ON AN EIGENVALUE AND EIGENFUNCTION PROBLEM
OF THE EQUATION \( \Delta u + \lambda u = 0 \)

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First of all, we shall observe a phenomenon on a two-dimensional eigenvalue problem of the equation \( \Delta u + \lambda u = 0 \) about the fixed boundary condition for a special domain whose boundary consists of a circumference of a circle and its centre. Let us denote by \( D \) such a domain, by \( C \) its circular boundary with radius \( R \) and by \( C^* \) its centre which is also a boundary point of \( D \).

Next we take a sequence of annuli

\[
D_1 \subset D_2 \subset \ldots \subset D_n \subset \ldots
\]

exhausting the domain \( D \). The boundary of the annulus \( D_n \) consists of two circular components. Let one of them be \( C \) which is the circular boundary of \( D \), and another be \( C_n \) of radius \( R_n \) where \( \lim_{n \to \infty} R_n = 0 \).

Now consider an eigenvalue problem for the domain \( D_n \):

\[
\begin{align*}
\Delta u + \lambda u &= 0 \quad \text{in} \quad D_n, \\
u &= 0 \quad \text{on} \quad C + C_n, \\
n &= 1, 2, \ldots
\end{align*}
\]

Let the first eigenvalue and the first eigenfunction be \( \lambda_n \) and \( u_n \), respectively. Then we can readily show that the following phenomenon occurs:

When \( n \) tends to \( \infty \), the sequence \( \{u_n\} \) satisfying a suitable normalization converges to the first eigenfunction of the whole circular domain together with its centre, namely \( D + C^* \) and the same is true for the eigenvalue, i.e. \( \lambda_n \) tends to the first eigenvalue of \( D + C^* \).

In fact, let the polar coordinates be denoted by \((r, \theta)\). Then \( u = a_1 J_0(\sqrt{\lambda} r) + a_2 Y_0(\sqrt{\lambda} r) \) is a general solution for \( \Delta u + \lambda u = 0 \) which is independent of \( \theta \), where \( J_0 \) and \( Y_0 \) denote the Bessel functions of the zero-th order, and \( a_1 \) and \( a_2 \) are any constants. As \( u = 0 \) on \( C_n \) and \( C \), so we have the relations

\[
\begin{align*}
a_1 J_0(\sqrt{\lambda} R_n) + a_2 Y_0(\sqrt{\lambda} R_n) &= 0, \\
a_1 J_0(\sqrt{\lambda} R) + a_2 Y_0(\sqrt{\lambda} R) &= 0.
\end{align*}
\]

Since \( J_0(\sqrt{\lambda} R_n) \to 1 \) and \( Y_0(\sqrt{\lambda} R_n) \to -\infty \) for \( R_n \to 0 \), the first equation implies that \( a_2 \) must tend to zero, and hence the limit function of \( u_n \) becomes

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\( a_1 J_0(\sqrt{\lambda} r) \) where \( a_1 \) is to be determined by a certain normalization. From the second equation there follows \( J_0(\sqrt{\lambda} R) = 0 \), so that the limit function satisfies the fixed boundary condition on the circular boundary \( C \).

Since we see that \( J_0(0) = 1 \neq 0 \), the first eigenfunction of the same problem for a circle never vanishes at its centre. Therefore, \( C^* \) can not be regarded as a boundary point of our limit function. But, we may explain this phenomenon in the following way:

Our limit function satisfies the boundary condition except for a single isolated boundary point \( C^* \) (of course, of capacity zero).

Thus this suggests us to investigate the following more general problem:

Let \( C \) be a smooth closed curve, \( D' \) be the bounded domain surrounded by \( C \), and \( C^* \) be a closed set lying entirely in the interior of \( D' \). We now consider the domain \( D \) whose boundary consists of \( C \) and \( C^* \). We take a sequence of domains

\[
D_1 \subset D_2 \subset \ldots \subset D_n \subset \ldots
\]

exhausting \( D \), i.e. a sequence \( \{D_n\} \) such that \( \lim_{n\to\infty} D_n = D \). Let the boundary of the domain \( D_n \) consist of \( C \) and \( C_n \) where \( C_n \) consists of a finite number of smooth curves tending to \( C^* \) as \( n \to \infty \).

