is true with the equation $\Delta u+f(x, y)u=0$.

relation $u_{1,n} \leqq w$ in K. Therefore, using the Weber's mean-value theorem

$$u_{1,n}(p) \leqq w(p) = \frac{1}{2\pi J_0(\sqrt{M}\varepsilon)} \int_0^{2\pi} w\, d\theta = \frac{1}{2\pi J_0(\sqrt{M}\varepsilon)} \int_0^{2\pi} u_{1,n}\, d\theta$$

where the integration is taken over the boundary of the circle K. Putting $J_0(\sqrt{M}\varepsilon) \equiv k$ and taking ε small enough such that $k > 0$. Then we get

$$u_{1,n}(p) \leqq \frac{1}{2\pi k} \int_0^{2\pi} u_n\, d\theta.$$

Multiplying both sides by r and integrating with respect to from 0 to ε,

$$u_{1,n}(p) \int_0^{\varepsilon} r dr \leqq \frac{1}{2\pi k} \int_0^{\varepsilon}\int_0^{2\pi} u_{1,n} r\, d\theta\, dr,$$

whence follows by the Schwarz's inequality and the normalization of $u_{1,n}$,

$$u_{1,n}(p) \leqq \frac{1}{k\pi\varepsilon^2} \frac{\sqrt{\pi\varepsilon^2}}{\mu} = \frac{1}{k\mu\sqrt{a}}$$

for any $p \in A$ and any $n \geqq m$, where $\sqrt{a} \equiv \sqrt{\pi\varepsilon^2}$. Thus $\{u_{1,n}\}$ is uniformly bounded in A.

Next remembering that a solution of the equation $\Delta u_{1,n} + \lambda_{1,n} f(x, y) u_{1,n} = 0$ with $\lambda_{1n} f(x, y) > 0$, $u_{1,n}$ is a superharmonic function. Therefore, we get

$$u_{1,n}(p) \geqq \alpha_r u_{1,n}(p)$$

where the right-hand member denotes the areal mean of the $u_{1,n}$ over the disk about p with radius r. On the other hand, we have a relation

$$u_{1,n}(p) \leqq \frac{1}{k} \cdot \frac{1}{\pi r^2} \iint u_{1,n}\, d\sigma$$

where $k \equiv J_0(\sqrt{M}r)$.

From these two relations we obtain

$$\alpha_r u_{1,n}(p) \leqq u_{1,n}(p) \leqq \frac{1}{J_0(\sqrt{M}r)} \alpha_r u_{1,n}(p),$$

and, by virtue of $J_0(0) = 1$,

$$\text{(b)} \qquad \lim_{r\to 0} \{\sup_{p\in A} | u_{1,n}(p) - \alpha_r u_{1,n}(p) |\} = 0.$$

(a) and (b) show that the Arsove's condition [1] for normal family is satisfied by $\{u_{1,n}\}$. Therefore we can select from $\{u_{1,n}\}$ a subsequence $\{u_{1,n'}\}$ uniformly convergent throughout A. Let its limit function be u_1, i. e.

$$\lim_{n'\to\infty} u_{1,n'} = u_1 \qquad \text{in } A.$$

§4. For later purpose we study on the uniform boundedness of the sequence $\{u_{1,n}\}$ in D. Let the smooth boundary curves of the domain D_N for a fixed N be

$$C_N = \{C_N^{(0)}, C_N^{(1)}, \cdots, C_N^{(s-1)}\}$$

where $C_N^{(0)}$ denotes the outer boundary of D_N and the others the inner boundaries.

