ON AN APPROXIMATION THEOREM IN A FAMILY OF QUASICONFORMAL MAPPINGS

BY MITSURU OZAWA

Let T be a topological mapping of the unit disc $|z|<1$ onto the unit disc $|w|<1$ which is quasiconformal in the sense of Pfluger-Ahlfors-Mori, then by a theorem due to Ahlfors or Mori T can be extended to a topological mapping of the closed unit disc $|z| \leqq 1$ onto the closed unit disc $|w| \leqq 1$. Let D be a domain in the z-plane surrounded by a finite number of mutually disjoint Jordan curves and Δ a likewise domain in the w-plane. Let T be a quasiconformal mapping of D onto Δ, then T can be extended to a topological mapping of $\bar{D}$ onto $\bar{\Delta}$ by the above-mentioned fact and the Carathéodory's theorem on the boundary correspondence in the Riemann's mapping theorem. Let $\mathfrak{F}(T)$ be the family of quasiconformal mappings any member of which gives the same boundary correspondence as T. In the present note we shall give a very simple approximation theorem in the family $\mathfrak{F}(T)$ and then apply it in order to establish a wider extremality of an extremal quasiconformal mapping in the Teichmüller problem. In the way we shall give some theorems guaranteeing the extremality of a given mapping.

LEMMA 1. *Let D and Δ be the unit discs and I the identity mapping $w=z$. Then in $\mathfrak{F}(I)$ there exists a one-parameter family of quasiconformal mappings $U_r(z)$ for any given $U(z) \in \mathfrak{F}(I)$ such that* (i) *$U_r(z) \equiv I$ for $|z| \geqq r$,* (ii) *$\lim_{r \to 1} U_r(z) = U(z)$ uniformly in $|z| \leqq 1$ and* (iii) *$K_{U_r} = K_U$, where $K_\Box$ denotes the maximal dilatation of mapping $\Box$, that is, the supremum of the dilatation of $\Box$.*

Proof. Let

$$U_r(z) = \begin{cases} I(z) & \text{for } |z| \geqq r, \\ rU(z/r) & \text{for } |z| \leqq r, \end{cases}$$

then $U_r(z)$ is a topological mapping of the closed unit disc $|z| \leqq 1$ onto the closed disc $|w| \leqq 1$, whose dilatation is equal to that of $U(z/r)$ in $|z|<r$ and to 1 in $r<|z|<1$, and hence $U_r(z)$ is a quasiconformal mapping belonging to $\mathfrak{F}(I)$. For any point z of $|z| \leqq R < 1$, we can choose r such that $|z|/r \leqq 1$. Then, by Mori's theorem,

$$\left| M\left(\frac{z}{r}\right) - M(z) \right| \leqq 16 \left\{ |z| \left(\frac{1}{r} - 1 \right) \right\}^{1/K_U}, \quad M(z) = \frac{U(z) - U(0)}{1 - \overline{U(0)}\, U(z)},$$

Received January 17, 1959.

holds for any z, since $M(0)=0$, and hence

$$\left|U\left(\frac{z}{r}\right)-U(z)\right|\leqq 16\left\{|z|\left(\frac{1}{r}-1\right)\right\}^{1/K}\frac{|1-\overline{U(0)}U(z/r)|\,|1-\overline{U(0)}U(z)|}{1-|U(0)|^2}$$
$$\leqq 64\sqrt[K]{\frac{1-r}{r}}\cdot\frac{1}{1-|U(0)|^2},$$

where K is any number not less than the maximal dilatation K_U of $U(z)$. Therefore, we have for $|z|\leqq R$

$$|U_r(z)-U(z)|\equiv\left|rU\left(\frac{z}{r}\right)-U(z)\right|\leqq r\left|U\left(\frac{z}{r}\right)-U(z)\right|+|U(z)|(1-r)$$
$$\leqq 64r\sqrt[K]{\frac{1-r}{r}}\frac{1}{1-|U(0)|^2}+1-r,$$

from which we can conclude that $U_r(z)$ tends to $U(z)$ uniformly in the wider sense in $|z|<1$, when r tends to 1. Let $M_r(z)$ be the composed map defined by

$$M_r(z)=\frac{U_r(z)-rU(0)}{1-r\overline{U(0)}U_r(z)},$$

then we can again use the Mori's theorem and obtain

$$|M_r(z_1)-M_r(z_2)|\leqq 16\sqrt[K]{|z_1-z_2|}$$

for any z_1, z_2 in $|z|\leqq 1$. Hence we have the Hölder continuity of $U_r(z)$:

$$|U_r(z_1)-U_r(z_2)|\leqq 64\sqrt[K]{|z_1-z_2|}\frac{1}{1-|U(0)|^2},$$

which shows that $\{U_r(z)\}$ is an equicontinuous family and hence there is a subsequence $\{U_{r_\nu}(z)\}$ converging uniformly in $|z|\leqq 1$. Since the limit mapping $\lim_{\nu\to\infty}U_{r_\nu}(z)$ coincides with $U(z)$ for any convergent subsequence, the original sequence itself converges to $U(z)$ uniformly in $|z|\leqq 1$.

