ON COEFFICIENT-REGIONS OF LAURENT SERIES WITH POSITIVE REAL PART

BY HAN NISHIMIYA

1. Introduction.

Let $\mathfrak{R}_0=\{\Phi(z)\}$ be the class of analytic functions which are regular and of positive real part in the unit circle $|z|<1$ and normalized by $\Phi(0)=1$. It is well known that Carathéodory [1, 2] has established a result on the variability-region of Taylor coefficients for $\Phi(z)\in\mathfrak{R}_0$; cf. also Rogosinski [6].

We now consider, besides $\mathfrak{R}_0$, the class $\mathfrak{R}_q=\{\Phi(z)\}$ consisting of analytic functions which are single-valued, regular and of positive real part in an annulus $(0<)\,q<|z|<1$ and normalized by the conditions

$$\Re\Phi(z)=1 \text{ along } |z|=q \quad\text{and}\quad \frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi(qe^{i\theta})\,d\theta=1.$$

Let the Laurent expansion of a function $\Phi(z)\in\mathfrak{R}_q$ be

$$\Phi(z)=1+\sum_{\nu=-\infty}^{\infty}{}' c_\nu z^\nu \qquad (q<|z|<1)$$

where the prime means that the summation extends over all integers except $\nu=0$. Then, for any two positive integers m and n, the point $\mathrm{P}=\mathrm{P}^n_{-m}[\Phi]$ with the coordinates $\{c_\nu;\ -m\leqq\nu\leqq n,\ \nu\neq0\}$ in the complex $(n+m)$-dimensional space is called the $(-m,\ n)$*th coefficient-point* of $\Phi(z)$. The purpose of the present paper is to determine precisely the *variability-region*, that is, the range of the point-set consisting of all possible points $\mathrm{P}^n_{-m}[\Phi]$ when $\Phi(z)$ extends over the class $\mathfrak{R}_q$. The class $\mathfrak{R}_0$ is, of course, regarded as a limiting case where the interior boundary component of the annulus degenerates to a single point, i.e. the origin.

On the other hand, if the first normalization for $\mathfrak{R}_q$ that $|z|=q$ corresponds to a segment parallel to the imaginary axis is rejected, there occurs an extended class $\hat{\mathfrak{R}}_q$ which includes $\mathfrak{R}_q$ as a subclass. Namely, let $\hat{\mathfrak{R}}_q=\{\Phi(z)\}$ denote the class of analytic functions which are single-valued, regular and of positive real part in the annulus $(0<)\,q<|z|<1$ and normalized by the condition

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi(re^{i\theta})\,d\theta=1 \qquad (q<r<1).$$

Corresponding to the case of $\mathfrak{R}_q$, we shall consider also an analogous problem with respect to this extended class.

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2. Lemmas on representation formulas.

We enumerate integral representations of Stieltjes type valid for any function of the respective classes under consideration. Though they are consequences of a more general representation formula previously given by Komatu [3] and, in particular, the representation for $\mathfrak{R}_q$ has been re-formulated by the same author [4], we first state it below as lemma 1 for the sake of convenience, since it will play important roles in subsequent discussions.

Let $\Phi^*(z)$ denote an analytic function defined by

$$\Phi^*(z)=\frac{2}{i}\left(\zeta(i\lg z)-\frac{\eta_1}{\pi}i\lg z\right)=1+\sum_{\nu=-\infty}^{\infty}{}' \frac{2}{1-q^{2\nu}}z^\nu$$

where the elliptic zeta-function concerns Weierstrassian theory with primitive periods

$$2\omega_1=2\pi \qquad \text{and} \qquad 2\omega_3=-2i\lg q.$$

Since $\Phi^*(z)$ maps $q<|z|<1$ onto the right half-plane cut along a rectilinear segment parallel to the imaginary axis, it belongs surely to $\mathfrak{R}_q$. It will play the role of kernel of the integral representation and also, combined with a rotation of the z-plane, the role of extremal function.

LEMMA 1. *For any $\Phi(z)\in\mathfrak{R}_q$ we have an integral representation*

$$\Phi(z)=\int_{-\pi}^{\pi}\Phi^*(ze^{-i\varphi})\,d\rho(\varphi)$$

where $\rho(\varphi)=\rho_\Phi(\varphi)$ is a real-valued function satisfying the conditions

$$d\rho(\varphi)\geqq 0 \qquad \textit{and} \qquad \int_{-\pi}^{\pi}d\rho(\varphi)=1.$$

From lemma 1, by expanding the kernel $\Phi^*(ze^{-i\varphi})$ into the Laurent series, there follows readily an integral representation for the coefficients of $\Phi(z)\in\mathfrak{R}_q$.

LEMMA 2. *For any $\Phi(z)\in\mathfrak{R}_q$ the Laurent coefficients are represented by*

$$c_\nu=\frac{2}{1-q^{2\nu}}\int_{-\pi}^{\pi}e^{-i\nu\varphi}d\rho(\varphi) \qquad (\nu\neq 0)$$

where $\rho(\varphi)$ is the function associated to $\Phi(z)$ in lemma 1.

The representation formula given in lemma 1 for $\mathfrak{R}_q$ has been generalized by Komatu [5] to that for $\hat{\mathfrak{R}}_q$ which may be re-stated as follows.

LEMMA 3. *For any $\Phi(z)\in\hat{\mathfrak{R}}_q$ we have an integral representation*

$$\Phi(z)=\int_{-\pi}^{\pi}\Phi^*(ze^{-i\varphi})\,d\rho(\varphi)+\int_{-\pi}^{\pi}\Psi^*(ze^{-i\varphi})\,d\tau(\varphi)-1$$

where $\Phi^*(z)$ *and* $\Psi^*(z)$ *are analytic functions defined by*

$$\Phi^*(z) = \frac{2}{i}\left(\zeta(i \lg z) - \frac{\eta_1}{\pi} i \lg z\right),$$

$$\Psi^*(z) = \Phi^*\left(\frac{q}{z}\right) = 1 - \frac{2}{i}\left(\zeta_3(i \lg z) - \frac{\eta_1}{\pi} i \lg z\right),$$

and $\rho(\varphi) = \rho_\Phi(\varphi)$ *and* $\tau(\varphi) = \tau_\Phi(\varphi)$ *are real-valued increasing functions both defined for* $-\pi < \varphi \leqq \pi$ *and with the total variation equal to unity.*

By expanding the kernels of the representation given in lemma 3, we get the representation for the Laurent cofficients of $\Phi(z) \in \hat{\mathfrak{R}}_q$.

LEMMA 4. *Let the Laurent expansion of* $\Phi(z) \in \hat{\mathfrak{R}}_q$ *be*

$$\Phi(z) = 1 + \sum_{\nu=-\infty}^{\infty}{}' \hat{c}_\nu z^\nu.$$

Then we have

$$\hat{c}_\nu = \frac{2}{1-q^{2\nu}} \int_{-\pi}^{\pi} e^{-i\nu\varphi} d\rho(\varphi) - \frac{2q^\nu}{1-q^{2\nu}} \int_{-\pi}^{\pi} e^{-i\nu\varphi} d\tau(\varphi) \qquad (\nu \neq 0)$$

where $\rho(\varphi)$ *and* $\tau(\varphi)$ *are the functions associated to* $\Phi(z)$ *in lemma* 3.