For sufficiently large N, the area of the part of the domain D cut off by $C_N^{(0)}$ and lying outside of the bounded domain enclosed by $C_N^{(0)}$ is as small as we wish. The area of the part of D cut off by $C_N^{(i)}$ $(i=1, 2, \cdots, s-1)$ and lying inside of the bounded domain enclosed by $C_N^{(i)}$ then becomes also small, we decompose the domain D_N into $s+1$ subdomains not overlapping each other:

$$D_N=\widetilde{D}_N+D_N^{(0)}+D_N^{(1)}+\cdots+D_N^{(s-1)},$$

where the domain $D_N^{(i)}$ is surrounded by $C_N^{(i)}$ and a smooth closed curve $B^{(i)}$ with positive distance from boundaries of $D_N^{(j)}$ $(j \neq i)$, and $\widetilde{D}_N$ denotes $D_N-\{D_N^{(0)}+D_N^{(1)}+\cdots+D_N^{(s-1)}\}$ and hence it is contained completely in the interior of D_N and also of A.

Let the area of the part of the domain D cut off by the curve $B^{(0)}$ and lying outside of the bounded domain enclosed by $B^{(0)}$ be smaller than the area of a circle R with $M\lambda_{1,1}$ as the first eigenvalue for the same boundary value problem. Let the area of the part of the domain D cut off by the curves $B^{(i)}$ $(i=1, 2, \cdots, s-1)$ and lying inside of the bounded domain enclosed by $B^{(i)}$ be also smaller than that of R. For $n>N$, denote by $D_n^{(0)}$ the part of the domain D_n cut off by the same curve $B^{(0)}$ mentioned above and lying outside of the bounded domain enclosed by $B^{(0)}$. Then the area of $D_n^{(0)}$ is, of course, smaller than that of R. Applying the same procedure to each curves $B^{(i)}$ $(i=1, 2, \cdots, s-1)$ mentioned above, we have the domain $D_n^{(i)}$ which is the part of the domain D_n cut off by $B^{(i)}$ and lying inside of the bounded domain enclosed by $B^{(i)}$. Then the area of $D_n^{(i)}$ is also smaller than that of R. Let $C_n^{(i)}$ be the boundary part of D_n contained in $D_n^{(i)}$ which consists of a finite number of smooth curves. For such a domain $D_n^{(i)}$ $i=0, 1, \cdots, s-1)$ the Green's function of the equation $\Delta u+\lambda_{1,n}f(x, y)u=0$ for fixed boundary condition is uniquely determined, and it can be represented by

$$\Omega_n^{(i)}(p, q)=\frac{1}{2\pi}\log\frac{1}{r}+H_n^{(i)}(p, q)$$

where $H_n^{(i)}$ denotes a bounded function in $D_n^{(i)}$. By Green's formula, we have

$$u_{1,n}(p)=\int_{B^{(i)}}u_{1,n}\frac{\partial\Omega_n^{(i)}}{\partial\nu}ds \qquad \text{for} \quad p\in D_n^{(i)},$$

where ν denotes the inner normal of the boundary.

In order to obtain an estimation for $u_{1,n}$ in $D_n^{(i)}$, we introduce an auxiliary harmonic function such that

$$\begin{cases} \Delta\varphi_n^{(i)}=0 & \text{in } D_n^{(i)}, \\ \varphi_n^{(i)}=u_{1,n} & \text{on } B^{(i)}, \\ \varphi_n^{(i)}=0 & \text{on } C_n^{(i)}. \end{cases}$$

By Green's formula, we have

$$(3)\qquad \int_{B^{(i)}+C_n^{(i)}+\kappa}\left\{\varphi_n^{(i)}\frac{\partial\Omega_n^{(i)}}{\partial\nu}-\Omega_n\frac{\partial\varphi_n^{(i)}}{\partial\nu}\right\}ds$$
$$=-\iint_{D_n^{(i)}-E}\{\varphi_n^{(i)}\Delta\Omega_n^{(i)}-\Omega_n^{(i)}\Delta\varphi_n^{(i)}\}d\sigma(q)$$

where E denotes a small circular domain about p and κ its boundary. By the definition of $D_n^{(i)}$ all the eigenvalues of the problem of

$$(4)\qquad \begin{cases}\Delta u+\lambda Mu=0 & \text{in } D_n^{(i)},\\ u=0 & \text{on } \beta^{(i)}+C_n^{(i)}\end{cases}$$

