ON FUNCTIONS OF BOUNDED DIRICHLET INTEGRAL

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Ozawa [1] has derived a perfect criterion in terms of local coefficients in order that a single-valued regular function has an image domain whose area does not exceed π. In the present paper we shall notice that the criterion may be obtained as a particular case of a topological theorem due to Helly [2]. We shall further show that from this point of view it can be extended functions defined on a Riemann surface.

1. Basic notations. Let B be a planar n-ply connected schlicht domain with a boundary Γ consisting of analytic curves Γ_ν ($\nu=1,\dots,n$). For the sake of simplicity, suppose that B contains the origin.

Let z_0 be any assigned point in B and $P(z,z_0)$ be a polynomial with respect to $t \equiv 1/(z-z_0)$:

$$P(z,z_0)=\sum_{m=1}^{N} x_m t^m.$$

Let further α be a real parameter and $f_P(z,z_0;\alpha)$ be a single-valued meromorphic function characterized by the following conditions:

i. $f_P(z,z_0,\alpha) - P(z,z_0)$ is regular in B and vanishes at z_0;

ii. all the images of Γ_ν ($\nu=1,\dots,n$) by $f_P(z,z_0,\alpha)$ are segments with inclination α to the real axis.

Existence of $f_P(z,z_0,\alpha)$ for any given P and α, together with its uniqueness, is well-known; cf. Grunsky [3].

Put

$$F_P(z,z_0;\alpha)=\frac{1}{2}\left(f_P(z,z_0,\alpha)-f_P(z,z_0,\alpha+\tfrac{\pi}{2})\right),$$

then we have

$$F_P(z,z_0,\alpha)=e^{2i\alpha}\sum_{m=1}^{N}\bar{x}_m\,\varphi_m(z,z_0),$$

where

$$\varphi_m(z,z_0)=\frac{1}{2}\left(f_{t^m}(z,z_0,0)-f_{t^m}(z,z_0,\tfrac{\pi}{2})\right).$$

Put further

$$\Phi_m(z,z_0)=\frac{1}{2}\left(f_{t^m}(z,z_0,0)+f_{t^m}(z,z_0,\tfrac{\pi}{2})\right).$$

The local expansions of Φ_m and φ_m ($m \geq 1$) about z_0 are obviously of the forms

$$\Phi_m(z,z_0)=\frac{1}{(z-z_0)^m}+\sum_{\nu=1}^{\infty}B_{m\nu}(z-z_0)^\nu,$$

and

$$\varphi_m(z,z_0)=\sum_{\nu=1}^{\infty}S_{m\nu}(z-z_0)^\nu,$$

respectively. There holds

$$d\varphi_m = d\overline{\Phi}_m \qquad \text{along } \Gamma.$$

Let $L^2(B)$ be a family of functions $\psi(z)$ satisfying the following conditions:

i. $\Psi(z)\equiv\int^z\psi(z)\,dz$ is a single-valued function regular in B;

ii. $\iint_B|\psi(z)|^2\,d\sigma_z<\infty$, $d\sigma_z$ denoting the areal element.

In $L^2(B)$ we define the Dirichlet norm by

$$\|\psi\|_B=D_B(\Psi,\Psi)^{\frac{1}{2}}\equiv\left(\iint_B|\psi(z)|^2\,d\sigma_z\right)^{1/2},$$

and denote by $D_B(\Psi_1,\Psi_2)$ the associated bilinear integral form:

$$D_B(\Psi_1,\Psi_2)=\iint_B\Psi_1'(z)\overline{\Psi_2'(z)}\,d\sigma_z.$$

Introducing further the inner product by

$$(\psi_1,\psi_2)=D_B(\Psi_1,\Psi_2),$$

$$\Psi_j(z)\equiv\int^z\psi_j(z)\,dz \quad (j=1,2),$$

$L^2(B)$ becomes a Hilbert space.