3. The class $\mathfrak{R}_q$.

We first observe a point $\mathrm{C}(\varphi)$ with the coordinates

$$\left\{\frac{2}{1-q^{2\nu}} e^{-i\nu\varphi}, \ -m \leqq \nu \leqq n, \ \nu \neq 0\right\},$$

φ being a real parameter. It is obviously the coefficient-point of the function $\Phi^*(ze^{-i\varphi})$. Let $\mathfrak{C}$ denote the locus of $\mathrm{C}(\varphi)$ when φ varies from $-\pi$ to π:

$$\mathfrak{C} = \mathfrak{C}_{-m}^{n} = \{\mathrm{C}(\varphi);\ -\pi < \varphi \leqq \pi\}.$$

Now, we can characterize the set of coefficient-points for $\mathfrak{R}_q$ in terms of $\mathfrak{C}$.

THEOREM 1. *Let* $K = K_{-m}^{n}$ *denote the variability-region consisting of all possible coefficient-points* $\mathrm{P} = \mathrm{P}_{-m}^{n}[\Phi] = \{c_\nu;\ -m \leqq \nu \leqq n,\ \nu \neq 0\}$ *when* $\Phi(z)$ *extends over the class* $\mathfrak{R}_q$. *Let* $\mathfrak{K} = \mathfrak{K}_{-m}^{n}$ *denote the smallest closed convex hull which contains* $\mathfrak{C}$. *Then we have*

$$K = \mathfrak{K}.$$

Any point $\mathrm{P} = \{c_\nu\} \in \mathfrak{K}$ *can be expressed in the form*

$$c_\nu = \frac{2}{1-q^{2\nu}} \sum_{j=1}^{p} \lambda_j e^{-i\nu\varphi_j} \qquad (-m \leqq \nu \leqq n,\ \nu \neq 0);$$

$$\lambda_j > 0\ (j = 1, \cdots, p;\ 0 \leqq p \leqq n+m), \quad \sum_{j=1}^{p} \lambda_j \leqq 1, \quad -\pi < \varphi_1 < \cdots < \varphi_p \leqq \pi.$$

The representation of coefficients in this form is unique. In other words, *under the conditions imposed on* (λ, φ, p), $\mathfrak{K}$ *is covered without repetition by the set of points with the coordinates* $\{c_\nu\} = \{c_\nu(\lambda, \varphi, p)\}$ *when* (λ, φ, p) *extends over the whole range. The boundary* $\mathfrak{D} = \mathfrak{D}^n_{-m}$ *of* $\mathfrak{K}$ *is characterized by* $\sum_{j=1}^{p} \lambda_j = 1$ *and every point of* $\mathfrak{D}$ *is attained only by a function of the form*

$$\Phi(z) = \sum_{j=1}^{p} \lambda_j \Phi^*(ze^{-i\varphi_j}),$$

where $\{\lambda_j\}$, $\{\varphi_j\}$ *and* p *correspond uniquely to any assigned point of* $\mathfrak{D}$.

Proof. Any point $P = tP_1 + (1-t)P_2$ with $P_1, P_2 \in K$ and $0 \leqq t \leqq 1$ belongs to K, since a function $\Phi = t\Phi_1 + (1-t)\Phi_2$ belongs to $\mathfrak{R}_q$ for any pair $\Phi_1, \Phi_2 \in \mathfrak{R}_q$. Hence, the set K is convex. The set K is closed, because the family $\mathfrak{R}_q$ is compact and normal. Since $C(\varphi)$ is the coefficient-point of $\Phi^*(ze^{-i\varphi}) \in \mathfrak{R}_q$, we have $C(\varphi) \in K$ for any φ, whence follows $\mathfrak{C} \subset K$. Therefore, we conclude that

$$\mathfrak{K} \subset K.$$

On the other hand, the coordinates of the center of gravity consisting of $2(n+m)$ points $C(\mu\pi/(n+m)) \in \mathfrak{K}$ $(\mu = 1, \cdots, 2(n+m))$ are given by

$$\frac{1}{2(n+m)} \cdot \frac{2}{1-q^{2\nu}} \sum_{\mu=1}^{2(n+m)} e^{-i\nu\mu\pi/(n+m)} = 0 \quad (-m \leqq \nu \leqq n,\ \nu \neq 0).$$

Accordingly, $\mathfrak{K}$ contains the origin $O^n_{-m} = \{0, \cdots, 0\}$. Let Π denote the supporting hyperplane of $\mathfrak{K}$ through any point $\{\gamma_\nu\} \in \mathfrak{D}$. Let its equation be

$$\Pi: \qquad \Re \sum_{\nu=-m}^{n}{}' u_\nu(z_\nu - \gamma_\nu) = 0,$$

where the complex constants u_ν are supposed to satisfy the normalizations

$$\sum_{\nu=-m}^{n}{}' |u_\nu|^2 = 1 \qquad \text{and} \qquad \Re \sum_{\nu=-m}^{n}{}' u_\nu \gamma_\nu \geqq 0.$$

Since the left-hand member of the equation of Π is non-positive at the origin $O \in \mathfrak{K}$, we see that $\mathfrak{K}$ belongs to its non-positive region. In particular, for any point $C(\varphi)$ which surely belongs to $\mathfrak{K}$, we have

$$\Re \sum_{\nu=-m}^{n}{}' u_\nu \left(\frac{2}{1-q^{2\nu}} e^{-i\nu\varphi} - \gamma_\nu \right) \leqq 0.$$

Consequently, in view of the representation given in lemma 2, any point $P = \{c_\nu\} \in K$ satisfies

$$\Re \sum_{\nu=-m}^{n}{}' u_\nu(c_\nu - \gamma_\nu) = \int_{-\pi}^{\pi} \Re \sum_{\nu=-m}^{n}{}' u_\nu \left(\frac{2}{1-q^{2\nu}} e^{-i\nu\varphi} - \gamma_\nu \right) d\rho(\varphi) \leqq 0.$$

This shows that P belongs to $\mathfrak{K}$, i.e.

$$K \subset \mathfrak{K}.$$

Thus, it has been verified that K coincides with $\mathfrak{K}$.