are greater than $\lambda_{1,1}$ and also greater than $\lambda_{1,n}$ since $D_{1,1}\subset D_{1,n}$. Hence, there exists in $D_n^{(i)}$ the Green's function of $\Delta u+\lambda_{1,n}Mu=0$. Moreover, by lemma 1, the first eigenvalue of the equation $\Delta u+\lambda_{1,n}f(x, y)u=0$ under fixed boundary condition is greater than that of $\Delta u+\lambda Mu=0$ and also is greater than $\lambda_{1,n}$. Therefore, there does exist in $D_n^{(i)}$ the Green's function of the equation $\Delta u+\lambda_{1,n}f(x, y)u=0$.

By making the radius of E tend to zero, and we see from the boundary condition for $\varphi_n^{(i)}$ and $\Omega_n^{(i)}$ that the left-hand member of (3) becomes

$$\int_{B^i}\varphi_n^{(i)}\frac{\partial\Omega_n^{(i)}}{\partial\nu}ds=u_{1,n}(p).$$

As $\Delta\varphi_n^{(i)}=0$ and $\Delta\Omega_n^{(i)}+f(x, y)\lambda_{1,n}\Omega_n^{(i)}=0$ in $D_n^{(i)}$, so the right-hand member of (3) is equal to

$$\lambda_{1,n}\iint f(q)\varphi_n^{(i)}\Omega_n^{(i)}\,d\sigma(q).$$

Therefore, by making the radius of E tend to zero, we get

$$u_{1,n}(p)=\lambda_{1,n}\iint_{D_n^{(i)}}f(q)\varphi_n^{(i)}\Omega_n^{(i)}(p, q)\,d\sigma(q).$$

From this and the maximum principle for the harmonic functions, we obtain an inequality

$$|u_{1,n}(p)|\leqq M\lambda_{1,n}\max_{q\in B^{(i)}}|\varphi_n^{(i)}(q)|\iint_{D_n^{(i)}}\Omega_n^{(i)}(p, q)\,d\sigma(q).$$

Next we shall show that

$$\int_{D_n^{(i)}}\Omega_n^{(i)}(p, q)\,d\sigma(q)$$

is bounded independently of the suffices i and n.

For this purpose we make the following observations. Consider the Green's function of the equation with constant coefficient $\Delta v+\lambda_{1,1}Mv=0$ in $D_n^{(i)}$. By the assumption about the area of $D_n^{(i)}$ there exists such a Green's function $\Omega_n'^{(i)}$. After some calculations we have the relation

$$\Omega_n'^{(i)}\geqq\Omega_n^{(i)}.$$

That is $\Omega_n^{(i)}$ is majorated by $\Omega_n'^{(i)}$. From our previous paper [2] (see pp.

184–195) we know that

$$\iint_{D_n^{(i)}} \Omega_n'^{(i)}\, d\sigma(q)$$

is bounded independently of suffices i and n. Hence we get

$$u_{1,n}(p) \leqq \lambda_{1,n} k' \max_{q\in B^{(i)}} u_{1,n}(q).$$

On the other hand, from §3, $B^{(i)}$ can be taken in A such that $\max_{q\in B^{(i)}} u_{1,n}(q)$ does not exceed a fixed constant and $\lambda_{1,n}<\lambda_{1,1}$ as $D_1\subset D_n$. Thus we obtain

$$u_{1,n}(p) \leqq \lambda_1 k' \mathfrak{M} \quad \text{in} \quad D_n^{(i)}$$

for any i and n, where λ_1, k', $\mathfrak{M}$ in the right-hand member are constants independent of i and n. Moreover, as $\tilde{D}_n \subset A$ from §3, $\{u_{1,n}\}$ is uniformly bounded in $\tilde{D}$. Thus the uniform boundedness of $\{u_{1,n}\}$ in D has been established.

§ **5.** Here we show that the normalization condition

$$\iint_D f(x, y) u_1^2\, d\sigma = 1$$

holds.