Now consider the eigenvalue problem:

\[
\Delta u + \lambda u = 0 \quad \text{in} \quad D_n, \\
u = 0 \quad \text{on} \quad C + C_n.
\]

Denote by \( \lambda_n \) the first eigenvalue and by \( u_n \) the first eigenfunction normalized by

\[
\int_{D_n} u_n^2 d\sigma = 1 \quad \text{and} \quad u_n > 0,
\]

where \( d\sigma \) denotes the area element. Then, what would be the behavior of \( u_n \) and \( \lambda_n \) as \( n \) tends to \( \infty \)? It was Prof. M. Tsuji who has kindly recommended this problem for study.

In the present paper, investigating this problem, we obtain the following result:

**Theorem.** \( \lim_{n\to\infty} u_n = u \) and \( \lim_{n\to\infty} \lambda_n = \rho \) exist and are determined independently of the choice of exhausting sequence. Moreover, the limit function \( u \) and the limit value \( \rho \) satisfy the equation \( \Delta u + \rho u = 0 \) in \( D \) together with the condition \( u = 0 \) on the boundary of \( D \) except for a set of capacity zero, where the exceptional points are identical with those of Green's function for the same domain \( D \).

1) In particular, if \( C^* \) consists of a finite number of smooth curves, then the limit function \( u \) and the limit value \( \rho \) are the first eigenfunction and the first eigenvalue, respectively, for the domain \( D \) itself.
Especially if $C^*$ is of capacity zero, the limit function $v$ and the limit value $\rho$ coincide with the first eigenfunction and the first eigenvalue for the domain $D'$, i.e. $D + C^*$, respectively.

The proof of this theorem will be given in the several steps as follows.

§ 1. Since our eigenvalue is a monotone domain function decreasing in the strict sense, so we have

$$\lambda_n > \lambda_{n+1}$$

for every $n$.

Moreover

$$\lambda_n > \mu;$$

here $\mu$ denotes the first eigenvalue of the domain $D'$. Hence, there exists a limit value of $\lambda_n$ as $n \to \infty$.

Put

$$\lim_{n \to \infty} \lambda_n = \rho,$$

then $\rho \geq \mu$. This limit value $\rho$ is uniquely determined no matter how the sequence of exhausting domains is chosen. In fact, take another sequence of exhausting domains

$$D_1 \subset D_2 \subset \cdots \subset D_n \subset \cdots.$$ Let the boundary of $D_n$ be $C$ and $\overline{C}_n$ where $\overline{C}_n$ consists of a finite number of smooth closed curves. Let corresponding eigenvalues be

$$\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$$

and set

$$\lim_{n \to \infty} \lambda_n = \overline{\rho}.$$ Since $D_n$ tends to $D$, there exists a $D_\infty$ such that $D_b \subset D_\infty$ for a fixed $D_b$, hence $\overline{\lambda}_b > \lambda_\infty$. Therefore $\overline{\rho} \geq \rho$. On the other hand, fixing $D_b$ we get $\rho \geq \overline{\rho}$, just in the same way. Thus $\rho = \overline{\rho}$, so that $\lim_{n \to \infty} \lambda_n = \rho$ is determined independently of the choice of exhausting sequence.

§ 2. In order to investigate the behavior of the function $u_n$, we take in $D$ a domain $A$ bounded by $\Gamma$ and $C_\varepsilon$ where $\Gamma$ consists of a finite number of closed smooth curves which enclose $C^*$ and have the distance $2\varepsilon$ from $C^*$, while $C_\varepsilon$ is a curve with the distance $\varepsilon$ from $C$. Moreover, we take in the domain which is surrounded by $\Gamma$ and includes $C^*$, a finite number of curves $\Gamma'$ with the distance $\varepsilon$ from $\Gamma$. Then $C_\varepsilon$ and $\Gamma'$ bound another domain $A'$. If we take an integer $m$ large enough, then all of the $C_n$ with $n > m$ become to lie outside of the domain $A'$.