APPROXIMATION THEOREM. *Let D be the unit disc $|z|<1$ and Δ any simply-connected domain. Let S belong to the family $\mathfrak{F}(T)$, then there is a one-parameter family of quasiconformal mappings $S_r\in\mathfrak{F}(T)$ such that* (i) *$S_r\equiv T$ for any $|z|\geqq r$ and* (ii) *$\lim_{r\to 1}S_r=S$ uniformly in $|z|\leqq 1$.*

Proof. Let U be $T^{-1}S$, then $U\in\mathfrak{F}(I)$. Let U_r be a mapping constructed in lemma 1 and S_r be TU_r, then this S_r satisfies evidently two conditions of the theorem by lemma 1.

We shall give a precision of the above theorem, which is, however, not used in the sequel.

COROLLARY. *Let T satisfy a differential equation $cp_T=q_T$ almost everywhere in $|z|<1$ with a suitable constant c and $p_T=\partial T/\partial z$, $q_T=\partial T/\partial\bar{z}$, then there exists a one-parameter family of quasiconformal mappings S_r in $\mathfrak{F}(T)$ such that* (i) *and* (ii) *in approximation theorem are satisfied and* (iii) $K_{S_r}=\max(K_S,(1+|c|)/(1-|c|))$.

Proof. S_r defined in the approximation theorem is a composed mapping: $w=T(rT^{-1}S(z/r))$. Here we shall put $w=T(z_2)$, $z_2=rz_1$, $z_1=T^{-1}(w_1)$, $w_1=S(z')$ and $z'=z/r$. A simple calculation then leads to

$$\begin{aligned} p_{S_r}(z) &= \frac{\partial w}{\partial z} \\ &= \frac{1}{|p_T(z_1)|^2-|q_T(z_1)|^2}\Big[p_S\Big(\frac{z}{r}\Big)\{p_T(rz_1)\bar{p}_T(z_1)-q_T(rz_1)\bar{q}_T(z_1)\} \\ &\qquad + \bar{q}_S\Big(\frac{z}{r}\Big)\{q_T(rz_1)p_T(z_1)-p_T(rz_1)q_T(z_1)\}\Big], \end{aligned}$$

$$\begin{aligned} q_{S_r}(z) &= \frac{\partial w}{\partial \bar{z}} \\ &= \frac{1}{|p_T(z_1)|^2-|q_T(z_1)|^2}\Big[p_S\Big(\frac{z}{r}\Big)\{p_T(rz_1)\bar{p}_T(z_1)-q_T(rz_1)\bar{q}_T(z_1)\} \\ &\qquad - \bar{q}_S\Big(\frac{z}{r}\Big)\{p_T(rz_1)q_T(z_1)-q_T(rz_1)p_T(z_1)\}\Big], \end{aligned}$$

where $p_\square(t)=\partial\square/\partial t$ and $q_\square(t)=\partial\square/\partial\bar{t}$. Therefore we have

$$\frac{q_{S_r}(z)}{p_{S_r}(z)}=\frac{\bar{p}_S(z/r)}{p_S(z/r)}\,\frac{\dfrac{q_S(z/r)}{\bar{p}_S(z/r)}-\dfrac{B}{A}}{1-\dfrac{B}{A}\dfrac{\bar{q}_S(z/r)}{p_S(z/r)}},\quad \frac{B}{A}=\frac{p_T(z_1)}{\bar{p}_T(z_1)}\,\frac{\dfrac{q_T(z_1)}{p_T(z_1)}-\dfrac{q_T(rz_1)}{p_T(rz_1)}}{1-\dfrac{q_T(rz_1)}{p_T(rz_1)}\dfrac{\bar{q}_T(z_1)}{\bar{p}_T(z_1)}}.$$

By the assumption $q_T(z)=cp_T(z)$, $B/A\equiv 0$ holds and hence

$$\frac{q_{S_r}(z)}{p_{S_r}(z)}=\frac{q_S(z/r)}{p_S(z/r)}$$

remains valid at almost all points in $|z|<r$. Since $S_r\equiv T$ holds in $|z|>r$, we have

$$\text{sup dilatation of } S_r=\max(\text{sup dilatation of } S,\ (1+|c|)/(1-|c|)),$$

which is our desired result.

Let D be a Riemann domain spread over the z-plane and Δ the image domain of D by a sense-preserving affine mapping T: $w=w(z)=Az+B\bar{z}+C$, $0<B/A<1$.