By putting

$$U_{1,n}(p) = \begin{cases} u_{1,n}(p) & \text{in} \quad D_n, \\ 0 & \text{in} \quad D-D_n, \end{cases}$$

the function $u_{1,n}$ defined in D_n is extended into the whole domoin D. Since $U_{1,n}$ is uniformly bounded in D by §3,

$$\lim_{n\to\infty}\iint_D f(x, y) U_{1,n}^2\, d\sigma = \iint_D \lim_{n\to\infty} f(x, y) U_{1,n}^2\, d\sigma = \iint_D f(x, y) u_1^2\, d\sigma.$$

But

$$\lim_{n\to\infty}\iint_D f(x, y) U_{1,n}^2\, d\sigma = \lim_{n\to\infty}\iint_{D_n} f(x, y) u_{1,n}^2\, d\sigma = 1.$$

Hence

$$\iint_D f(x, y) u_1^2\, d\sigma = 1.$$

§ **6.** Next we show that the function u_1 and the limit value λ_1 satisfy the integral equation

$$u_1(p) = \lambda_1 \iint_D G(p, q) f(q) u_1(q)\, d\sigma(q)$$

where $2\pi G(p, q)$ denotes the ordinary Green's function for the domain D, i. e.

$$G(p, q) = \frac{1}{2\pi}\log\frac{1}{r} + H(p, q),$$

H being a regular harmonic function in D.

In the first step, let p be a fixed interior point of D, then $p \in D_n$ for sufficiently large n. It is well known that our assertion is true for $\lambda_{1,n}$, $u_{1,n}$ and D_n, namely

$$u_{1,n}(p) = \lambda_{1,n} \iint_{D_n} G_n(p, q) f(q) u_{1,n}(q) \, d\sigma(q);$$

here $2\pi G_n(p, q)$ denotes the ordinary Green's function for the domain D_n. Now by putting

$$\mathfrak{G}_n(p, q) = \begin{cases} G_n(p, q) & \text{in } D_n, \\ 0 & \text{in } D - D_n, \end{cases}$$

we have

$$U_{1,n}(p) = \lambda_{1,n} \iint_D \mathfrak{G}_n(p, q) f(q) U_{1,n}(q) \, d\sigma(q)$$

and, by using the uniform boundedness of $U_{1,n}$ in D.

$$\mathfrak{G}_n(p, q) f(q) U_{1,n}(q) \leqq \mathfrak{M} \mathfrak{G}_n(p, q) \leqq \mathfrak{M} G(p, q)$$

where $\mathfrak{M}$ is a constant.

By the Lebesgue's convergence theorem, we get

$$\begin{aligned} \lim_{n\to\infty} \iint_D \mathfrak{G}_n(p, q) f(q) U_{1,n}(q) \, d\sigma(q) &= \iint_D \lim_{n\to\infty} \mathfrak{G}_n(p, q) f(q) U_{1,n}(q) \, d\sigma(q) \\ &= \iint_D G(p, q) f(q) u_1(q) \, d\sigma(q), \end{aligned}$$

since $f(q) G(p, q)$ is integrable. Thus we have

$$u_1(p) = \lambda_1 \iint_D G(p, q) f(q) u_1(q) \, d\sigma(q). \tag{5}$$

By this relation together with the boundedness of $f(q)$ and $u_1(q)$, we obtain the relations

$$\frac{\partial u_1}{\partial x} = \lambda_1 \iint_D \frac{\partial G(p, q)}{\partial x} f(q) u_1(q) \, d\sigma(q),$$

$$\frac{\partial u_1}{\partial y} = \lambda_1 \iint_D \frac{\partial G(p, q)}{\partial y} f(q) u_1(q) \, d\sigma(q)$$

and these derivatives are continuous in any interior point p of the domain. Therefore, we can conclude that our function u_1 satisfied the equation

$$\Delta u_1 + \lambda_1 f(x, y) u_1 = 0$$

in any interior point of the domain.