Therefore

$$A' \subset D_n$$

for $n \geq m$. 

Now, we will show that the function \( u_n \) is uniformly bounded in \( A \) with respect to \( n \).

To prove it, let us take an arbitrary point \( p \) in \( A \) and a circle \( K \) of radius \( \varepsilon \) about \( p \). Since evidently \( K \) is contained in \( A' \), and also in \( D_n \), we get, by Schwarz's inequality

\[
\int_{K} |u_n| d\sigma \leq \left( \int_{K} |u_n|^2 d\sigma \right)^{1/2} \leq \sqrt{\pi \varepsilon a} = \sqrt{a}
\]

where \( a \) denotes the area of the circle \( K \).

On the other hand, we have an equality

\[
\int_{0}^{2\pi} J_0(\sqrt{\lambda_n r}) u_n(p) \frac{d\theta}{2\pi} = \frac{1}{2\pi} \int_{0}^{\varepsilon} u_n(r) dr,
\]

where \( J_0 \) is the Bessel function of the zero-th order and the integral in the right hand member is taken over the circumference of the circle about \( p \) with radius \( r \). Multiplying both sides of (2) by \( r \) and integrating, we have

\[
\int_{0}^{\varepsilon} J_0(\sqrt{\lambda_n r}) u_n(r) r dr = \frac{1}{2\pi} \int_{0}^{\varepsilon} u_n(r) r dr.
\]

From the beginning, let \( \varepsilon \) be small enough such that \( 0 < \varepsilon < j_0 / \sqrt{\lambda_1} \) where \( j_0 \) is the first positive zero of the function \( J_0 \). From the property of \( J_0 \) we have, for \( \varepsilon > 0 \),

\[
J_0(\sqrt{\lambda_n r}) \geq J_0(\sqrt{\lambda_1 r}) \geq J_0(\sqrt{\lambda_1 \varepsilon}) \equiv k > 0.
\]

Then from (3)

\[
\left| u_n(p) \right| \int_{0}^{\varepsilon} kr dr \leq \frac{1}{2\pi} \int_{K} |u_n| d\sigma,
\]

and therefore

\[
\left| u_n(p) \right| \frac{k \varepsilon}{2} \leq \frac{1}{2\pi} \int_{K} |u_n| d\sigma,
\]

which, together with (1), implies

\[
\left| u_n(p) \right| k \leq \frac{1}{2\pi \varepsilon^2} \int_{K} |u_n| d\sigma < \frac{1}{a} \sqrt{a}.
\]

Hence

\[
\left| u_n(p) \right| < \frac{1}{k \sqrt{a}} \quad \text{for every } n \geq m.
\]

Thus \( \{u_n(p)\} \) is uniformly bounded in \( A \).

\[\text{§ 3.} \quad \{\partial u_n(p)/\partial x\} \text{ and } \{\partial u_n(p)/\partial y\} \text{ are also uniformly bounded in } A. \quad \text{In fact, } \partial u_n/\partial x \text{ also satisfies our differential equation, i.e.}
\]

\[
A \left( \frac{\partial u_n}{\partial x} \right) + \lambda_n \left( \frac{\partial u_n}{\partial x} \right) = 0 \quad \text{in } D
\]
where \((x, y)\) denote the coordinates of \(p\), i.e. \(p(x, y)\).

From the same reasoning as in the preceding paragraph, we have the same kind of inequality for \(n \geq m\), as (4) in § 2,

\[
\left| \frac{\partial u_n}{\partial x} \right| \leq \frac{1}{2\pi} \int_{C} \left| \frac{\partial u_n}{\partial x} \right| d\sigma.
\]

On the other hand, we have the relation

\[
\int_{D_n} \left( \left( \frac{\partial u_n}{\partial x} \right)^2 + \left( \frac{\partial u_n}{\partial y} \right)^2 \right) d\sigma = \lambda_n < \lambda_1,
\]

so that

\[
\left( \int_{C} \left| \frac{\partial u_n}{\partial x} \right| d\sigma \right)^2 \leq \int_{C} \left( \frac{\partial u_n}{\partial x} \right)^2 d\sigma \cdot \int_{C} d\sigma < a \int_{C} \left( \frac{\partial u_n}{\partial x} \right)^2 d\sigma < a \lambda_1.
\]

This implies

\[
(6) \quad \left| \frac{\partial u_n(p)}{\partial x} \right| < \frac{1}{k} \sqrt{\frac{\lambda_1}{a}} \quad \text{for} \quad n \geq m.
\]

Thus \(\partial u_n(p)/\partial x\) is also uniformly bounded in \(A\).