We next consider a function defined by

$$S(\varphi;\, u) = \sum_{\nu=-m}^{n}{}' u_\nu \frac{2}{1-q^{2\nu}} e^{-i\nu\varphi},$$

u_ν being the coefficients of Π introduced above, and put

$$M(u) = \operatorname*{Max}_{-\pi<\varphi\leqq\pi} \Re S(\varphi;\, u).$$

Let

$$\varphi_j = \varphi_j(u) \qquad (j=1, \cdots, p;\ 1 \leqq p \leqq n+m)$$

denote the values for which the maximum $M(u)$ is attained. Here the integer p which may depend on $\{\gamma_\nu\}$ does not exceed $n+m$, since $\Re S(\varphi;\, u)$ is a trigonometric polynomial of degree at most $n+m$. By means of the formula given in lemma 2, we get for any point $\{\gamma_\nu\} \in \mathfrak{D}(\subset \mathfrak{K})$ the inequality

$$\Re \sum_{\nu=-m}^{n}{}' u_\nu \gamma_\nu = \Re \sum_{\nu=-m}^{n}{}' u_\nu \frac{2}{1-q^{2\nu}} \int_{-\pi}^{\pi} e^{-i\nu\varphi} d\rho(\varphi)$$
$$= \int_{-\pi}^{\pi} \Re S(\varphi;\, u)\, d\rho(\varphi) \leqq M(u).$$

Now, based on the definition of $M(u)$, we get an inequality

$$M(u) = \operatorname*{Max}_{-\pi<\varphi\leqq\pi} \Re \sum_{\nu=-m}^{n}{}' u_\nu \frac{2}{1-q^{2\nu}} e^{-i\nu\varphi} \leqq \operatorname*{Max}_{\mathrm{P}\in\mathfrak{K}} \Re \sum_{\nu=-m}^{n}{}' u_\nu c_\nu .$$

On the other hand, since $\{\gamma_\nu\}$ lies on the supporting hyperplane Π, we have

$$\Re \sum_{\nu=-m}^{n}{}' u_\nu c_\nu \leqq \Re \sum_{\nu=-m}^{n}{}' u_\nu \gamma_\nu .$$

Thus, we obtain the relation

$$M(u) = \operatorname*{Max}_{\mathrm{P}\in\mathfrak{K}} \Re \sum_{\nu=-m}^{n}{}' u_\nu c_\nu = \Re \sum_{\nu=-m}^{n}{}' u_\nu \gamma_\nu .$$

By virtue of lemma 2, this equality implies

$$M(u) = \Re \sum_{\nu=-m}^{n}{}' u_\nu \frac{2}{1-q^{2\nu}} \int_{-\pi}^{\pi} e^{-i\nu\varphi} d\rho_\Phi(\varphi) = \int_{-\pi}^{\pi} \Re S(\varphi;\, u)\, d\rho_\Phi(\varphi),$$

i.e.

$$\int_{-\pi}^{\pi} (M(u) - \Re S(\varphi;\, u))\, d\rho_\Phi(\varphi) = 0,$$

where $\rho_\Phi(\varphi)$ is the function associated to $\Phi(z)$ with $\{\gamma_\nu\}$ as its coefficient-point. Now, the difference $M(u) - \Re S(\varphi;\, u)$ becomes never negative and vanishes only for $\varphi = \varphi_j$ $(j=1, \cdots, p)$. On the other hand, $\rho(\varphi)$ is an increasing function with the total variation equal to unity. Hence the last equation holds if and only if $d\rho_\Phi(\varphi) = 0$ except at $\varphi = \varphi_j$ $(j=1, \cdots, p)$ where $\rho_\Phi(\varphi)$ possesses jumps with the heights λ_j, respectively, λ_j being any constants with $\lambda_j \geqq 0$ and $\sum_{j=1}^{p} \lambda_j = 1$. Consequently, the coordinates $\{\gamma_\nu\}$ of a boundary point of $\mathfrak{K}$ and its corresponding function $\Phi(z)$ are given by

$$\gamma_\nu = \frac{2}{1-q^{2\nu}} \sum_{j=1}^{p} \lambda_j e^{-i\nu\varphi_j} \qquad (-m \leqq \nu \leqq n,\ \nu \neq 0),$$

and

$$\Phi(z) = \sum_{j=1}^{p} \lambda_j \Phi^*(ze^{-i\varphi_j}),$$

respectively, where

$$\lambda_j \geqq 0 \ (j=1, \cdots, p;\ 1 \leqq p \leqq n+m), \quad \sum_{j=1}^{p} \lambda_j = 1, \quad -\pi < \varphi_1 < \cdots < \varphi_p \leqq \pi.$$

Conversely, for any (λ, φ, p) satisfying the requirements just above, we have to prove that the point P with the coordinates $\{\gamma_\nu\}$ of the last-mentioned form is a boundary point of $\mathfrak{K}$. Evidently, P belongs to $\mathfrak{K}$ as a coefficient-point of $\Phi(z) \in \mathfrak{N}_q$. Moreover, we shall show that P is contained in a supporting hyrerplane of $\mathfrak{C}$, i.e. that of $\mathfrak{K}$. For this purpose, we fill up the sequence $\{\varphi_j\}_{j=1}^{p}$ to an arbitrary wider sequence $\{\varphi_j\}_{j=1}^{n+m}$ such that it consists of $n+m$ different values. Every hyperplane through $n+m$ points

$$C(\varphi_j) = \left\{ \frac{2}{1-q^{2\nu}} e^{-i\nu\varphi_j};\ -m \leqq \nu \leqq n,\ \nu \neq 0 \right\} \in \mathfrak{C} \quad (j=1, \cdots, n+m)$$

always contains P. By a system of $2(n+m)$ linear equations

$$\Re \sum_{\nu=-m}^{n}{}' u_\nu \frac{2}{1-q^{2\nu}} e^{-i\nu\varphi_j} - d = 0, \qquad \Im \sum_{\nu=-m}^{n}{}' \nu u_\nu \frac{2}{1-q^{2\nu}} e^{-i\nu\varphi_j} = 0$$
$$(j=1, \cdots, n+m),$$

we can determine the ratio of $n+m+1$ values u_ν $(-m \leqq \nu \leqq n,\ \nu \neq 0)$ and $d \geqq 0$. Then the equation

$$\Pi: \qquad \Re \sum_{\nu=-m}^{n}{}' u_\nu z_\nu = d$$

represents a tangent hyperplane of $\mathfrak{C}$ through $n+m$ points $C(\varphi_j)$. On the hand, the trigonometric polynomial defined by

$$T(\varphi) = \Re S(\varphi; u) - d = \Re \sum_{\nu=-m}^{n}{}' u_\nu \frac{2}{1-q^{2\nu}} e^{-i\nu\varphi} - d$$

has at most $2(n+m)$ irreducible zero-points, u_ν and d being the constants determined above. The above system of $2(n+m)$ linear equations assures

$$T(\varphi_j) = 0, \quad \frac{d}{d\varphi} T(\varphi_j) = 0 \qquad (j=1, \cdots, n+m).$$

Hence, every φ_j is an at least double zero of $T(\varphi)$. Thus, since $d \geqq 0$, we have $T(\varphi) \leqq 0$ throughout $-\pi < \varphi \leqq \pi$, that is, Π is a supporting hyperplane of $\mathfrak{C}$ containing P. Now, we have to prove the uniqueness of the representation of $\{\gamma_\nu\} \in \mathfrak{D}$ in the form stated in the theorem. Suppose that a point $\{\gamma_\nu\}$ is expressed in that form by (λ, φ, p) and (μ, ψ, q), respectively. Namely, let the equality

$$\sum_{j=1}^{p} \lambda_j \Phi^*(ze^{-i\varphi_j}) = \sum_{j=1}^{q} \mu_j \Phi^*(ze^{-i\psi_j})$$

holds identically with respect to z. Comparing the poles of both sides of the equation together with their residues, we readily conclude that

$$(\lambda, \varphi, p) \equiv (\mu, \psi, q).$$

We finally determine the representation of any interior point $\mathrm{P} = \{c_\nu\} \in \mathfrak{K}$. In case $\mathrm{P} = \mathrm{O}$, i.e. $c_\nu = 0$ $(\Phi(z) \equiv 1)$, we have $p = 0$ which corresponds to the vacuous sum. In case $\mathrm{P} \neq \mathrm{O}$, the ray OP issuing from O intersects the boundary $\mathfrak{D}$ of $\mathfrak{K}$ at a single point, $\mathrm{Q} \in \mathfrak{D}$ say. According to the representation of boundary point, let $\mathrm{Q} = \{\gamma_\nu\}$ be expressed by

$$\gamma_\nu = \frac{2}{1 - q^{2\nu}} \sum_{j=1}^{p} \mu_j e^{-i\nu\varphi_j}; \qquad \mu_j > 0, \ \sum_{j=1}^{p} \mu_j = 1.$$