THEOREM 2. *If D is of finite area, the affine mapping T is a unique extremal quasiconformal mapping in $\mathfrak{F}(T)$.*

Proof. The proof depends upon the usually availed length-area principle. The common set of the straight line $y=y_0$ and D consists of a finite or an infinite number of connected components each of which consists of a finite or an infinite number of segments connected at a finite or an infinite number of points at which relative ramification points of D over the z-plane lie. The image set of this set by T is of the same character. Let $l_j^k(y_0)$ be the jth segment of the kth component, lying in D over $y=y_0$, and $L_j^k(y_0)$ its image by T. The lengths of $l_j^k(y_0)$ and $L_j^k(y_0)$ are denoted by $|l_j^k(y_0)|$ and $|L_j^k(y_0)|$, respectively. Then, since D is of finite area over the z-plane,

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}|l_j^k(y_0)|<\infty$$

for almost all y_0. Evidently we have

$$|L_j^k(y_0)|=\int_{l_j^k(y_0)}\left|\frac{\partial T}{\partial x}\right|dx=|A+B|\ |l_j^k(y_0)|$$

and

$$A(\varDelta)=(|A|^2-|B|^2)A(D)$$

where $A(\varDelta)$ and $A(D)$ are the areas of $\varDelta$ and D, respectively, each of which is of finite value. Let S belong to $\mathfrak{F}(T)$, then by Mori's theorem the length of the curve $S(l_j^k(y^0))$ is equal to

$$\int_{l_j^k(y_0)}|dS(\mathfrak{z}_j)|=\int_{l_j^k(y_0)}\left|\frac{\partial S(x_j,y_0)}{\partial x_j}\right||dx_j|$$

for almost all y_0, where $\mathfrak{z}_j=(x_j,y_0)$ is a local parameter at a point lying on $l_j^k(y_0)$. Then we have for almost all y_0

$$|L_j^k(y_0)|\leqq\int_{l_j^k(y_0)}\left|\frac{\partial S(x_j,y_0)}{\partial x_j}\right||dx_j|.$$

By the Schwarz' inequality we have

$$|L_j^k(y_0)|^2\leqq\int_{l_j^k(y_0)}\left|\frac{\partial S(x_j,y_0)}{\partial x_j}\right|^2|dx_j|\int_{l_j^k(y_0)}|dx_j|,$$

whence follows that

$$|A+B|^2\,|l_j^k(y_0)|\leqq\int_{l_j^k(y_0)}\left|\frac{\partial S(x_j,y_0)}{\partial x_j}\right|^2|dx_j|.$$

By Mori's theorem there holds a relation

$$\begin{aligned}\left|\frac{\partial S}{\partial x_j}\right|^2&=|p_S+q_S|^2\leqq(|p_S|+|q_S|)^2\\&=\frac{|p_S|+|q_S|}{|p_S|-|q_S|}(|p_S|^2-|q_S|^2)\leqq K_S(|p_S|^2-|q_S|^2)\end{aligned}$$

for almost all points in D. Therefore we have

$$|A+B|^2\,|l_j^k(y_0)|\leqq K_S\int_{l_j^k(y_0)}(|p_S|^2-|q_S|^2)\,|dx_j|.$$

Summing up this inequality for j and k and then integrating with respect to y_0 in such a manner that $\sum\sum l_j^k(y_0)$ covers just D, we obtain

$$|A+B|^2\int\sum\sum|l_j^k(y_0)|\,dy_0\leqq K_S\iint_{\Sigma\Sigma l_j^k(y_0)}(|p_S|^2-|q_S|^2)\,|dx_j|\,dy_0.$$

By Fubini's theorem the right-hand member is equal to

$$K_S\iint_D(|p_S|^2-|q_S|^2)\,dxdy=K_SA(\Delta)$$
$$=K_S(|A|^2-|B|^2)A(D)<\infty,$$

from which we have

$$|A+B|^2\leqq K_S(|A|^2-|B|^2).$$

Since B/A is real positive and is less than one, we have

$$K_T=\frac{|A|+|B|}{|A|-|B|}\leqq K_S.$$

Equality sign occurs if and only if $S\equiv T$. This completes the proof of our theorem.

Let W and W' be two closed Riemann surfaces of genus $g>1$ and H a homotopy class of topological sense-preserving mappings of W onto W', then in H there exists either a conformal mapping or an extremal quasiconformal mapping $T(z)$. In the latter case there correspond a regular quadratic differential fdz^2 uniquely determined up to a positive constant factor and a positive constant k $(0<k<1)$, and $T(z)$ is differentiable and has complex derivatives at all points with exception of $4g-4$ zeros of fdz^2 on W and satisfies a differential equation of Beltrami type

$$\frac{q_T}{p_T}=k\frac{\bar{f}}{|f|}$$

at all such non-exceptional points.

