THEOREMS ON SUBHARMONIC FUNCTIONS IN THE UNIT CIRCLE

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1. Let l_φ be a line through $e^{i\theta}$, making an angle φ $(-\pi/2 < \varphi < \pi/2)$ with the inner normal of $|z| = 1$ at $e^{i\theta}$. Then M. Tsuji [1] proved the following theorem.

THEOREM. *Let*

$$w(z) = \int_{|a|<1} \log\left|\frac{1-\bar{a}z}{z-a}\right| d\mu(a),$$

where

$$\Omega(r) = \int_{|a|<r} d\mu(a) = O\left(\frac{1}{(1-r)^{\lambda}}\right), \qquad 0 < \lambda < 1.$$

Then there exists a set E of measure 2π on $|z| = 1$, such that if $e^{i\theta} \in E$, then for almost all ψ,

$$\lim_{z\to e^{i\theta}} w(z) = 0,$$

when $z \to e^{i\theta}$ along $l_\psi(e^{i\theta})$.

Let $u(z)$ be a subharmonic function in $|z| < 1$ such that

$$\int_0^{2\pi} |u(re^{i\theta})|\, d\theta = O(1), \qquad 0 \leqq r < 1,$$

and put

$$L(u, r) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\, d\theta,$$

then $L(u, r)$ is an increasing convex function of $\log r$, and Tsuji proved the following theorem.

THEOREM. *Let $u(z)$ be a subharmonic function in $|z| < 1$, such that*

$$\int_0^{2\pi} |u(re^{i\theta})|\, d\theta = O(1), \quad \frac{d}{dr} L(u, r) = O\left(\frac{1}{(1-r)^{\lambda}}\right), \qquad 0 < \lambda < 1.$$

Then there exists a set E of measure 2π on $|z| = 1$, such that if $e^{i\theta} \in E$, then for almost all ψ,

$$\lim_{z\to e^{i\theta}} u(z) = u(e^{i\theta}) \neq \infty$$

exists, when $z \to e^{i\theta}$ along $l_\psi(e^{i\theta})$.

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In this note we shall prove the following theorems using Hayman's method [2].

THEOREM 1. *Let*

$$w(z) = \int_{|a|<1} \log \left| \frac{1-\bar{a}z}{z-a} \right| d\mu(a),$$

where

$$\Omega(r) = \int_{|a|<r} d\mu(a) = O\left(\frac{1}{(1-r)^{\lambda}}\right), \qquad 0<\lambda<1.$$

Then there exists a set E of measure 2π on $|z|=1$, such that for $e^{i\theta} \in E$, there corresponds a ρ-set $\Delta_{\theta,\varphi_0}$ of finite logarithmic length, such that

$$\lim_{\rho\to 0} w(z) = w(e^{i\theta} - \rho e^{i(\theta-\varphi)}) = 0,$$

uniformly for $|\varphi| \leqq \varphi_0$ as $\rho \to 0$ outside $\Delta_{\theta,\varphi_0}$, where $0<\varphi_0<\pi/2$.

THEOREM 2. *Let $u(z)$ be a subharmonic function in $|z|<1$, such that*

$$\int_0^{2\pi} |u(re^{i\theta})| d\theta = O(1), \qquad \frac{d}{dr} L(u,r) = O\left(\frac{1}{(1-r)^{\lambda}}\right), \qquad 0<\lambda<1.$$

Then there exists a set E of measure 2π on $|z|=1$, such that for $e^{i\theta}\in E$, there corresponds a ρ-set $\Delta_{\theta,\varphi_0}$ of finite logarithmic length, such that

$$\lim_{\rho\to 0} u(z) = u(e^{i\theta} - \rho e^{i(\theta-\varphi)}) = u(e^{i\theta}),$$

uniformly for $|\varphi| \leqq \varphi_0$ as $\rho \to 0$ outside $\Delta_{\theta,\varphi_0}$.

For the proof we use the following Lemma. We put

$$d\sigma(a) = (1-|a|)\, d\mu(a),$$

and let Δ_t be the common part of $|z|<1$ and $|z-e^{i\theta}| \leqq t$, then

LEMMA 1. (TSUJI). *If $\Omega(r) = O(1/(1-r)^{\lambda})$, $0<\lambda<1$, then there exists a set E of measure 2π on $|z|=1$, such that if $e^{i\theta}\in E$, then for some positive t_0,*

$$\nu(t) \equiv \sigma(\Delta_t) = O(t^{1+\delta}), \qquad \textit{where} \quad 0<\delta<1, \quad t\leqq t_0.$$

Proof of this Lemma is contained in the proof of Theorem 3 of Tsuji's paper [1].