The same is true for \(\partial u_n(p)/\partial y\).

After all, by the theorem of Ascoli–Arzelà, we can select a uniformly convergent subsequence \({u_n'}\) from \({u_n}\). Let us denote its limit function by \(v\), i.e.

\[
\lim_{n \to \infty} u_n' = v \quad \text{in} \quad A.
\]

§ 4. Here we study furthermore about the uniform boundedness of the sequence \({u_n}\) in \(D\). Let \(\bar{D}_n\) be a domain whose boundary consists of \(C_n\) and \(C'\) where \(C_n\) represents the boundary of \(D_n\) as we defined already, while \(C'\) does a closed smooth curve lying in the interior of the domain \(D_n\) and enclosing \(C_n\).

Since \(\bar{D}_n \supseteq D_n\), all of the eigenvalues of the same problem for \(\bar{D}_n\) are greater than the first eigenvalue \(\lambda_n\) for \(D_n\). Therefore the Green’s function \(\Gamma_n(p, q)\) of the equation \(\Delta u + \lambda_n u = 0\) for \(\bar{D}_n\) is uniquely determined; here

\[
\Gamma_n(p, q) = \frac{1}{2\pi} \log \frac{1}{r} + H_n(p, q)
\]

where \(H_n\) is the regular solution of \(\Delta u + \lambda_n u = 0\) for \(\bar{D}_n\).

By Green’s formula

\[
\int_{C' + C_n + s} \left( u_n(q) \frac{\partial \Gamma_n}{\partial v} - \Gamma_n(q) \frac{\partial u_n}{\partial v} \right) ds = - \int_{\bar{D}_n - E} (u_n d\Gamma_n - \Gamma_n du_n) d\sigma(q),
\]

where \(\nu\) denotes the inner normal of the boundary of \(\bar{D}_n - E\), and \(E\) is a small circular domain around \(p\) and \(K\) is the boundary of \(E\). By making the radius of the circle \(E\) tend to zero, we get
\[ u_n(p) = \int_{C_n} u_n \frac{\partial \Gamma_n}{\partial v} ds \quad \text{for} \quad p \in \bar{D}_n, \]

as \( u_n = 0 \) on \( C_n \) and \( \Gamma_n = 0 \) on \( C_n + C' \).

In order to obtain an estimation for \( u_n \) in the domain \( \bar{D}_n \), we introduce an auxiliary harmonic function \( \varphi_n \) such that

\[
\begin{align*}
\Delta \varphi_n &= 0 \quad \text{in} \quad \bar{D}_n, \\
\varphi_n &= u_n \quad \text{on} \quad C', \\
\varphi_n &= 0 \quad \text{on} \quad C_n.
\end{align*}
\]

By Green's formula

\[
\int_{C' + C_n + \epsilon} \left\{ \varphi_n \frac{\partial \Gamma_n}{\partial v} - \Gamma_n \frac{\partial \varphi_n}{\partial v} \right\} ds = -\int_{\bar{D}_n - E} \{ \varphi_n \Delta \Gamma_n - \Gamma_n \Delta \varphi_n \} d\sigma(q).
\]

By making the radius of \( E \) tend to zero and from the boundary condition for \( \varphi_n \) and \( \Gamma_n \), the left hand side of (8) becomes

\[
\int_{C'} \varphi_n \frac{\partial \Gamma_n}{\partial v} ds (= u_n(p)).
\]