In the second step, we show that the relation (5) remains to hold even when p tends to a boundary point. What is to be shown is that

$$\begin{aligned} \lim_{p\to p_0} u_1(p) &= \lambda_1 \lim_{p\to p_0} \iint_D G(p, q) f(q) u_1(q) \, d\sigma(q) \\ &= \lambda_1 \iint_D \lim_{p\to p_0} G(p, q) f(q) u(q) \, d\sigma(q) = \lambda_1 \iint_D G(p_0, q) u(q) \, d\sigma(q) \end{aligned}$$

where

$$G(p_0, q)=\lim_{p\to p_0} G(p, q).$$

In fact,

$$G(p, q)<\text{const.}\log\frac{1}{r}, \qquad r=\overline{pq},$$

$$f(q)u_1(q)\leqq \mathfrak{M} \qquad \text{in} \quad D$$

$$G(p, q)f(q)u_1(q)\leqq \text{const.}\log\frac{1}{r} \qquad \text{in} \quad D.$$

By the Lebesgue's convergence theorem, we obtain the required relation

$$\lim_{p\to p_0} u_1(p)=\lambda_1\iint_D G(p_0, q)u_1(q)d\sigma(q).$$

Since we know that $\lim_{p\to p_0}G(p, q)$ becomes zero except for a set of capacity zero, $u_1(p)$ also becomes zero, as p tends to a boundary point. Taking account of $u_1(p)>0$ in D, our exceptional points are identical with those of of the ordinary Green's function $G(p, q)$ for the same domain. Hence we might regard our limit value λ_1 and limit function u_1 as the first eigenvalue and the first eigenfunction, respectively, of general bounded domain D.

§7. What is left to be shown in theorem 1 is that the limit function u_1 is determined independently of a choice of exhausting sequence. For this purpose, we take another exhausting sequence

$$\bar{D}_1\subset\bar{D}_2\subset\cdots\subset\bar{D}_n\subset\cdots.$$

Let the corresponding first eigenfunctions be

$$\bar{u}_{1,1},\ \bar{u}_{1,2},\ \cdots,\ \bar{u}_{1,n},\ \cdots$$

and $\bar{u}_1$ be its limit function. We shall show that $u_1=\bar{u}_1$.

First we consider a particular case where the boundary of D consists of a finite number of smooth closed curves. Then, $u_1(p)$ is expressed by

$$u_1(p)=\lambda_1\iint_D G(p, q)u_1(q)d\sigma(q).$$

In this case, since $G(p_0, q)=0$ for all p_0 on the boundary of the domain, the limit value λ_1 and the limit function u_1 are, respectively, the first eigenvalue and the first eigenfunction. From the well known property of such a domain, the first eigenvalue λ_1 must be simple, and the first eigenfunction must be unique. Consequently, we have $u_1=\bar{u}_1$.

It is further shown that the first eigenvalue and the eigenfunction have the continuity with respect to the domain provided its boundary consists of a finite number of smooth curves.

Now we return to the general case, where the boundary does not necessarily consist of smooth curves only. Suppose contrarily we had $u_1\neq\bar{u}_1$ in D. Then there would be a point p such that

$$|u_1(p)-\bar{u}_1(p)|=a>0$$

and, for sufficiently large integers m, n and a small positive number ε,

$$|u_{1,m}(p)-u_1(p)|<\frac{\varepsilon}{2}, \qquad |\bar{u}_{1,n}(p)-\bar{u}_1(p)|<\frac{\varepsilon}{2}.$$

So we would get

$$(6) \qquad |u_{1,m}(p)-\bar{u}_{1,n}(p)|>a-\varepsilon$$

But on the other hand, we have

$$(7) \qquad |u_{1,m}(p)-\bar{u}_{1,n}(p)|<\eta$$

where η may be small positive number by making m and n large enough, based on the above mentioned continuity relation between the first eigenfunction and the domain since the boundaries of D_m and D_n consist of smooth curves. Then (6) contradicts (7). Therefore $u_1=\bar{u}_1$ should hold, which proves that our limit functions u_1 is determined independently of a choice of exhausting sequence.

Thus our theorem 1 has been proved.