This theorem, originally due to Teichmüller and reproved by Ahlfors, should constitute a climax theorem in the global theory of quasiconformal mappings and offers a foundation to attack the modulus problem in the theory of Riemann surfaces.

To establish the extremality and uniqueness of the desired mapping, Ahlfors has made use of the uniformization theory. Let C and C' be the universal covering surfaces of W and W' mapping conformally onto the unit discs $|z|<1$ and $|\zeta|<1$, respectively. Corresponding cover-transformation groups $\mathfrak{G}$ and $\mathfrak{G}'$ consist of an infinite number of linear transformations $\{L\}$ and $\{L'\}$, respectively. A topological mapping of W onto W' gives rise to a likewise topological mapping $\zeta=\zeta(z)$ of C onto C' or rather to infinitely many such mappings $L'\zeta(z)$ with $L'\in\mathfrak{G}'$. To any $L\in\mathfrak{G}$ there corresponds a unique $L^\alpha\in\mathfrak{G}'$ such that $\zeta(Lz)=L^\alpha\zeta(z)$ and, as an isomorphism, α is determined up to an inner automorphism of $\mathfrak{G}$ or $\mathfrak{G}'$. Then two topological mappings of W onto W' are homotop if and only if they determine isomorphisms which differ only by an inner automorphism, and hence the homotopy classes are in one-to-one correspondence with a set of outer automorphisms. Thus H corresponds to a fixed outer automorphism α. Under these preparations, Teichmüller's theorem can be reformulated in terms of uniformization.

Any member S of H gives rise to a mapping $\zeta_S(z)$: $C \to C'$ satisfying a functional equation $\zeta_S(Lz) = L^\alpha \zeta_S(z)$. Now we shall prove that $\zeta_S(z)$ has the same boundary correspondence $\{|z|=1\} \to \{|\zeta|=1\}$ for any $S \in H$ under the fixed choice of fundamental regions in C and C'. Let $\prod_1^\infty L_n$ with $L_n \in \mathfrak{G}$ be an infinite product for which any point z lying in any compact set in $|z|<1$ converges to a point $e^{i\theta}$. The set $\{e^{i\theta}\}$ of such points is dense on the unit circumference $|z|=1$. Let ρ_{12} be a non-euclidean distance $d(\zeta_{S_1}(z), \zeta_{S_2}(z))$ in $|\zeta|<1$ for $S_1, S_2 \in H$. Both $\zeta_{S_1}(z)$ and $\zeta_{S_2}(z)$ are homeomorphisms of $\bar{C}$ onto $\bar{C}'$. Therefore we have

$$\zeta_{S_1}\left(\prod_1^\infty L_n z\right) = \zeta_{S_1}(e^{i\theta}) = e^{i\varphi_1}$$

and

$$\zeta_{S_2}\left(\prod_1^\infty L_n z\right) = \zeta_{S_1}(e^{i\theta}) = e^{i\varphi_2}$$

for any $|z| \leqq r < 1$. By the functional relations we have

$$\prod_1^\infty L_n{}^\alpha \zeta_{S_1}(z) = e^{i\varphi_1} \quad \text{and} \quad \prod_1^\infty L_n{}^\alpha \zeta_{S_2}(z) = e^{i\varphi_2},$$

and the invariance of the non-euclidean distance for $\mathfrak{G}'$ implies that

$$d\left(\prod_1^\infty L_n{}^\alpha \zeta_{S_1}(z),\ \prod_1^\infty L_n{}^\alpha \zeta_{S_2}(z)\right) = \rho_{12} < \infty.$$

Hence we have

$$d(e^{i\varphi_1}, e^{i\varphi_2}) = \rho_{12} < \infty,$$

which shows that $e^{i\varphi_1}$ coincides with $e^{i\varphi_2}$. Since the set $\{e^{i\theta}\}$ is dense on $|z|=1$ and the homeomorphisms of ζ_{S_1} and ζ_{S_2} hold on $|z| \leqq 1$, $\zeta_{S_1}(e^{i\chi}) = \zeta_{S_2}(e^{i\chi})$ remains valid for any χ. This shows that ζ_{S_1} and ζ_{S_2} have the same boundary correspondence if S_1 and S_2 belong to H under the fixed choice of fundamental regions in C and C'.