2. Estimation of $w(z)$.

We assume that $z=1$ belongs to E and put $1-a=\zeta$, $1-z=\xi$, $|z|=r$, $|1-z|=\rho$, $|1-a|=t$. We suppose that z lies beween $l_{-\varphi_0}$ and l_{φ_0}, and if we denote the complement of Δ_{t_0} with respect to $|z|<1$ by Δ^* and $\Delta_{t_1,t_2} = \Delta_{t_1} - \Delta_{t_2}$ $(t_1>t_2)$, then

$$\begin{aligned} w(z) &= \int_{\Delta^*} \log \left| \frac{1-\bar{a}z}{z-a} \right| \frac{d\sigma(a)}{1-|a|} + \int_{\Delta_{t_0,2\rho}} + \int_{\Delta_{2\rho,\frac{\rho}{2}}} + \int_{\Delta_{\frac{1}{2}\rho}}. \\ &= I_1 + I_2 + I_3 + I_4, \quad \text{say.} \end{aligned}$$

For arbitrary t_0 we have evidently

$$\lim_{z\to 1} I_1 = 0. \tag{1}$$

Since

$$\log\left|\frac{1-\bar a z}{z-a}\right| \leqq 2\frac{(1-|a|)(1-|z|)}{|z-a|^2},$$

we have

$$I_2 \leqq 2\int \frac{1-|z|}{|z-a|^2}d\sigma(a).$$

Since z lies in the domain bounded by l_{φ_0} and $l_{-\varphi_0}$, $1-|z|\leqq \rho$, and if $a\in \Delta_{t_0,2\rho}$, $|z-a|=|\xi-\zeta|\geqq|\zeta|-|\xi|$, we have, putting $\nu_0=[\log(t_0/\rho)]$,

$$I_2 \leqq 2\int_{\Delta t_0,2\rho} \frac{1-|z|}{|z-a|^2}d\sigma(a)$$

$$\leqq \text{const.}\sum_{j=1}^{\nu_0-1}\int_{2^j\rho\leqq|\zeta|\leqq 2^{j+1}\rho}\frac{1}{|z-a|^2}d\sigma(a)$$

$$+\text{const.}\,\rho\int_{2^{\nu_0}\rho\leqq|\zeta|\leqq t_0}\frac{1}{|z-a|^2}d\sigma(a).$$

Since in $2^j\rho\leqq|\zeta|\leqq 2^{j+1}\rho$, $|\xi-\zeta|\geqq|\zeta|-|\xi|\geqq 2^j\rho-\rho\geqq \text{const}\ 2^j\rho$,

$$I_2\leqq \text{const.}\,\rho\sum_{j=1}^{\nu_0-1}\frac{1}{2^{2j}\rho^2}\nu(2^{j+1}\rho)+\text{const.}\,\rho\frac{\nu(t_0)}{2^{2\nu_0}\rho^2}$$

$$\leqq \text{const.}\,\rho\sum_{j=1}^{\nu_0-1}\frac{1}{2^{2j}\rho^2}2^{(j+1)(1+\delta)}\rho^{1+\delta}+\text{const.}\,\rho\frac{t_0^{(1+\delta)}}{2^{2\nu_0}\rho^2}$$

$$\leqq \text{const.}\,\rho^\delta\sum_{j=1}^{\infty}\frac{1}{2^{j(1-\delta)}}\leqq \text{const.}\,\rho^\delta,$$

so that

$$I_2\leqq \text{const.}\,\rho^\delta. \tag{2}$$

In I_4, $|z-a|\geqq|\xi|-|\zeta|\geqq \text{const.}\,\rho$, so that similarly we have

$$I_4\leqq \text{const.}\int_{\Delta_{\frac{1}{2}\rho}}\frac{1-|z|}{|z-a|^2}d\sigma(a)$$

$$\leqq \text{const.}\frac{1}{\rho}\nu\left(\frac{1}{2}\rho\right)\leqq \text{const.}\,\rho^\delta,$$