Consequently, putting

$$\frac{\overline{\mathrm{PQ}}}{\overline{\mathrm{OQ}}} = \tau \qquad (0 < \tau < 1),$$

we see that P is expressed by

$$c_\nu = \frac{2}{1 - q^{2\nu}} \sum_{j=1}^{p} \lambda_j e^{-i\nu\varphi_j}; \qquad \lambda_j = (1 - \tau)\mu_j, \ \sum_{j=1}^{p} \lambda_j = (1 - \tau) < 1.$$

The uniqueness of the representation of the interior point P in this form follows from that of the boundary point Q already established. Thus the proof of theorem 1 has been completed.

In theorem 1, we have observed the $n + m$-dimensional coefficient-point of the form

$$\mathrm{P}_{-m}^{n}[\Phi] = \{c_\nu; \nu = -m, \cdots, -1, 1, \cdots, n\}.$$

We could, however, consider more generally the point of the form

$$\mathscr{P}_N[\Phi] = \{c_\nu; \nu = \nu_1, \nu_2, \cdots, \nu_l\}$$

where $N = \{\nu_k\}_{k=1}^{l}$ is any assigned increasing sequence of integers, negative or positive. Let $\mathscr{K} = \mathscr{K}_N$ denote the variability-region consisting of such $\mathscr{P}_N[\Phi]$ for $\Phi(z) \in \mathfrak{R}_q$. Theorem 1 can be generalized by means of readily comprehensible modifications. Here we mention, as an example, a result for the most particular case; cf. [4].

COROLLARY. *The Laurent coefficient of $\Phi(z) \in \mathfrak{R}_q$ are estimated by*

$$|c_n| \leqq \frac{2}{|1 - q^{2n}|} \qquad (n \neq 0).$$

For any assigned α the equation

$$c_n = \frac{2e^{-i\alpha}}{1 - q^{2n}} \qquad (n \neq 0)$$

holds if and only if $\Phi(z)$ is of the form

$$\Phi(z) = \sum_{j=1}^{|n|} \rho_j \Phi^*(z e^{-i(\alpha + 2j\pi)/n}),$$

$$\rho_j \geqq 0 \ (j=1, 2, \cdots, |n|), \qquad \sum_{j=1}^{|n|} \rho_j = 1.$$

Proof. Though this is a particular case of theorem 1, it can be proved directly and very briefly. In fact, by means of the representation stated in lemma 2, we get

$$|c_n| = \left| \frac{2}{1-q^{2n}} \int_{-\pi}^{\pi} e^{-in\varphi} d\rho(\varphi) \right| \leqq \frac{2}{|1-q^{2n}|} \qquad (n \neq 0).$$

In order to show the last part of corollary, we have only to notice that an extremal $\Phi(z)$ is characterized by $d\rho_\Phi(\varphi)=0$ except for values of φ such that

$$n\varphi \equiv \alpha \ (\text{mod } 2\pi), \quad \text{i.e.} \quad \varphi = (\alpha + 2j\pi)/n \qquad (j=1, 2, \cdots, |n|).$$

For later purpose, it will become necessary to consider a class closely related to $\mathfrak{R}_q$. Namely, let $\mathfrak{R}_q' = \{\Psi(z)\}$ be the class of analytic functions $\Psi(z)$ such that $\Psi(q/z) \in \mathfrak{R}_q$. A result on $\mathfrak{R}_q'$ analogous to theorem 1 can be readily obtained. For the purpose, let Laurent expansion of a function $\Psi(z) \in \mathfrak{R}_q'$ be

$$\Psi(z) = 1 + \sum_{\nu=-\infty}^{\infty}{}' c_\nu' z^\nu \qquad (q < |z| < 1).$$

Corresponding to the point $\mathrm{C}(\varphi)$ of $\Phi^*(ze^{-i\varphi}) \in \mathfrak{R}_q$, we observe the coefficient-point of the function $\Psi^*(ze^{-i\varphi}) \in \mathfrak{R}_q'$:

$$\mathrm{C}'(\varphi) \equiv \mathrm{P}_{-m}^{n}[\Psi^*(ze^{-i\varphi})] = \left\{ -\frac{2q^\nu}{1-q^{2\nu}} e^{-i\nu\varphi}; \ -m \leqq \nu \leqq n, \ \nu \neq 0 \right\},$$

φ being a real parameter. Let $\mathfrak{C}'$ denote the locus of $\mathrm{C}'(\varphi)$ when φ varies from $-\pi$ to π:

$$\mathfrak{C}' = \mathfrak{C}'^{n}_{-m} = \{\mathrm{C}'(\varphi); \ -\pi < \varphi \leqq \pi\}.$$

THEOREM 2. *Let $K' = K'^{n}_{-m}$ denote the variability-region consisting of all possible coefficient-points $\mathrm{P} = \mathrm{P}^{n}_{-m}[\Psi] = \{c_\nu'; \ -m \leqq \nu \leqq n, \ \nu \neq 0\}$ when $\Psi(z)$ extends over the class $\mathfrak{R}_q'$. Let $\mathfrak{K}' = \mathfrak{K}'^{n}_{-m}$ denote the smallest closed convex hull which contains $\mathfrak{C}'$. Then, we have*

$$K' = \mathfrak{K}'.$$

Any point $\mathrm{P} = \{c_\nu'\} \in \mathfrak{K}'$ can be expressed in the form

$$c_\nu' = \frac{-2q^\nu}{1-q^{2\nu}} \sum_{j=1}^{p'} \lambda_j' e^{-i\nu\varphi_j'} \qquad (-m \leqq \nu \leqq n, \ \nu \neq 0);$$

$$\lambda_j' > 0 \ (j=1, \cdots, p'; \ 1 \leqq p' \leqq n+m), \quad \sum_{j=1}^{p'} \lambda_j' \leqq 1, \quad -\pi < \varphi_1' < \cdots < \varphi_{p'}' \leqq \pi.$$

The representation of coefficients in this form is unique. The boundary $\mathfrak{D}' = \mathfrak{D}'^{n}_{-m}$ of $\mathfrak{K}'$ is characterized by $\sum_{j=1}^{p'} \lambda_j' = 1$ and every point of $\mathfrak{D}'$ is attained only by a function of the form

$$\Psi(z) = \sum_{j=1}^{p'} \lambda_j' \Psi^*(ze^{-i\varphi_j'}),$$

where $\{\lambda_j'\}$, $\{\varphi_j'\}$ *and* p' *correspond uniquely to any assigned point of* $\mathfrak{D}'$.