As \( \Delta \varphi_n = 0 \), \( \Delta \Gamma_n + \lambda_n \Gamma_n = 0 \) in \( \bar{D}_n - E \), so the right hand side of (8) is equal to \( \lambda_n \int_{\bar{D}_n} \varphi_n \Gamma_n d\sigma(q) \). Therefore by making the radius of \( E \) tend to zero, we get

\[
u_n(p) = \lambda_n \int_{\bar{D}_n} \varphi_n \Gamma_n d\sigma.
\]

From this and the maximum principle for harmonic functions, we can obtain an inequality

\[ |u_n(p)| \leq \lambda_n \max_{q \in C'} |\varphi_n(q)| \int_{\bar{D}_n} \Gamma_n(p, q) d\sigma(q). \]

Next we shall estimate \( \Gamma_n(p, q) \) which is represented as

\[ \Gamma_n(p, q) = -\frac{1}{4} Y_0(\sqrt{\lambda_n r_{pq}}) + B - \gamma_n(p, q), \]

where \( B \) represents the first maximum value of \( Y_0/4 \) and \( \gamma_n(p, q) \) a function which satisfies the conditions

\[ \Delta u + \lambda_n u = 0 \quad \text{in} \quad \bar{D}_n \]

\[ u = B - \frac{1}{4} Y_0(\sqrt{\lambda_n r_{pq}}) \quad \text{on the boundary of} \quad \bar{D}_n. \]

\( \bar{D}_n \subseteq D_n \) assures the unique determination of \( \gamma_n \).

Since \( \gamma_n \) is a superharmonic function, its minimum is attained on the
boundary and hence \( \gamma_n(p, q) \geq 0 \).

Therefore we get

\[
0 \leq \Gamma_n(p, q) \leq B - \frac{1}{4} Y_0(\sqrt{\lambda_n} r_{pq})
\]

which has \(- \log r_{pq}/2\pi\) as the main term. Then we have

\[
\iint_{\Sigma} \Gamma_n d\sigma \leq \iint_{\tilde{D}_n} \left( B - \frac{1}{4} Y_0(\sqrt{\lambda_n} r_{pq}) \right) d\sigma.
\]

Finally we will give an estimation for the right hand member. If we take a circle \( d \) which contains the domain \( D' \) in its interior, then on account of the property of \( Y_0 \), we have

\[
\int_{\tilde{D}_n} \{ B - \frac{1}{4} Y_0(\sqrt{\lambda_n} r_{pq}) \} \, d\sigma \leq \int_{d} \{ B - \frac{1}{4} Y_0(\sqrt{\lambda_1} r_{pq}) \} \, d\sigma \equiv k' \leq \infty
\]

where \( k' \) is independent of \( n \). So we get

\[
u_n(p) \leq \lambda_n k' \max_{q \in \partial'} u_n(q).
\]

Moreover, from § 2, \( C' \) can be taken in \( A \) such that

\[
\max_{q \in \partial'} u_n(q) < M, \quad \text{and} \quad \lambda_1 > \lambda_n.
\]

Thus we obtain an estimation

\[
u_n(p) \leq \lambda_1 k' M \quad \text{in} \quad \tilde{D}_n \quad \text{for any} \quad n,
\]

where the right hand side is independent of \( n \).

Now the domain defined by

\[
\tilde{D} = D_n - \tilde{D}_n
\]

is surrounded by \( C \) and \( C' \) only. Then the uniform boundedness of \( \{ u_n \} \) in \( \tilde{D} \) can be shown just in the same way as the above. Thus, combining the results of § 2 and this paragraph, the uniform boundedness of \( \{ u_n \} \) in \( D \) has been established.