Let $\{\alpha\}$ be a class of quasiconformal mappings of C onto C' satisfying the functional relation $\zeta(Lz) = L^\alpha \zeta(z)$ for any $|z| \leqq 1$. Let $\mathfrak{F}\{\alpha\}$ be a class of quasiconformal mappings of C onto C' with the same boundary correspondence as that of any member of $\{\alpha\}$. In the above definitions we assume that the fixation of fundamental regions in C and C' is made, that is, we fix a boundary correspondence by a suitable linear transformation. Then $\{\alpha\}$ is a proper subclass of $\mathfrak{F}\{\alpha\}$. Teichmüller's theorem shows that in $\{\alpha\}$ there exists either a conformal mapping or a unique extremal quasiconformal mapping T which is differentiable and has non-vanishing complex derivatives with the exception of a countably infinite number of points in $|z|<1$ whose accumulation points lie only on $|z|=1$, and further satisfies with these exceptions a differential equation of Beltrami type

$$\frac{q}{p} = k\frac{\bar{f}}{|f|}.$$

Here $f(z)$ ($\not\equiv 0$) is a regular function determined uniquely up to a positive constant factor and k is a uniquely determined constant with $0 < k < 1$.

The inverse mapping T^{-1} is also a unique extremal mapping in $\{\alpha\}^{-1}$ and hence it satisfies a differential equation with a suitable function $\varphi(\zeta)$ regular in $|\zeta| < 1$

$$\frac{q}{p} = k\frac{\varphi}{|\varphi|}.$$

Let z_0 and ζ_0 be corresponding points by T for which $f(z_0) \neq 0$ and $\varphi(\zeta_0) \neq 0$. We introduce new local parameters by

$$z^* = \int \sqrt{f}\, dz, \qquad \zeta^* = \int \sqrt{\varphi}\, d\zeta$$

with fixed branches of the square roots and arbitrary integration constants. Then the composed mapping T: $z^* \to z \overset{T}{\to} \zeta \to \zeta^*$ is an affine mapping of a Riemann domain D_{z^*} onto a Riemann domain Δ_{ζ^*}. This is easily seen and has been actually proved in Ahlfors' paper. Let S be any mapping in $\mathfrak{F}\{\alpha\}$, then by the approximation theorem there exists a sequence of quasiconformal mappings $\{S_r\}$ for which (i) $S_r \equiv T$ in $|z| \geqq r$, (ii) $\lim_{r\to 1} S_r = S$ uniformly in $|z| \leqq 1$. Let the image of $|z| \leqq r$ on the z^*-plane D_r and its image by T^* be Δ_r, then D_r and Δ_r are bounded and finitely-sheeted covering surfaces over the z^*-plane and the ζ^*-plane, respectively. Let S_r^* be an induced mapping of S_r by introducing new variables z^* and ζ^*. By the theorem 2, we have that T^* is a unique extremal quasiconformal mapping of D_r onto Δ_r in $\mathfrak{F}(T^*)$. Evidently S_r^* belongs to $\mathfrak{F}(T^*)$. Thus we have

$$K_{S_r} = K_{S_r^*} > K_{T^*} = \frac{1+k}{1-k},$$

whence follows

$$K_S \geqq \frac{1+k}{1-k}.$$

THEOREM 3. *In $\mathfrak{F}\{\alpha\}$, T is also an extremal quasiconformal mapping.*

It is our final conjecture that T is a unique extremal quasiconformal in $\mathfrak{F}\{\alpha\}$. However, we cannot yet settle this problem.

In connection with the theorem 2 we shall give another quite different method of proof in a slightly restricted case. Let D be a simply-connected Jordan domain in the z-plane and Δ a likewise convex domain in the w-plane. Let T be a sense-preserving topological mapping of D onto Δ which satisfies a functional relation $p_T \bar{q}_T = F(z)$ almost everywhere in D with a certain regular function $F(z)$ in D. Let $I_B(S)$ and $A_B(S)$ be the Dirichlet and area integrals over a domain B, respectively, that is,

$$I_B(S)=\iint_B(|p_S|^2+|q_S|^2)dxdy, \quad A_B(S)=\iint_B(|p_S|^2-|q_S|^2)dxdy.$$

We shall prove the following lemma.

LEMMA 2. *Let B be a subdomain of D such that $\bar{B}\subset D$ and S a continuous mapping differentiable almost everywhere in D which coincides with T in $D-B$. If $I_B(T)<\infty$, then $I_B(T)<I_B(S)$, unless $S=T$.*

Proof. We have already given a proof of this lemma when both B and $T(B)$ are rectangles and T is an affine mapping. However, for completeness, we shall give here a full proof of this lemma. Suppose that $I_B(S)\leqq I_B(T)$ occurs. S_t defined by $tS+(1-t)T$ has its image $S_t(B)$ of B in $\varDelta$. The Dirichlet integral of $S_t(B)$ is computed as follows:

$$\begin{aligned} I_B(S_t)&=\iint_B(|p_t|^2+|q_t|^2)dxdy \\ &=t^2I_B(S)+(1-t)^2I_B(T)+2t(1-t)\iint_B \operatorname{Re}(p_T\bar{p}_S+q_T\bar{q}_S)dxdy. \end{aligned}$$

Hence we have

$$\begin{aligned} \lim_{t\to 0}\frac{I_B(S_t)-I_B(T)}{t}&=-2I_B(T)+2\operatorname{Re}\iint_B(p_T\bar{p}_S+q_T\bar{q}_S)dxdy \\ &\leqq -2I_B(T)+2\sqrt{I_B(T)I_B(S)}\leqq 0. \end{aligned}$$

Equality sign occurs if and only if $S=T$.