Proof. For any function $\Psi(z)\in\mathfrak{R}_q'$ with the Laurent coefficients $\{c_\nu'\}$, the integral representations corresponding to those of $\mathfrak{R}_q$ in lemmas 1 and 2 become

$$\Psi(z)=\int_{-\pi}^{\pi}\Psi^*(ze^{-i\varphi})\,d\tau(\varphi),$$

$$c_\nu'=\frac{-2q^\nu}{1-q^{2\nu}}\int_{-\pi}^{\pi}e^{-i\nu\varphi}d\tau(\varphi) \qquad (\nu\neq 0)$$

where $\tau(\varphi)=\tau_\Psi(\varphi)$ is a real-valued function associated to $\Psi(z)$ satisfying the conditions

$$d\tau(\varphi)\geqq 0 \quad \text{and} \quad \int_{-\pi}^{\pi}d\tau(\varphi)=1.$$

Making use of these representations, we can prove the present theorem quite similarly as theorem 1.

4. The limit case $\mathfrak{R}_0$.

The classical theorem due to Carathéodory may be regarded as a limit case of theorem 1 as $q\to 0$. In fact, let

$$\Phi(z)=1+\sum_{\nu=1}^{\infty}c_\nu z^\nu \qquad (|z|<1)$$

be a function from $\mathfrak{R}_0$ with the coefficient-point

$$\mathrm{P}=\mathrm{P}_1^n[\Phi]=\{c_\nu;\ 1\leqq\nu\leqq n\}.$$

For the class $\mathfrak{R}_0$, the linear function

$$\Phi_0^*(z)\equiv\frac{1+z}{1-z}=1+2\sum_{\nu=1}^{\infty}z^\nu$$

which maps the unit circle onto the right half-plane plays a distinguished role. Let the coefficient-point of $\Phi_0^*(ze^{-i\varphi})$ be defined by

$$\mathrm{C}(\varphi)=\{2e^{-i\nu\varphi};\ 1\leqq\nu\leqq n\},$$

φ being a real parameter, and the curve $\mathfrak{C}$ defined by

$$\mathfrak{C}=\{\mathrm{C}(\varphi);\ -\pi<\varphi\leqq\pi\}.$$

Then Carathéodory's theorem may be re-stated as below. Here we shall give an alternative proof based on analogues for $\mathfrak{R}_0$ of lemmas 1 and 2.

THEOREM 3. *Let* $K=K_1^n$ *denote the variablility-region consisting of all possible coefficient-points* $\mathrm{P}=\mathrm{P}_1^n[\Phi]=\{c_\nu\}$ *when* $\Phi(z)$ *extends over the class* $\mathfrak{R}_0$. *Let* $\mathfrak{K}=\mathfrak{K}_1^n$ *denote the smallest closed convex hull which contains* $\mathfrak{C}$. *Then, we have*

$$K=\mathfrak{K}.$$

Any point $\mathrm{P}=\{c_\nu\}\in\mathfrak{K}$ *can be expressed in the form*

$$c_\nu = 2\sum_{j=1}^{p}\lambda_j e^{-i\nu\varphi_j} \qquad (1\leqq \nu \leqq n);$$

$$\lambda_j>0\ (j=1,\cdots,p;\ 0\leqq p\leqq n),\qquad \sum_{j=1}^{p}\lambda_j\leqq 1,\quad -\pi<\varphi_1<\cdots<\varphi_p\leqq\pi.$$

The representation of coefficients in this form is unique. The boundary $\mathfrak{D}=\mathfrak{D}_1^n$ *of* $\mathfrak{K}$ *is characterized by* $\sum_{j=1}^{p}\lambda_j=1$ *and every point of* $\mathfrak{D}$ *is attained by the rational function of the form*

$$\Phi(z)=\sum_{j=1}^{p}\lambda_j\Phi_0^*(ze^{-i\varphi_j})\equiv\sum_{j=1}^{p}\lambda_j\frac{e^{i\varphi_j}+z}{e^{i\varphi_j}-z},$$

where $\{\lambda_j\}$, $\{\varphi_j\}$ *and* p *correspond uniquely to any assigned point of* $\mathfrak{D}$.

Proof. Lemma 1 for $\mathfrak{R}_q$ is to be replaced for $\mathfrak{R}_0$ by a classical representation due to Herglotz. Accordingly, we have only to make use of the linear function $\Phi_0^*(z)$ instead of the kernel $\Phi^*(z)$ in case of $\mathfrak{R}_q$.

5. The class $\hat{\mathfrak{R}}_q$.

We now proceed to consider the extended class $\hat{\mathfrak{R}}_q$. The previous classes $\mathfrak{R}_q$ and $\mathfrak{R}_q'$ are involved in $\hat{\mathfrak{R}}_q$ as subclasses. However, it will be seen that these subclasses play distinguished roles in the following discussions.

THEOREM 4. *Let* $\hat{K}=\hat{K}_{-m}^n$ *denote the variability-region consisting of all possible coefficient-points* $\hat{\mathrm{P}}=\hat{\mathrm{P}}_{-m}^n[\Phi]=\{\hat{c}_\nu;\ -m\leqq\nu\leqq n,\ \nu\neq 0\}$ *when* $\Phi(z)$ *extends over the class* $\hat{\mathfrak{R}}_q$. *Let* $\hat{\mathfrak{K}}=\hat{\mathfrak{K}}_{-m}^n$ *denote the point set consisting of all the points* $\hat{\mathrm{P}}$ *which are of the form* $\hat{\mathrm{P}}=\mathrm{P}+\mathrm{P}'$ *with* $\mathrm{P}\in\mathfrak{K}$ *and* $\mathrm{P}'\in\mathfrak{K}'$. *Then, we have*

$$\hat{K}=\hat{\mathfrak{K}}.$$

Any point $\hat{\mathrm{P}}=\{\hat{c}_\nu\}\in\hat{\mathfrak{K}}$ *can be expressed in the form*

$$\hat{c}_\nu=\frac{2}{1-q^{2\nu}}\sum_{j=1}^{p}\lambda_j e^{-i\nu\varphi_j}-\frac{2q^\nu}{1-q^{2\nu}}\sum_{j=1}^{p'}\lambda_j' e^{-i\nu\varphi_j'}\quad(-m\leqq\nu\leqq n,\ \nu\neq 0);$$

$$\lambda_j>0\ (j=1,\cdots,p;\ 0\leqq p\leqq n+m),\quad \sum_{j=1}^{p}\lambda_j\leqq 1,\qquad -\pi<\varphi_1<\cdots<\varphi_p\leqq\pi,$$

$$\lambda_j'>0\ (j=1,\cdots,p';\ 0\leqq p'\leqq n+m),\quad \sum_{j=1}^{p'}\lambda_j'\leqq 1,\qquad -\pi<\varphi_1'<\cdots<\varphi_{p'}\leqq\pi.$$

The representation of coefficients in this form is unique. The boundary $\hat{\mathfrak{D}}=\hat{\mathfrak{D}}_{-m}^n$ *of* $\hat{\mathfrak{K}}$ *is characterized by* $\sum_{j=1}^{p}\lambda_j=\sum_{j=1}^{p'}\lambda_j'=1$ *and every point of* $\hat{\mathfrak{D}}$ *is attained only by a function of the form*

$$\Phi(z)=\sum_{j=1}^{p}\lambda_j\Phi^*(ze^{-i\varphi_j})+\sum_{j=1}^{p'}\lambda_j'\Psi^*(ze^{-i\varphi_j'})-1,$$

where $\{\lambda_j\}$, $\{\lambda_j'\}$, $\{\varphi_j\}$, $\{\varphi_j'\}$, p *and* p' *correspond uniquely to any assigned* *point of* $\hat{\mathfrak{D}}$.