so that

$$I_4\leqq \text{const.}\,\rho^\delta. \tag{3}$$

3. Estimation of I_3.

Let $\Delta'_{2\rho,\frac{1}{2}\rho}$ be the part of $\Delta_{\frac{1}{2}\rho,\rho}$ which is outside the circle Γ_ξ: $|z-a|\leqq k\rho$, where $k=\min(1/2, \sin|\varphi-\varphi_0|)$ and φ_1 is a constant such that $\varphi_0<\varphi_1<\pi/2$, then Γ_ξ is contained in the common part of $\Delta_{2\rho,\frac{1}{2}\rho}$ and the domain which lies between $l_{-\varphi_1}$, l_{φ_1}. Then

$$I_3 = \int_{\Delta'_{2\rho,\frac{1}{2}\rho}} + \int_{\Gamma_\xi} = I'_3 + I''_3, \qquad \text{say.}$$

For I_3',

$$I'_3 \leqq \text{const.} \int_{\Delta'_{2\rho,\frac{1}{2}\rho}} \frac{1-|z|}{|z-a|^2} d\sigma(a)$$

$$\leqq \text{const.} \int_{\Delta'_{2\rho,\frac{1}{2}\rho}} \frac{\rho}{\rho^2} d\sigma(a)$$

$$\leqq \text{const.} \frac{1}{\rho} \nu(2\rho) \leqq \text{const.}\ \rho^\delta.$$

Hence

$$I'_3 \leqq \text{const.}\ \rho^\delta. \tag{4}$$

Since in Γ_ξ, $1-|a| \geqq \text{const.}\ |1-a| = \text{const.}\ t$, we have

$$I''_3 \leqq \text{const.} \int_{\Gamma_\xi} \log \left| \frac{1-\bar{a}z}{z-a} \right| \frac{d\sigma(a)}{t}.$$

To prove theorem 1 we need further to estimate I_3''. For this purpose we use the following Lemmas, which are similar to Hayman's Lemmas [2].

DEFINITION. Let ε be a fixed number. We shall say that a number $\rho < t_0$ is *normal* (ε), if for $0 < h < \rho/2$ we have

$$\int_{\rho-h\leq|\zeta|\leq\rho+h} \frac{d\sigma(a)}{t} = \int_{\rho-h\leq t\leq\rho+h} \frac{d\nu(t)}{t} < \varepsilon\frac{h}{\rho}.$$

LEMMA 2. *The set of all values $\rho < t_0$, which are not normal (ε), has finite logarithmic length.*

Proof. We put $d\omega(t) = d\nu(t)/t$, then since $\nu(t) = O(t^{1+\delta})$, for $t \leqq t_0$, $\int_{t_0}^1 d\omega(t) < \infty$. Suppose that the Lemma is false for some positive ε, then for any given constant $G > 0$, we can find a closed set F of values ρ not normal (ε), which is contained in the open interval $(0,1)$, and such that

$$\int_F \frac{d\rho}{\rho} > G.$$

For each ρ in F, there exists an open interval $I(\rho - h, \rho + h)$ with $0 < h < \rho/2$, such that

$$\int_{\rho-h<|\zeta|<\rho+h} d\omega(t) \geqq \frac{\varepsilon h}{\rho} > \frac{\varepsilon}{4}\int_{\rho-h}^{\rho+h} \frac{dt}{t}. \tag{5}$$

By the Heine–Borel theorem there exists a finite set $I_1, I_2, \cdots, I_n$ of such intervals covering F. We may assume that none of these intervals is entirely contained in the union of the others. Let I_ν be (ρ'_ν, ρ_ν) where $1/\rho_\nu$ increases with ν. Then if $\mu > \nu$, $\rho'_\mu < \rho'_\nu$ since otherwise I_μ would be contained in $I_{\nu+1}$. Also $\rho_{\nu+2} \leqq \rho'_\nu$, since otherwise $I_{\nu+1}$ would be contained in the union of I_ν and $I_{\nu+2}$. Thus each of the set F_1 of intervals $I_1, I_3, \cdots$, and F_2 of intervals $I_2, I_4, \cdots$ are non-overlapping, and since they together cover F, at least one, F_1 say, has the logarithmic length at least $G/2$. From (5) we have

$$\int_{t\in F_1} d\omega(t) > \frac{\varepsilon}{4}\int_{F_1}\frac{dt}{t} \geqq \frac{G}{8},$$

which is a contradiction.