§ 5. For our limit function \( v \) the normalization condition \( \iint_{\tilde{D}} \nu^2 d\sigma = 1 \) holds as shown below. By setting

\[
U_n(p) = \begin{cases} 
u_n(p) & \text{in} \quad D_n, \\ 0 & \text{in} \quad D - D_n \end{cases}
\]
the definition of \( u_n \) in \( D_n \) is extended into the whole domain \( D \).
Because \( U_n \) is uniformly bounded in \( D \) by § 4,

\[
\lim_{n \to \infty} \int_D U_n^2 d\sigma = \int_D \lim_{n \to \infty} U_n^2 d\sigma = \int_D v^2 d\sigma.
\]

But

\[
\lim_{n \to \infty} \int_D U_n^2 d\sigma = \lim_{n \to \infty} \int_{D_n} u_n^2 d\sigma = 1.
\]

Hence

(12) \[
\int_D v^2 d\sigma = 1.
\]

§6. Next we shall show that the limit function \( v \) and the limit value \( \rho \) satisfy the integral equation

(13) \[
u(p) = \lambda \int_D G(p, q) u(q) d\sigma(q)
\]

where \( 2\pi G(p, q) \) denotes the ordinary Green's function for the domain \( D \), and hence

\[
G(p, q) = \frac{1}{2\pi} \log \frac{1}{r} + H(p, q),
\]

where \( H \) denotes a regular harmonic function in \( D \).

In the first step, let \( p \) be a fixed interior point of \( D \), then \( p \in D_n \) for sufficiently large \( n \). It is known that (13) holds true for \( u_n, \lambda_n \) and \( D_n \), namely

\[
u_n(p) = \lambda_n \int_{D_n} G_n(p, q) u_n(q) d\sigma(q);
\]

here \( 2\pi G_n(p, q) \) denotes the ordinary Green's function for the domain \( D_n \).

Now by setting

\[
\Theta_n(p, q) = \begin{cases} G_n(p, q) & \text{in } D_n, \\ 0 & \text{in } D - D_n, \end{cases}
\]

we have

\[
U_n = \lambda_n \int_D \Theta_n(p, q) U_n(q) d\sigma(q),
\]

and

\[
\Theta_n(p, q) U_n(q) \leq M \Theta_n(p, q) \leq MG(p, q),
\]
by using the uniform boundedness of $U_n$ in $D$.

Because $G(p,q)$ is integrable, so by Lebesgue's bounded convergence theorem

$$\lim_{n \to \infty} \int_D \mathcal{G}(p,q) U_n(q) \, d\sigma(q) = \int_D \lim_{n \to \infty} \mathcal{G}(p,q) U_n(q) \, d\sigma(q)$$

$$= \int_D G(p,q) u(q) \, d\sigma(q).$$

Thus we have

$$v(p) = \rho \int_D G(p,q) v(q) \, d\sigma(q).$$

In the second step, it will be seen that the relation still remains to hold even if $p$ tends to a boundary point. What is to be shown is:

$$\lim_{p \to p_0} u(p) = \lambda \lim_{p \to p_0} \int_D G(p,q) u(q) \, d\sigma(q)$$

$$= \lambda \int_D \lim_{p \to p_0} G(p,q) u(q) \, d\sigma(q) = \lambda \int_D G(p_0,q) u(q) \, d\sigma(q)$$

where $p_0$ is a boundary point of $D$ and

$$G(p_0,q) \equiv \lim_{p \to p_0} G(p,q).$$

In fact

$$G(p,q) \leq \text{const} \cdot \log \frac{1}{r}, \quad r = \overline{pq},$$

$$u(q) \leq M \quad \text{in} \quad D,$$

$$G(p,q) u(q) \leq \text{const} \cdot \log \frac{1}{r} \quad \text{in} \quad D.$$

By Lebesgue's bounded convergence theorem, we get the result required.

Thus for any point in $D$ and on the boundary point of $D$,

$$v(p) = \rho \int_D G(p,q) v(q) \, d\sigma(q)$$

(15)

does hold.

§ 7. For any interior point $p$ of the given domain $D$, our limit function and limit value satisfy the equation
\[ \Delta u + \lambda u = 0. \]

In fact, from § 6, for \( v \) and \( \rho \)

\[ v(p) = \rho \int \int_p G(p, q) v(q) d\sigma(q). \]

Let \( p(x, y) \) be an arbitrary interior point of \( D \) and \( K \) a circle about \( p \) small enough to be contained in \( D \). Then

\[ \rho \int \int_p G(p, q) v(q) d\sigma(q) = \rho \int \int_{D-K} G(p, q) v(q) d\sigma(q) + \rho \int \int_K G(p, q) v(q) d\sigma(q). \]