Proof. Since the sets $\mathfrak{K}$ and $\mathfrak{K}'$ are both closed and convex, the set $\hat{\mathfrak{K}}$ is also closed and convex. The set $\hat{\mathfrak{K}}$ coincides with the smallest closed convex hull which contains the set $\hat{\mathfrak{C}}=\{\hat{\mathrm{C}}\}$ is defined by

$$\hat{\mathfrak{C}}=\{\hat{\mathrm{C}}=\mathrm{C}(\varphi)+\mathrm{C}'(\varphi');\ \mathrm{C}(\varphi)\in\mathfrak{C},\ \mathrm{C}'(\varphi')\in\mathfrak{C}'\}.$$

On the other hand, based on the definition of $\hat{\mathfrak{R}}_q$, the set $\hat{K}$ is closed and convex. Since, for any φ and φ', $\hat{\mathrm{C}}=\mathrm{C}(\varphi)+\mathrm{C}'(\varphi')$ is the coefficient-point of $\Phi(z)=\Phi^*(ze^{-i\varphi})+\Psi^*(ze^{-i\varphi'})-1\in\hat{\mathfrak{R}}_q$ we have $\hat{\mathrm{C}}\in\hat{K}$, whence follows $\hat{\mathfrak{C}}\subset\hat{K}$. Therefore, we conclude that

$$\hat{\mathfrak{K}}\subset\hat{K}.$$

Let $\hat{\Pi}$ denote the supporting hyperplane of $\hat{\mathfrak{K}}$ through a given point $\{\hat{\gamma}_\nu\}\in\hat{\mathfrak{D}}$. Let the equation of $\hat{\Pi}$ be

$$\hat{\Pi}:\qquad \Re\sum_{\nu=-m}^{n}{}' u_\nu(z_\nu-\hat{\gamma}_\nu)=0,$$

where the complex constants u_ν are supposed to satisfy the normalizations

$$\sum_{\nu=-m}^{n}{}' |u_\nu|^2=1 \qquad\text{and}\qquad \Re\sum_{\nu=-m}^{n}{}' u_\nu\hat{\gamma}_\nu\geqq 0.$$

Since the left-hand member of the equation of $\hat{\Pi}$ is non-positive at the origin $\mathrm{O}\in\hat{\mathfrak{K}}$, we see that $\hat{\mathfrak{K}}$ belongs to its non-positive region. For any point $\hat{\mathrm{P}}=\{\hat{c}_\nu\}\in\hat{\mathfrak{K}}$, there exist some points $\mathrm{P}=\{c_\nu\}\in\mathfrak{K}$ and $\mathrm{P}'=\{c_\nu'\}\in\mathfrak{K}'$ such that $\hat{\mathrm{P}}=\mathrm{P}+\mathrm{P}'$. Since $\{\hat{\gamma}_\nu\}$ lies on $\hat{\Pi}$, and P and P′ extend independently over $\mathfrak{K}$ and $\mathfrak{K}'$, respctively, we obtain

$$\Re\sum_{\nu=-m}^{n}{}' u_\nu\hat{\gamma}_\nu=\operatorname*{Max}_{\hat{\mathrm{P}}\in\hat{\mathfrak{K}}}\Re\sum_{\nu=-m}^{n}{}' u_\nu\hat{c}_\nu=\operatorname*{Max}_{\mathrm{P}\in\mathfrak{K}}\Re\sum_{\nu=-m}^{n}{}' u_\nu c_\nu+\operatorname*{Max}_{\mathrm{P}'\in\mathfrak{K}'}\Re\sum_{\nu=-m}^{n}{}' u_\nu c_\nu'.$$

Now, put

$$\operatorname*{Max}_{\mathrm{P}\in\mathfrak{K}}\Re\sum_{\nu=-m}^{n}{}' u_\nu c_\nu=\Re\sum_{\nu=-m}^{n}{}' u_\nu\gamma_\nu,\qquad \operatorname*{Max}_{\mathrm{P}'\in\mathfrak{K}'}\Re\sum_{\nu=-m}^{n}{}' u_\nu c_\nu'=\Re\sum_{\nu=-m}^{n}{}' u_\nu\gamma_\nu'.$$

Since $\mathfrak{K}$ and $\mathfrak{K}'$ are both closed, we get

$$\Re\sum_{\nu=-m}^{n}{}' u_\nu\hat{\gamma}_\nu=\Re\sum_{\nu=-m}^{n}{}' u_\nu\gamma_\nu+\Re\sum_{\nu=-m}^{n}{}' u_\nu\gamma_\nu';\qquad \{\gamma_\nu\}\in\mathfrak{K},\ \{\gamma_\nu'\}\in\mathfrak{K}'.$$

The equations

$$\Pi:\quad \Re\sum_{\nu=-m}^{n}{}' u_\nu(z_\nu-\gamma_\nu)=0,\qquad\qquad \Pi':\quad \Re\sum_{\nu=-m}^{n}{}' u_\nu(z_\nu-\gamma_\nu')=0,$$

represent supporting hyperplanes of $\mathfrak{K}$ and $\mathfrak{K}'$ through the points $\{\gamma_\nu\}$ and $\{\gamma_\nu'\}$, respectively. We see that $\mathfrak{K}$ and $\mathfrak{K}'$ belongs to the non-positive regions of the equations of Π and Π', respectively. In particular, we have

$$\Re\sum_{\nu=-m}^{n}{}' u_\nu\left(\frac{2}{1-q^{2\nu}}e^{-i\nu\varphi}-\gamma_\nu\right)\leqq 0,\qquad \Re\sum_{\nu=-m}^{n}{}' u_\nu\left(\frac{-2q^\nu}{1-q^{2\nu}}e^{-i\nu\varphi}-\gamma_\nu'\right)\leqq 0.$$

Consequently, in view of the representation given in lemma 4, any point $\hat{\mathrm{P}}=\{\hat{c}_\nu\}\in\hat{K}$ satisfies

$$\Re\sum_{\nu=-m}^{n}{}' u_\nu(\hat{c}_\nu-\hat{\gamma}_\nu)=\Re\sum_{\nu=-m}^{n}{}' u_\nu(\hat{c}_\nu-(\gamma_\nu+\gamma_\nu'))$$

$$=\int_{-\pi}^{\pi}\Re\sum_{\nu=-m}^{n}{}' u_\nu\left(\frac{2}{1-q^{2\nu}}e^{-i\nu\varphi}-\gamma_\nu\right)d\rho(\varphi)+\int_{-\pi}^{\pi}\Re\sum_{\nu=-m}^{n}{}' u_\nu\left(\frac{-2q^\nu}{1-q^{2\nu}}e^{-i\nu\varphi}-\gamma_\nu'\right)d\tau(\varphi)\leqq 0.$$

This shows that any point $\hat{\mathrm{P}}\in\hat{K}$ belongs to $\hat{\mathfrak{K}}$, i.e.

$$\hat{K}\subset\hat{\mathfrak{K}}.$$

Thus, it has been verified that $\hat{K}$ coincides with $\hat{\mathfrak{K}}$.