Let $H_\varepsilon(w)=w+\varepsilon h_1(w)$ with $h_1(w)=0$ on $D-B$, then $H_\varepsilon T=T(z)+\varepsilon h_1(T(z))$ $=T(z+\varepsilon h(z))$. Since $dw_S(z)=p_Sdz+q_Sd\bar{z}$, we have

$$dw_S(z)\overline{dw_S(z)}=(|p_S|^2+|q_S|^2)dzd\bar{z}+p_S\bar{q}_Sdz^2+\bar{p}_Sq_Sd\bar{z}^2$$

and we shall put the right-hand member of the above identity

$$a_Sdz^2+\bar{a}_Sd\bar{z}^2+b_Sdzd\bar{z}.$$

For $H_\varepsilon T$, we have

$$\begin{aligned} dw_{H_\varepsilon T}(z)\,dw_{H_\varepsilon T}(z)&=a_{H_\varepsilon T}(z)\,dz_{H_\varepsilon}{}^2+\bar{a}_{H_\varepsilon T}(z)\,d\bar{z}_{H_\varepsilon}{}^2+b_{H_\varepsilon T}dz_{H_\varepsilon}d\bar{z}_{H_\varepsilon} \\ &=a_T(z+\varepsilon h(z))\,d(z+\varepsilon h(z))^2+\bar{a}_T(z+\varepsilon h(z))\,d\overline{(z+\varepsilon h(z))}^2 \\ &\quad+b_T(z+\varepsilon h(z))\,d(z+\varepsilon h(z))\,d\overline{(z+\varepsilon h(z))} \\ &=2\operatorname{Re}\left[a_T(z+\varepsilon h(z))\{(1+\varepsilon h_z)^2dz^2+\varepsilon^2h_{\bar{z}}{}^2\}+2a_T(z+\varepsilon h)(1+\varepsilon h_z)\,\varepsilon h_{\bar{z}}dzd\bar{z}\right] \\ &\quad+b_T(z+\varepsilon h(z))\,d(z+\varepsilon h(z))\,d\overline{(z+\varepsilon h(z))}, \end{aligned}$$

where $d(z+\varepsilon h(z))=dz(1+\varepsilon h_z)+\varepsilon h_{\bar{z}}d\bar{z}$. Since the Dirichlet differential of $dw_Td\bar{w}_T$ is equal to

$$\frac{1}{2}dw_T\wedge d\bar{w}_T=\frac{i}{2}b_Tdz\wedge d\bar{z}=b_Tdxdy,$$

that of $dw_{H_\varepsilon T}(z)\overline{dw_{H_\varepsilon T}(z)}$ is equal to

$$\frac{i}{2}b_T(z+\varepsilon h)d(z+\varepsilon h)\wedge \overline{d(z+\varepsilon h)}+2i(\operatorname{Re} a_T(z+\varepsilon h)(1+\varepsilon h_z)\varepsilon h_{\bar{z}})\,dz\wedge d\bar{z},$$

and therefore we have

$$\begin{aligned} I_B(H_\varepsilon T) &= \frac{i}{2}\iint_B b_T(z+\varepsilon h)\,d(z+\varepsilon h)\wedge \overline{d(z+\varepsilon h)} \\ &\quad + 2i\iint_B (\operatorname{Re} a_T(z+\varepsilon h)(1+\varepsilon h_z)\varepsilon h_{\bar{z}})\,dz\wedge d\bar{z} \\ &= I_B(T) + 4\operatorname{Re}\iint_B a_T(z+\varepsilon h)(1+\varepsilon h_z)\,\varepsilon h_{\bar{z}}\,dxdy. \end{aligned}$$

Since $a_T(z)=F(z)$, we have

$$\begin{aligned} \delta_\varepsilon I_B(T) &\equiv I_B(H_\varepsilon T) - I_B(T) \\ &= 4\operatorname{Re}\varepsilon\iint_B F(z)\,h_{\bar{z}}(z)\,dxdy + O(\varepsilon). \end{aligned}$$

On the other hand, by integration by parts we have

$$\iint_B F(z)h_{\bar{z}}(z)\,dxdy = -\iint_B h(z)F_{\bar{z}}(z)\,dxdy + \frac{i}{2}\int_{\partial B} h(z)F(z)\,dz,$$

from which we see that the left-hand member is equal to 0, since $F(z)$ is regular in D and $h(z)$ vanishes identically on $D-B$, especially on ∂B. Thus we have

$$\delta_\varepsilon I_B(T) = O(\varepsilon),$$

which shows that

$$\lim_{\varepsilon\to 0}\frac{\delta_\varepsilon I_B(T)}{\varepsilon} = 0.$$

If t is sufficiently small, then S_t can be obtained from T by a suitable infinitesimal deformation H_ε. Hence we arrive at a contradiction.