LEMMA 3. *Let $\rho < t_0$ and ρ be normal* (ε), *then we have*

$$\int_{\Gamma_\xi} \log\left|\frac{1-\bar{a}z}{z-a}\right| \frac{d\rho(a)}{1-|a|} \leqq \varepsilon A,$$

where A is an absolute constant.

Proof. Let C_n be the ring $\rho/2^{n+1} \leqq |\xi - \zeta| \leqq \rho/2^n$. We suppose that $|\xi|$ is normal (ε), so that $\sigma(\xi) = 0$. Then putting $C_n' = C_n \cap \{|z| < 1\}$,

$$I_3'' \leqq \sum_{n=1}^{\infty}\int_{C_n'} \log\left|\frac{1-\bar{a}z}{z-a}\right| \cdot \frac{d\sigma(a)}{t}.$$

Since in C_n'

$$\log\left|\frac{1-\bar{a}z}{z-a}\right| \leqq \log\left|\frac{\bar{\zeta}+\xi-\bar{\zeta}\xi}{\zeta-\xi}\right| \leqq \log\frac{\rho+t+\rho t}{|\zeta-\xi|}$$

$$\leqq \log\frac{4\rho 2^{n+1}}{\rho} = \log 2^{n+3},$$

$$\int_{C_n'}\frac{d\sigma(a)}{t} \leqq \int_{\rho-\rho/2^n<t\leqq\rho+\rho/2^n}\frac{d\nu(t)}{t} \leqq \frac{\varepsilon}{\rho}\cdot\frac{\rho}{2^n} = \varepsilon\frac{1}{2^n},$$

hence

$$I_3'' \leqq \varepsilon\sum_{n=1}^{\infty}\frac{1}{2^n}\log 2^{n+3} < \varepsilon A,$$

so that

$$I_3'' < \varepsilon A. \tag{6}$$

5. Proof of the theorem 1.

Let $\Delta(\varepsilon)$ be the set of all $\rho < t_0$, which are not normal (ε), then by Lemma 2,

$$\int_{\Delta(\varepsilon)}\frac{d\rho}{\rho} < \infty.$$

Hence if $\Delta(\rho, \varepsilon)$ denotes the part of $\Delta(\varepsilon)$ in $(0, \rho)$, we can choose sufficiently small ρ_n such that

$$\int_{\Delta(\rho_n, 1/n)}\frac{d\rho}{\rho} < \frac{1}{2^n}.$$

Let Δ_0 be the union of all the set $\Delta(\rho_n, 1/n)$, then

$$\int_{\Delta_0} \frac{d\rho}{\rho} < \sum_{n=1}^{\infty} \int_{\Delta(\rho_n, 1/n)} \frac{d\rho}{\rho} < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

On the other hand if ρ lies outside Δ_0 and $|\xi| = \rho > \rho_n$, Lemma 3 gives

$$\int_{\Gamma_\xi} \log \left| \frac{1 - \bar{a}z}{z - a} \right| \frac{d\sigma(\zeta)}{1 - |a|} \leqq \frac{A}{n}, \tag{7}$$

so that from (1), (2), (3), and (7) theorem 1 follows.

6. Proof of theorem 2.

Since $\int_0^{2\pi} |u(re^{i\theta})| d\theta = O(1)$, by Littlewood's theorem [3], $u(z)$ can be represented as

$$u(z) = v(z) - w(z),$$

where $v(z)$ is harmonic in $|z| < 1$, such that

$$\int_0^{2\pi} |v(re^{i\theta})| d\theta = O(1), \tag{8}$$

and

$$w(z) = \int_{|a|<1} \log \left| \frac{1 - \bar{a}z}{z - a} \right| d\mu(a),$$

where $d\mu(a)$ is a positive mass distribution in $|z| < 1$, such that

$$\int_{|a|<1} (1 - |a|) d\mu(a) < \infty.$$

By (8), for almost all $e^{i\theta}$,

$$\lim_{z \to e^{i\theta}} v(z) = v(e^{i\theta}) \tag{9}$$

exists, when $z \to e^{i\theta}$ from the inside of any Stolz domain, whose vertex is at $e^{i\theta}$. Since

$$r \frac{d}{dr} L(u, r) = \Omega(r) \equiv \int_{|a|<1} d\mu(a) = O\left(\frac{1}{(1 - r)^{\lambda}}\right), \quad 0 < \lambda < 1,$$

$w(z)$ satisfies the condition of theorem 1, whence theorem 2 follows.

References

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