Now, for any given point $\{\hat{\gamma}_\nu\}\in\hat{\mathfrak{D}}$, we consider the functions defined by

$$S(\varphi;\,u)=\sum_{\nu=-m}^{n}{}' u_\nu\frac{2}{1-q^{2\nu}}e^{-i\nu\varphi},\qquad T(\varphi;\,u)=\sum_{\nu=-m}^{n}{}' u_\nu\frac{-2q^\nu}{1-q^{2\nu}}e^{-i\nu\varphi},$$

u_ν being the coefficients of the equation of the supporting hyperplane $\hat{\Pi}$ at $\{\hat{\gamma}_\nu\}$ introduced above, and put

$$M(u)=\operatorname*{Max}_{-\pi<\varphi\leqq\pi}\Re S(\varphi;\,u),\qquad N(u)=\operatorname*{Max}_{-\pi<\varphi\leqq\pi}\Re T(\varphi;\,u).$$

Let further

$$\varphi_j=\varphi_j(u)\qquad (j=1,\cdots,p;\ 1\leqq p\leqq n+m),$$
$$\varphi_j'=\varphi_j'(u)\quad (j=1,\cdots,p';\ 1\leqq p'\leqq n+m)$$

denote the values for which the maxima $M(u)$ and $N(u)$ are attained, respectively; both integers p and p' not exceeding $n+m$ may depend on $\{\hat{\gamma}_\nu\}$. By means of the formula given in lemma 4, we get

$$M(u)+N(u)=\operatorname*{Max}_{\mathrm{P}\in\mathfrak{K}}\Re\sum_{\nu=-m}^{n}{}' u_\nu c_\nu+\operatorname*{Max}_{\mathrm{P}'\in\mathfrak{K}'}\Re\sum_{\nu=-m}^{n}{}' u_\nu c_\nu'=\Re\sum_{\nu=-m}^{n}{}' u_\nu\hat{\gamma}_\nu$$

$$=\int_{-\pi}^{\pi}\Re\sum_{\nu=-m}^{n}{}' u_\nu\frac{2}{1-q^{2\nu}}e^{-i\nu\varphi}d\rho_\Phi(\varphi)+\int_{-\pi}^{\pi}\Re\sum_{\nu=-m}^{n}{}' u_\nu\frac{-2q^\nu}{1-q^{2\nu}}e^{-i\nu\varphi}d\tau_\Phi(\varphi)$$

$$=\int_{-\pi}^{\pi}\Re S(\varphi;\,u)d\rho_\Phi(\varphi)+\int_{-\pi}^{\pi}\Re T(\varphi;\,u)\,d\tau_\Phi(\varphi),$$

i.e.

$$\int_{-\pi}^{\pi}(M(u)-\Re S(\varphi;\,u))\,d\rho_\Phi(\varphi)+\int_{-\pi}^{\pi}(N(u)-\Re T(\varphi;\,u))\,d\tau_\Phi(\varphi)=0,$$

where $\rho_\Phi(\varphi)$ and $\tau_\Phi(\varphi)$ are functions associated to $\Phi(z)$ with $\{\hat{\gamma}_\nu\}$ as its coefficient-point. The differences $M(u)-\Re S(\varphi;\,u)$ and $N(u)-\Re T(\varphi;\,u)$ become never negative and vanish only at $\varphi=\varphi_j$ $(j=1,\cdots,p)$ and $\varphi=\varphi_j'$ $(j=1,\cdots,p')$, respectively. On the other hand, $\rho(\varphi)$ and $\tau(\varphi)$ are both increasing functions with the total variation equal to unity. Hence the last equation holds if and only if $d\rho_\Phi(\varphi)=0$ except at $\varphi=\varphi_j$ $(j=1,\cdots,p)$ and $d\tau_\Phi(\varphi)=0$ except at $\varphi=\varphi_j'$ $(j=1,\cdots,p')$ where $\rho_\Phi(\varphi)$ and $\tau_\Phi(\varphi)$ possess the jumps with the heights λ_j and λ_j', respectively, λ_j and λ_j' being any constants with $\lambda_j\geqq 0$, $\lambda_j'\geqq 0$,

$\sum_{j=1}^{p}\lambda_j=1$ and $\sum_{j=1}^{p'}\lambda_j'=1$. Consequently, the coordinates $\{\hat{\gamma}_\nu\}$ of a boundary point of $\hat{\mathfrak{R}}$ and its corresponding function $\Phi(z)$ are expressed by

$$\hat{\gamma}_\nu=\frac{2}{1-q^{2\nu}}\sum_{j=1}^{p}\lambda_j e^{-i\nu\varphi_j}-\frac{2q^\nu}{1-q^{2\nu}}\sum_{j=1}^{p'}\lambda_j' e^{-i\nu\varphi_j'}\quad(-m\leqq\nu\leqq n;\ \nu\neq0),$$

and

$$\Phi(z)=\sum_{j=1}^{p}\lambda_j\Phi^*(ze^{-i\varphi_j})+\sum_{j=1}^{p'}\lambda_j'\Psi^*(ze^{-i\varphi_j'})-1,$$

respectively, where

$$\lambda_j\geqq0\ (j=1,\cdots,p;\ 1\leqq p\leqq n+m),\quad \sum_{j=1}^{p}\lambda_j=1,\quad -\pi<\varphi_1<\cdots<\varphi_p\leqq\pi;$$

$$\lambda_j'\geqq0\ (j=1,\cdots,p';\ 1\leqq p'\leqq n+m),\quad \sum_{j=1}^{p'}\lambda_j'=1,\quad -\pi<\varphi_1'<\cdots<\varphi_{p'}'\leqq\pi.$$

Conversely, as shown in theorem 1, for any (λ, φ, p) satisfying the requirements just above, we can determine a boundary point P of $\mathfrak{R}$ with the coordinates

$$\gamma_\nu=\frac{2}{1-q^{2\nu}}\sum_{j=1}^{p}\lambda_j e^{-i\nu\varphi_j}\qquad(-m\leqq\nu\leqq n,\ \nu\neq0).$$

Let the equation of a supporting hyperplane of $\mathfrak{R}$ through the point P $=\{\gamma_\nu\}$ be

$$\Pi:\qquad \Re\sum_{\nu=-m}^{n}{}' u_\nu(z_\nu-\gamma_\nu)=0;\qquad \sum_{\nu=-m}^{n}{}'|u_\nu|^2=1,\quad \Re\sum_{\nu=-m}^{n}{}' u_\nu\gamma_\nu\geqq0,$$

where the values $\{u_\nu\}$ depend on φ_j $(j=1,\cdots,p)$. We then observe the trigonometric polynomial defined by

$$\Re T(\varphi;u)=\Re\sum_{\nu=-m}^{n}{}' u_\nu\frac{-2q^\nu}{1-q^{2\nu}}e^{-i\nu\varphi},$$

and put

$$N(u)=\operatorname*{Max}_{-\pi<\varphi\leqq\pi}\Re T(\varphi;u).$$

Let the maximum $N(u)$ be attained for

$$\varphi_j'=\varphi_j'(u)\qquad(j=1,\cdots,p';\ 1\leqq p'\leqq n+m).$$

As shown in theorem 2, for these values φ_j' $(j=1,\cdots,p')$ and any $\{\lambda_j'\}$ with $\lambda_j'\geqq0$, $\sum_{j=1}^{p'}\lambda_j'=1$, we can determine a boundary point P$'$ of $\mathfrak{R}'$ with the coordinates

$$\gamma_\nu'=\frac{-2q^\nu}{1-q^{2\nu}}\sum_{j=1}^{p'}\lambda_j' e^{-i\nu\varphi_j'}\qquad(-m\leqq\nu\leqq n,\ \nu\neq0).$$