THEOREM 2′. *If S is a homeomorphism and T further satisfies a functional relation $|q_T|/|p_T|=k$, $0<k<1$ almost everywhere in D besides the assumptions in lemma 2, then the extremality together with the uniqueness of T holds, that is,*

$$K_S > K_T = \frac{1+k}{1-k},$$

unless $S\equiv T$.

Proof. If $K_S \leqq K_T$ holds, then the excentricity of mapping S is not greater than that of mapping T almost everywhere in B, that is,

$$\frac{|q_S|}{|p_S|} \leqq k = \frac{|q_T|}{|p_T|}.$$

Thus we have $I_B(T)/A_B(T) \geqq I_B(S)/A_B(S)$. On the other hand, by lemma 2, we have $I_B(S)/A_B(S) \geqq I_B(T)/A_B(T)$, which leads to $I_B(S) = I_B(T)$. This implies that $S \equiv T$, which contradicts our assumption.

We should remark that theorem 2′ is a simple corollary of theorem 2. Indeed, a pair of our assumptions $p_T\bar{q}_T = F$ and $|q_T|/|p_T| = k$ is equivalent to a pair $q_T/p_T = k\bar{F}/|F|$ and $|F| = k|p_T|^2$. Here we shall introduce a new local variable z^* by

$$z^* = \int \sqrt{F}\, dz,$$

then T^* induced from T by the change of variable $z \to z^*$ satisfies $p^*_{T^*}\bar{q}^*_{T^*} = 1$ and $q^*_{T^*}/p^*_{T^*} = k$. If we put $w = u + iv$ and $z^* = x + iy$, then there hold

$$u_x{}^2 - v_y{}^2 - v_x{}^2 + u_y{}^2 = 4, \qquad u_x v_x + u_y v_y = 0$$

and

$$u_x - v_y = k(u_x + v_y), \qquad v_x + u_y = k(v_x - u_y).$$

Let $\alpha(x, y)$ be $u_x/v_y = -u_y/v_x$. If $v_y = 0$ for a set E of positive measure, then $u_x = 0$ on E and hence $|p_T|^2 - |q_T|^2 = 0$ on E, which is absurd. Thus $k = (\alpha - 1)/(\alpha + 1)$ holds almost everywhere in D, and hence α is a constant almost everywhere in D. Let $(1-k)/(1+k)u = U$ and $v = V$, then the Cauchy-Riemann equations $U_x = V_y$, $U_y = -V_x$ hold almost everywhere in D. Thus $U + iV = (1-k)u/(1+k) + iv$ becomes an analytic function of z^* under a suitable modification of values on a point-set of measure zero. Putting this analytic function $g(z^*)$, we have

$$w - k\bar{w} = (1+k)g(z^*),$$

and hence we see that

$$(1-k^2)w = (1+k)g(z^*) + k(1+k)\overline{g(z^*)}.$$

Let z^{**} be $g(z^*)$, then we have

$$(1-k^2)w = (1+k)(z^{**} + k\overline{z^{**}}).$$

Thus T^{**} induced from T^* by the change of variable $z^* \to z^{**}$ is really an affine mapping with real positive ratio $B/A = k$, where we put $T = Az^{**} + B\overline{z^{**}}$. Since $I_B(T) = A_B(T)(1+k^2)/(1-k^2)$ is of finite value and the Jacobian of the change of variable $z \to z^{**}$ is bounded in B, $I_{B^*}(T^{**})$ is of finite value. Therefore theorem 2′ can be derived from theorem 2.

THEOREM 4. *Let Δ be a convex domain with infinite area. If a quasiconformal mapping T satisfies two functional relations $p_T\bar{q}_T = F(z)$ and $|q_T|/|p_T| = k$ almost everywhere in D for a suitable function $F(z)$ regular in D and a constant k $(0 < k < 1)$. Then, if $S \in \mathfrak{F}(T)$ we have*

$$K_S \geqq K_T = \frac{1+k}{1-k}.$$

Proof. After we map D onto $|z| < 1$ and make use of the approximation theorem as in theorem 3 and return to D, we have

$$K_{S_r} > K_T$$

unless $S_r \equiv T$ or equivalently $S \equiv T$ by theorem 2′, since the assumption in theorem 2′ $I_B(T) < \infty$ surely holds for an image domain B of $|z| \leqq r$ for any r $(0 < r < 1)$. Therefore we have

$$K_S = \lim_{r\to 1} K_{S_r} \geqq K_T.$$

Theorems 2′ and 4 can be extended to a little more general form.