Let now $f'(z) \in L^2(B)$, and $f(z) = \sum_{\nu=1}^{\infty} c_\nu (z-z_0)^\nu$.
We then have

$$D_B(f, \varphi_m) = \frac{i}{2}\int_\Gamma f\, d\overline{\varphi}_m = \frac{i}{2}\int_\Gamma f\, d\Phi_m$$

$$= \frac{i}{2}\int_\Gamma f(z)\frac{m}{(\bar z - \bar z_0)^{m+1}}d\bar z = \pi m C_m .$$

Especially we have

$$D_B(\varphi_\nu, \varphi_\mu) = \pi\mu S_{\nu\mu} = \pi\nu \overline{S}_{\mu\nu}.$$

2. Perfect condition for Dirichlet integral to be bounded.

Theorem 1. Let $f(z)$ be single-valued function regular in B and its power series expansion be

$$f(z) = \sum_{\mu=1}^{\infty} c_\mu z^\mu$$

valid in a neighborhood of the origin. In order that $D_B(f,f) \leq \pi$, it is necessary and sufficient that there hold the inequalities

$$\left|\sum_{\mu=1}^{N} \mu C_\mu x_\mu\right|^2 \leq \sum_{\mu,\nu=1}^{N} \nu S_{\mu\nu} x_\nu \overline{x}_\mu \quad (N=1,2,\cdots)$$

for any complex numbers x_μ.

Proof. We shall follow Helly's method. Since the necessity of the condition is trivial, we shall merely show the sufficiency.

We introduce a family

$$L_1 = \left\{ \psi(z),\ \psi = \sum_{\nu=1}^{N} \overline{x}_\nu \varphi_\nu',\ x_\nu \text{ and } N \text{ being arbitrary} \right\}.$$

By taking the norm in $L^2(B)$ as that in L_1, the latter becomes a normed subspace of the former. Put

$$\mathfrak{f}_1(\psi) = \sum_{\nu=1}^{N} \pi\nu\overline{c}_\nu \overline{x}_\nu \quad \text{for} \quad \psi = \sum_{\nu=1}^{N} \overline{x}_\nu \varphi_\nu' .$$

Then $\mathfrak{f}_1$ is a linear functional and $\mathfrak{f}_1 \in L_1^*$, L_1^* being the dual space of L_1. Based on the definition

$$\|\mathfrak{f}_1\|_{L_1} = \sup_{\psi\in L_1,\ \|\psi\|_B \leq 1} |\mathfrak{f}_1(\psi)|,$$

the assumed inequalities of the theorem imply

$$\|\mathfrak{f}_1\|_{L_1} \leq \sqrt{\pi}$$

since, for $\psi = \sum_{\nu=1}^{N} \overline{x}_\nu \varphi_\nu'$, we have

$$\|\psi\|^2 = \sum_{\nu,\mu=1}^{N} \overline{x}_\nu x_\mu D_B(\varphi_\nu, \varphi_\mu)$$

$$= \pi\sum_{\nu,\mu=1}^{N} \nu \overline{S}_{\mu\nu} \overline{x}_\nu x_\mu$$

$$= \pi\sum_{\mu,\nu=1}^{N} \nu S_{\mu\nu} x_\nu \overline{x}_\mu .$$

In view of Hahn-Banach's theorem, $\mathfrak{f}_1$ can be extended to a linear functional $\mathfrak{f}$ defined in the whole $L^2(B)$. Namely, there exists a linear functional $\mathfrak{f}$ defined in $L^2(B)$ such that

$$\mathfrak{f}(\varphi_\nu') = \mathfrak{f}_1(\varphi_\nu') \equiv \pi\nu c_\nu \quad (\nu = 1, 2, \cdots),$$

and

$$\|\mathfrak{f}\|_{L^2(B)} = \|\mathfrak{f}_1\|_{L_1} \leq \sqrt{\pi}.$$

Since, by Riesz' theorem, $(L^2(B))^* = L^2(B)$, we have $\mathfrak{f} \in L^2(B)$ so that

$$\mathfrak{f}(\varphi_\nu') = (\mathfrak{f}, \varphi_\nu') = \pi\nu c_\nu \quad (\nu=1,2, \cdots).$$