Further, the equation

$$\Pi':\qquad \Re\sum_{\nu=-m}^{n}{}'(u_\nu-\gamma_\nu')=0,$$

represents the supporting hyperplane of $\mathfrak{R}'$ through the point P$'$ which is parallel to Π. Consequently, we obtain a point

$$\hat{\mathrm{P}}=\mathrm{P}+\mathrm{P}'\in\hat{\mathfrak{R}}$$

with the coordidates $\{\hat{\gamma}_\nu\} = \{\gamma_\nu + \gamma_\nu'\}$ of the form stated in the theorem. It remains to prove that the point $\hat{\mathrm{P}}$ is a boundary point of $\hat{\mathfrak{K}}$. For this purpose, we observe the hyperplane $\hat{\Pi}$ through the point $\hat{\mathrm{P}}$ which is parallel to Π and Π', i.e.

$$\hat{\Pi}: \qquad \Re \sum_{\nu=-m}^{n}{}' u_\nu(z_\nu - \hat{\gamma}_\nu) = 0.$$

Now, in view of lemma 4, for any point $\hat{\mathrm{Q}} = \{\hat{c}_\nu\} \in \hat{\mathfrak{K}}$ we have

$$\Re \sum_{\nu=-m}^{n}{}' u_\nu(\hat{c}_\nu - \hat{\gamma}_\nu) = \Re \sum_{\nu=-m}^{n}{}' u_\nu(\hat{c}_\nu - (\gamma_\nu + \gamma_\nu'))$$

$$= \int_{-\pi}^{\pi} \Re \sum_{\nu=-m}^{n}{}' u_\nu\left(\frac{2}{1-q^{2\nu}} e^{-i\nu\varphi} - \gamma_\nu\right) d\rho(\varphi) + \int_{-\pi}^{\pi} \Re \sum_{\nu=-m}^{n}{}' u_\nu\left(\frac{-2q^\nu}{1-q^{2\nu}} e^{-i\nu\varphi} - \gamma_\nu'\right) d\tau(\varphi)$$

$$= \int_{-\pi}^{\pi} (\Re S(\varphi;\, u) - M(u))\, d\rho(\varphi) + \int_{-\pi}^{\pi} (\Re T(\varphi;\, u) - N(u))\, d\tau(\varphi) \leqq 0.$$

Thus, $\hat{\Pi}$ is a supporting hyperplane of $\hat{\mathfrak{K}}$ and hence $\hat{\mathrm{P}}$ lies on the boundary of $\hat{\mathfrak{K}}$. The uniqueness of the representation of a boundary point $\{\hat{\gamma}_\nu\}$ and of the distinguished representation of an interior point can be verified quite similarly as in the proof of theorem 1. Thus, the proof of the present theorem has been completed.

A remark similar to that mentioned subsequently to theorem 1 is valid here also. In particular, we can state the following corollary corresponding to the most simple case.

COROLLARY. *The Laurent coefficients of $\Phi(z) \in \hat{\mathfrak{R}}_q$ are estimated by*

$$|\hat{c}_n| \leqq \frac{2}{|1-q^n|} \qquad (n \neq 0).$$

For any assigned α, the equation

$$\hat{c}_n = \frac{2e^{-i\alpha}}{1-q^n} \qquad (n \neq 0)$$

holds if and only if $\Phi(z)$ is of the form

$$\Phi(z) = \sum_{j=1}^{|n|} (\rho_j \Phi^*(ze^{-i(\alpha+2j\pi)/n}) + \tau_j \Psi^*(ze^{-i(\alpha+(2j-1)\pi)/n})) - 1;$$

$$\rho_j \geqq 0,\ \tau_j \geqq 0\ (j = 1, 2, \cdots, |n|), \qquad \sum_{j=1}^{|n|} \rho_j = \sum_{j=1}^{|n|} \tau_j = 1.$$

Proof. Based on the representation stated in lemma 4, we get

$$|\hat{c}_n| = \left|\frac{2}{1-q^{2n}} \int_{-\pi}^{\pi} e^{-in\varphi} d\rho(\varphi) - \frac{2q^n}{1-q^{2n}} \int_{-\pi}^{\pi} e^{-in\varphi} d\tau(\varphi)\right|$$

$$\leqq \frac{2}{|1-q^n|} \qquad (n \neq 0).$$

In order to show the last part of the corollary, we have only to notice that an extremal $\Phi(z)$ is characterized by $d\rho(\varphi)=0$ and $d\tau(\varphi)=0$ except for values of φ such that

$$n\varphi \equiv \alpha \quad (\mathrm{mod}\ 2\pi) \qquad \text{i.e.}\quad \varphi=(\alpha+2j\pi)/n$$

and

$$n\varphi \equiv \alpha-\pi \ (\mathrm{mod}\ 2\pi) \qquad \text{i.e.}\quad \varphi=(\alpha+(2j-1)\pi)/n \qquad (j=1, 2, \cdots, |n|),$$

respectively.

Throughout the present paper we have considered the annulus $q<|z|<1$ as the basic domain. However, we could take instead the annulus $\sqrt{q}<|z|<1/\sqrt{q}$ as a basic domain. Then, the functions

$$F(z)=\Phi(\sqrt{q}\,z) \quad \text{and} \quad G(z)=\Psi(\sqrt{q}\,z)$$

would take place of $\Phi(z)$ and $\Psi(z)$, respectively. The relation $\Psi(z)=\Phi(q/z)$ implies that they are connected by $G(z)=F(1/z)$. In particular, the Laurent expansion of the distinguished functions corresponding $\Phi^*(z)$ and $\Psi^*(z)$ then become

$$F^*(z)\equiv \Phi^*(\sqrt{q}\,z)=1+\sum_{\nu=-\infty}^{\infty}{}' \frac{2q^{\nu/2}}{1-q^{2\nu}}z^{\nu} \qquad \left(\sqrt{q}<|z|<\frac{1}{\sqrt{q}}\right)$$

and

$$G^*(z)\equiv \Psi^*(\sqrt{q}\,z)=1+\sum_{\nu=-\infty}^{\infty}{}' \frac{2q^{\nu/2}}{1-q^{2\nu}}z^{-\nu}$$

$$=1+\sum_{\nu=-\infty}^{\infty}{}' \frac{2q^{-\nu/2}}{1-q^{-2\nu}}z^{\nu} \qquad \left(\sqrt{q}<|z|<\frac{1}{\sqrt{q}}\right).$$

Comparison of these expansions with those of $\Phi^*(z)$ and $\Psi^*(z)$ shows formally a rather symmetry character.

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