THEOREM 2″. *Let T satisfy two functional relations*

$$\frac{q_T}{p_T} = k\frac{\bar{f}}{|f|} \qquad \textit{and} \qquad \frac{\bar{f}}{p_T\bar{q}_T} = \lambda(z)$$

with $\lambda^*(w) \equiv \lambda^*(T(z)) = |F'(w)|^2$ evaluated $\lambda(z)$ at a corresponding point w of z by T, where $F(w)$ is a function regular in Δ and mapping Δ onto a convex domain Δ_{w^*}. If

$$I_D(T^*) = \iint_D (|p_{T^*}|^2 + |q_{T^*}|^2)\,dxdy = \iint_D |F'(w)|^2(|p_T|^2 + |q_T|^2)\,dxdy$$

is of finite value, then for any $S \in \mathfrak{F}(T)$

$$K_S > \frac{1+k}{1-k},$$

unless $S \equiv T$.

If $I_D(T^*) = \infty$, then for any $S \in \mathfrak{F}(T)$

$$K_S \geqq \frac{1+k}{1-k}.$$

Proof. Let w^* be a new variable defined by

$$w^* = \int F'(w)\,dw = F(w) + \text{const.},$$

then $T^*(z) \equiv F(T(z)) + \text{const.}$ satisfies two functional equations

$$\frac{q_{T^*}}{p_{T^*}} = \frac{q_T}{p_T} = k\frac{\bar{f}}{|f|}$$

and

$$p_{T^*}\bar{q}_{T^*} = |F'(w)|^2 p_T q_T = \frac{f(z)}{\lambda(z)}|F'(w)|^2 = f(z).$$

Thus we have our theorem.

We shall list here some cases to which our theorems are applicable.

1) Let D and Δ be the rectangles, then the extremal quasiconformal mapping T is the Grötzsch's affine mapping.

2) Let D and Δ be the unit discs. A subfamily of $\mathfrak{F}(I)$ such that $S(0) = a$ is our subject. This case was already considered by Teichmüller and he reduced this to a case where D and Δ are the ellipses and the extremal

mapping is affine. His reduction process consists in the construction of two sheeted coverings. Teichmüller's final result has some important significances. In this case we can prove directly, without constructing two sheeted covering surfaces, by making use of Dirichlet and area integrals.

3) Let D and $\varDelta$ be two strip domains $\{-\infty < x < \infty,\ |y| < 1/2\}$ and $\{-\infty < \operatorname{Re} w < \infty,\ |\operatorname{Im} w| < 1/2\}$ and T an affine mapping $w = Kx + iy$, then, in $\mathfrak{F}(T)$, T is a unique extremal mapping. This result has been stated recently by Strebel. By our theorem 4 we can only state that T is an extremal mapping in $\mathfrak{F}(T)$. To prove its uniqueness, it is necessary to proceed precisely as in Strebel's proof.

References

[1] AHLFORS, L., On quasiconformal mappings. Journ. Anal. Math. **3** (1953/4), 1–58.

[2] GERSTENHABER, M., and H. RAUCH, On extremal quasi-conformal mappings, I. Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 808–812.

[3] MORI, A., On quasi-conformality and pseudo-analyticity. Sûgaku **7** (1955), 75–89. (Japanese)

[4] MORI, A., On quasi-conformality and pseudo-analyticity. Trans. Amer. Math. Soc. **84** (1957), 56–77.

[5] MORI, A., On an absolute constant in the theory of quasi-conformal mappings. Jap. Journ. Math. 8 (1956), 156–166.

[6] OZAWA, M., On Grötzsch's extremal affine mapping. Kōdai Math. Sem. Rep. 8 (1956), 112–114.

[7] STREBEL, K., Eine Abschätzung der Länge gewisser Kurven bei quasikonformer Abbildung. Ann. Acad. Sci. Fenn. **243** (1957) 10 pp.

[8] TEICHMÜLLER, O., Extremale quasikonforme Abbildungen und quadratische Differentiale. Preuss. Acad. Math.-naturw. Kl. **22** (1940), 1–197.

[9] TEICHMÜLLER, O., Bestimmung der extremalen quasikonformen Abbildungen bei geschlossenen orientierten Riemannschen Flächen. Preuss. Acad. Math.-naturw. Kl. **4** (1943), 1–42.

[10] TEICHMÜLLER, O., Ein Verschiebungssatz der quasikonformen Abbildung. Deut. Math. **7** (1944), 336–343.

DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.