Denote the first and the second integrals in the right hand member by \( I_1 \) and \( I_2 \), respectively. Then

\[ \frac{\partial I_1}{\partial x} = \int_{D-K} \frac{\partial G}{\partial x} v(q) d\sigma(q), \quad \frac{\partial^2 I_1}{\partial x^2} = \int_{D-K} \frac{\partial^2 G}{\partial x^2} v(q) d\sigma(q). \]

The same is valid for \( \frac{\partial^2 I_1}{\partial y^2} \), i.e.

\[ \frac{\partial^2 I_1}{\partial y^2} = \int_{D-K} \frac{\partial^2 G}{\partial y^2} v(q) d\sigma(q). \]

Therefore

\[ \Delta I_1 = \frac{\partial^2 I_1}{\partial x^2} + \frac{\partial^2 I_1}{\partial y^2} = \int_{D-K} \Delta G v(q) d\sigma(q) = 0, \]

as \( \Delta G = 0 \) in \( D - K \).

As for \( I_2 \), it is well known

\[ \Delta \left( \rho \int \int_K G(p, q) v(q) d\sigma(q) \right) = -\rho v(p) \]

so that

\[ \Delta v(p) + \rho v(p) = 0 \quad \text{in} \ D. \]

Moreover, when \( p \) tends to a boundary point, \( v(p) \) tends to zero except for a set of capacity zero. In fact, from § 6

\[ \lim_{p \to p_0} v(p) = \rho \int \int_p G(p, q) v(q) d\sigma(q) \]

where \( \lim_{p \to p_0} G(p, q) = G(p_0, q) \), since we know that \( G(p_0, q) \) becomes zero except for a set of capacity zero.

For the special case where the closed set \( C^* \) is of capacity zero, the limit function \( v \) is identical with the first eigenfunction of \( D' = D + C^* \).
§ 8. What is left to be proved is the uniqueness for the limit function $v$. As in the last part of § 1 where the uniqueness of $\rho$ was proved, taking another sequence of exhausting domains

$$D_1 \subset D_2 \subset \ldots \subset D_n \subset \ldots.$$ 

Let the corresponding first eigenfunctions be

$$\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n, \ldots$$

and $\bar{v}$ be a limit function of it. We shall prove $v = \bar{v}$.

First consider the case where $C^*$ consists of a finite number of smooth closed curves. In this case, because the boundary curve is smooth everywhere, $G(p_0, q) = 0$ for all $p_0$ on the boundary. Therefore from (15), $v$ satisfies the fixed boundary condition. Hence $v$ and $\rho$ are the first eigenfunction and the first eigenvalue of $D$. From the well known property of the first eigenvalue of such a domain, $\rho$ must be simple, and the first eigenfunction must be unique, i.e. $v = \bar{v}$.

The above fact shows us that the first eigenfunction has the continuity relation on the domain provided the boundary of the domain consists of smooth curves.

Now we return to the general case, where the boundary does not need to consist of smooth curves only. Suppose that $v \neq \bar{v}$ in $D$. Then there would be a point $p$ such that

$$|v(p) - \bar{v}(p)| = a > 0$$

and for sufficiently large integers $m, n$ and a small positive number $\varepsilon$,

$$|u_m(p) - v(p)| < \frac{\varepsilon}{2},$$

$$|\bar{u}_n(p) - \bar{v}(p)| < \frac{\varepsilon}{2}.$$

So we would get

$$|u_m(p) - \bar{u}_n(p)| > a - \varepsilon > 0.$$  \hfill (16)

But, on the other hand, we have

$$|u_m(p) - \bar{u}_n(p)| < \eta,$$  \hfill (17)

where $\eta$ can be any small positive number making $m$ and $n$ large enough, by the above mentioned continuity relation between the first eigenfunction and the domain, as the boundaries of $D_m$ and $D_n$ consist of smooth curves. Then (16) contradicts (17).
Therefore \( v = \bar{v} \), which proves that our limit function \( v \) is determined independently of the choice of exhausting sequence. Thus our theorem has been proved.

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