§8. THEOREM 2. *Retaining the notations in theorem* 1, *let* $\lambda_{k,n}$ *and* $u_{k,n}$ *be the k-th eigenvalue and the k-th eigenfunction, respectively, of the problem*

$$\begin{cases} \Delta u+\lambda f(x, y)u=0 & \text{in } D_n, \\ u=0 & \text{on } C_n \end{cases}$$

and $u_{k,n}$ *be normalized by*

$$\iint_{D_n} f(x, y)u_{k,n}{}^2 d\sigma=1.$$

Then the following results similar to those of theorem 1 *remain to hold except the uniqueness of the limit function. Namely, the limit value*

$$(1) \qquad \lim_{n\to\infty}\lambda_{k,n}=\lambda_k$$

exists independently of a choice of exhausting sequence, and for any infinite subsequence $\{u_{k,n'}\}$ *of the corresponding sequence* $\{u_{k,n}\}$ *of the eigenfunctions, there exists a uniformly convergent subsequence* $\{u_{k,n''}\}$. *Put* $\lim_{n''\to\infty}u_{k,n''}=u_k$, *then the normalization condition*

$$(2) \qquad \iint_D f(x, y)\,u_k{}^2 d\sigma=1$$

holds.

(3) *The limit value* λ_k *and the limit function* u_k *satisfy the equation* $\Delta u_k+\lambda_k f(x, y)u_k=0$ *in* D *together with the condition* $u_k=0$ *on the boundary of* D *except at most for the set of capacity zero which is exceptional for the ordinary Green's function.*

(4) *Furthermore, if we have the relations*

$$\lim_{n\to\infty}\lambda_{k-1,n}<\lim_{n\to\infty}\lambda_{k,n}<\lim_{n\to\infty}\lambda_{k+1,n},$$

then u_k is determined independently of a choice of the exhausting sequence. Otherwise, that is, if we have

$$\lim_{n\to\infty}\lambda_{k-\nu-1,n}<\lim_{n\to\infty}\lambda_{k-\nu,n}=\lim_{n\to\infty}\lambda_{k-\nu+1,n}$$
$$=\lim_{n\to\infty}\lambda_{k-\nu+m-1,n}<\lim_{n\to\infty}\lambda_{k-\nu+m,n},$$

then u_k depends on the choice of subsequence of $\{u_{k,n}\}$. But among possible limit functions there exist only m linearly independent ones.

§**9.** The proof of (4) in theorem 2 is exactly same as that of the equation with constant coefficient given in a previous paper [3] (pp. 184–185).

The proof of the other results (1), (2) and (3) in theorem 2 can be given in the almost same way as that of the corresponding ones in theorem 1. But some reasoning in §3, where we have fully used the positive character of the first eigenfunction, must be modified.

To show that $\{u_{k,n}\}$ is uniformly bounded in A, let us take an arbitrary point p in A and a circle K of radius ε about p as we have done in §3. Here also K is contained in A' and in D_n. Again by taking ε sufficiently small, let K have the same property as R, that is, there exist in no non-trivial solutions of the equations $\Delta w+\lambda Mw=0$ and $\Delta u+\lambda f(x, y)u=0$ under the fixed boundary condition, where $M\geqq f(x, y)\geqq\mu>0$ as in §3.

For $k>2$, $u_{k,n}$ changes its sign. Consider another circle, $\mathfrak{K}$ with p as the center and with the radius $r<\varepsilon$. Now denote by C^+ the part of the circumference of the above circle on which $u_{k,n}$ takes non-negative values and by C^- the remaining part, i.e.

$$u_{k,n}=\begin{cases}\psi^+\geqq 0 & \text{on } C^+,\\ \psi^-<0 & \text{on } C^-.\end{cases}$$