Let now the local expansion of $\mathfrak{f}$ be

$$\mathfrak{f}(z) = \sum_{\nu=1}^{\infty} \nu d_\nu z^{\nu-1}.$$

Then, we get

$$(\mathfrak{f}, \varphi_\nu') = \frac{i}{2}\int_\Gamma \mathfrak{f}\, d\overline{\varphi}_\nu = \frac{i}{2}\int_\Gamma \mathfrak{f}\, d\Phi_\nu$$

$$= \frac{i}{2}\int_\Gamma \mathfrak{f}\, \frac{\nu}{\bar z^{\nu+1}} d\bar z$$

$$= \pi\nu d_\nu,$$

whence follows $d_\nu = c_\nu$ ($\nu = 1,2,\ldots$). Consequently, $\mathfrak{f}$ is coincident locally with f'. But, $\mathfrak{f}$ and f' being both analytic, it follows that there holds $\mathfrak{f} \equiv f'$ in the large. Thus, we have shown the L^2-continuability of f', which completes our proof.

3. Extension to Riemann surface.

Let R be a Riemann surface not of class O_{AD}. Introduce a local parameter z in a neighborhood of a point $P_0 \in R$ and suppose that $z = 0$ corresponds to P_0.

Our present purpose is now to obtain a perfect condition in terms of local coefficients in order that a single-valued and regular function defined in a neighborhood of P_0 is continuable to a single-valued function on the whole R and further its Dirichlet norm on R does not exceed an assigned bound.

Let $L^2(R)$ be a family of single-valued covariants $\psi(z)$ regular on R and satisfying the following conditions:

i. $\int^z \psi(z)\,dz$ is a single-valued function regular on R;

ii. $\iint_R |\psi(z)|^2 d\sigma_z < \infty$.

As shows the procedure of the proof of theorem 1, an analogous theorem will be obtained provided that there exists a system of functions corresponding to $\{\varphi_m\}$ introduced in § 1.

Such a system can be, indeed, formally defined in a following manner. In fact, let $\psi(z)$ be any covariant which is single-valued on R. Then, as shown by Virtanen [4], there exists a covariant $E(z,x)$ for which $\int^z E(z,x)\,dz$ is one-valued on R and

$$\iint_R \psi(z)\,\overline{E(z,x)}\,d\sigma_z = \psi(x)$$

Consequently, from the local expansion

$$\psi(z) = \sum_{\nu=0}^{\infty} d_\nu z^\nu ,$$

we get

$$\begin{aligned} d_\nu &= \frac{1}{\nu!}\psi^{(\nu)}(0) \\ &= \frac{1}{\nu!}\left[\frac{d^\nu}{dx^\nu}\iint_R \psi(z)\,\overline{E(z,x)}\,d\sigma_z\right]^{x=0} \\ &= \iint_R \psi(z)\left[\frac{1}{\nu!}\frac{\partial^\nu}{\partial x^\nu}\overline{E(z,x)}\right]^{x=0} d\sigma_z \end{aligned}$$

$$(\nu = 0, 1, \cdots).$$

Thus, a system defined by

$$\xi_\nu(z) = \frac{1}{\nu!}\left[\overline{\frac{\partial^\nu}{\partial x^\nu}E(z,x)}\right]^{x=0}$$

$$(\nu = 0, 1, \cdots)$$

consists of covariants single-valued, regular and quadratically integrable on R and every $\int^z \xi_\nu(z)\,dz$ is also single-valued on R. As shown above, there holds

$$d_\nu = \iint_R \psi(z)\,\overline{\xi_\nu(z)}\,d\sigma_z$$

for any ν . This system being complete on $L^2(R)$, $L^2(R)$ is a Hilbert space. Hence, we can conclude the following theorem.

Theorem 2. Let $f(z)$ be a single-valued function regular on R and its power series expansion be

$$f(z) = \sum_{\nu=1}^{\infty} c_\mu z^\mu$$

valid in a neighborhood of a parameter circle. In order that $D_R(f,f) \leqq \gamma$, it is necessary and sufficient that there hold the inequalities

$$\left|\sum_{\mu=1}^{N} \mu c_\mu x_\mu\right| \leqq \gamma \left\|\sum_{\mu=1}^{N} x_\mu \xi_\mu\right\|$$

$$(N = 1, 2, \cdots)$$

for any complex numbers x_μ .

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