By the principle of linear superposition, we can decompose $u_{k,n}$ into two parts:

$$u_{k,n}=u_{k,n}^++u_{k,n}^-$$

where $u_{k,n}^+$ satisfies the equation

$$\Delta u_{k,n}^++\lambda_{k,n}f(x, y)u_{k,n}^+=0$$

with

$$u_{k,n}^+=\begin{cases}\psi^+ & \text{on } C^+,\\ 0 & \text{on } C^-,\end{cases}$$

and likewise $u_{k,n}^-$ satisfies the equation

$$\Delta u_{k,n}^-+\lambda_{k,n}f(x, y)u_{k,n}^-=0$$

with

$$u_{k,n}^-=\begin{cases}0 & \text{on } C^+,\\ \psi^- & \text{on } C^-.\end{cases}$$

To obtain an upper bound for $u_{k,n}$, consider two functions w^+ and w^- defined as follows:

$$\Delta w^+ + \lambda_{k,1} M w^+ = 0 \quad \text{in } \mathfrak{K},$$

$$w^+ = \begin{cases} \psi^+ & \text{on } C^+, \\ 0 & \text{on } C^-, \end{cases}$$

and

$$\Delta w^- = 0 \quad \text{in } \mathfrak{K},$$

$$w^- = \begin{cases} 0 & \text{on } C^+, \\ \psi^- & \text{on } C^-. \end{cases}$$

By lemma 2 we have

$$w^+ \geqq u_{k,n}^+$$

and since $u_{k,n}^- < 0$, so $u_{k,n}^-$ is subharmonic, whence follows

$$w^- \geqq u_{k,n}^-.$$

Therefore, we get

$$u_{k,n} = u_{k,n}^+ + u_{k,n}^- \leqq w^+ + w^-.$$

Applying the mean-value theorems of Gauss and that of Weber, we have

$$(7) \qquad u_{k,n}(p) \leqq \frac{1}{2\pi}\left(\frac{1}{J_0(\sqrt{\lambda_{k,1}}r)}\int_0^{2\pi}\psi^+ d\theta + \int_0^{2\pi}\psi^- d\theta\right)$$

where the integration is taken over $\mathfrak{K}$.

Next to obtain a lower bound for $u_{k,n}$, consider two functions v^+ and v^- defined as follows:

$$\Delta v^+ = 0 \quad \text{in } \mathfrak{K},$$

$$v^+ = \begin{cases} \psi^+ & \text{on } C^+, \\ 0 & \text{on } C^- \end{cases}$$

and

$$\Delta v^- + \lambda_{k,1} M v^- = 0 \quad \text{in } \mathfrak{K},$$

$$v^- = \begin{cases} 0 & C^+, \\ \psi^- & C^-. \end{cases}$$

Since $u_{k,n}^+$ is superharmonic, so $u_{k,n}^+ \geqq v^+$, and by lemma 2, $-v^- \geqq -u_{k,n}^-$ i. e. $v^- \leqq u^-$. Hence

$$u_{k,n} = u_{k,n}^+ + u_{k,n}^- \geqq v^+ + v^-.$$

Applying again the above mean-value theorem, we have

$$(8) \qquad u_{k,n}(p) \geqq \frac{1}{2\pi}\left(\int_0^{2\pi}\psi^+ d\theta + \frac{1}{J_0(\sqrt{\lambda_{1,1}M}r)}\int_0^{2\pi}\psi^- d\theta\right).$$

From the relations (7) and (8), the uniformly boundedness of $\{u_{k,n}\}$ in A can be verified as follows. As $\psi^- \leqq 0$, we get, from (7),

$$u_{k,n}\leqq\frac{1}{2\pi}\frac{1}{J_0(\sqrt{\lambda_{k,1}}Mr)}\int_0^{2\pi}(\psi^+-\psi^-)d\theta \tag{9}$$

where $\psi^+-\psi^-$ is equal to that of $|u_{k,n}|$ on the boundary of our circle. Similarly from (8), we get

$$u_{k,n}\geqq\frac{1}{2\pi}\frac{1}{J_0(\sqrt{\lambda_{k,1}}Mr)}\int_0^{2\pi}(-\psi^++\psi^-)d\theta. \tag{10}$$

Combining (9) and (10), we obtain

$$|u_{k,n}(p)|\leqq\frac{1}{2\pi}\frac{1}{J_0(\sqrt{\lambda_{k,1}}r)}\int_0^{2\pi}|u_{k,n}|d\theta.$$

By taking ε small enough such that $\sqrt{\lambda_{k,1}}M\varepsilon$ becomes smaller than the first positive zero-point of the Bessel function J_0, we get

$$|u_{k,n}(p)|\leqq\frac{1}{2\pi k}\int_0^{2\pi}|u_{k,n}|d\theta$$

since

$$J_0(\sqrt{\lambda_{k,1}}Mr)\geqq J(\sqrt{\lambda_{k,1}}M\varepsilon)\equiv k$$

Consequently, by $f(x, y)\geqq\mu>0$, we get

$$|u_{k,n}(p)|\leqq\frac{1}{2\pi k\sqrt{\mu}}\int_0^{2\pi}\sqrt{f}\,|u_{k,n}|d\theta.$$

Multiplying both sides by r and integrating with respect to r from 0 to ε, we obtain

$$|u_{k,n}(p)|\int_0^{\varepsilon}rdr\leqq\frac{1}{2\pi k\sqrt{\mu}}\int_0^{\varepsilon}\int_0^{2\pi}\sqrt{f}\,|u_{k,n}|rd\theta dr$$

whence follows, by Schwarz's inequality,

$$|u_{k,n}(p)|\frac{\varepsilon^2}{2}<\frac{1}{2\pi k\sqrt{\mu}}\left(\left(\iint_K d\sigma\right)\left(\iint_K fu_{k,n}{}^2d\sigma\right)\right)^{1/2}$$
$$\leqq\frac{1}{2\pi k\sqrt{\mu}}\left(\frac{\pi\varepsilon^2}{2}\iint_{D_n}fu_{k,n}{}^2d\sigma\right)^{1/2}=\frac{1}{2\pi k\sqrt{\mu}}\left(\frac{\pi\varepsilon^2}{2}\right)^{1/2},$$

i. e.

$$|u_{k,n}(p)|\leqq\frac{1}{\sqrt{2\pi\mu}\,k\varepsilon},$$

the right-hand member being independent of n as well as p.

Next with respect to the Arsove's second condition of the normal family, we slightly modify (9) in the form

$$u_{k,n}(p)\leqq\frac{1}{2\pi}\left\{\int_0^{2\pi}(\psi^++\psi^-)d\theta+\frac{1}{2\pi}\left(\frac{1}{J_0(\sqrt{\lambda_{k,1}}Mr)}-1\right)\int_0^{2\pi}\psi^+d\theta\right\}. \tag{11}$$

This implies

$$u_{k,n}(p)\leqq\frac{1}{2\pi}\int_0^{2\pi}u_{k,n}d\theta+\frac{1}{2\pi}\left(\frac{1}{J_0(\sqrt{\lambda_{k,1}}Mr)}-1\right)\int_0^{2\pi}\psi^+d\theta.$$

Likewise, from (10), we get

$$(12) \qquad u_{k,n}(p) \geqq \frac{1}{2\pi}\int_0^{2\pi} u_{k,n}\, d\theta + \frac{1}{2\pi}\left(\frac{1}{J_0(\sqrt{\lambda_{k,1}}Mr)} - 1\right)\int_0^{2\pi} \psi^- \, d\theta.$$

Multiplying (11) and (12) by r, integrating with respect to r from 0 to ε, and taking account of the uniform boundedness of $\{u_{k,n}\}$ in A and the estimation $1/J_0(\sqrt{\lambda_{k,1}}Mr) - 1 = O(r^2)$, we obtain readily

$$| u_{k,n}(p) - \text{Areal mean of } u_{k,n} | = O(\varepsilon^2).$$

Therefore, $\{u_{k,n}\}$ satisfies the Arsove's conditions for normality [1]. The reasonings left for the proof of theorem 2 are exactly same as those of theorem 1 except for trivial modification.

Finally, it would be noted that our theorems could be transferred into the 3-dimensional case.

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