SOME CLASSES OF POSITIVE SOLUTIONS OF $\Delta u = Pu$

ON RIEMANN SURFACES, II

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§ 4. Dimensions of ideal boundary (continuation).

If $F \in O_\Omega$ and $\Gamma \in (b)$, then the situation is somewhat troublesome to handle. Let $\mathcal{S}_v$ be the least harmonic majorant of $v \in \mathcal{G}_{F-\bar{F}_0}$. Then $\mathcal{S}_v$ has its sense for any $v \in \mathcal{G}_{F-\bar{F}_0}$. Let S_v be a limit harmonic function $\lim_{n\to\infty} v^n$ defined as follows: v^n is harmonic on $F_n - \bar{F}_0$ such that $v^n = v$ on $\Gamma_n + \Gamma_0$. Then we have easily that $\mathcal{S}_v \equiv S_v$. Let $\mathcal{T}_u$ be the largest minorant of $u \in \mathbb{P}_{F-\bar{F}_0}$ belonging to $\mathcal{P}_{F-\bar{F}_0}$. Then $\mathcal{T}_u$ is equal to either constant zero or a solution of (A) such that $\mathcal{T}_u \not\equiv 0$. If $\mathcal{T}_u \not\equiv 0$, then $\mathcal{T}_u \in \mathcal{P}_{F-\bar{F}_0}$. Let u^n be a solution of (A) on $F_n - \bar{F}_0$ such that $u^n = u$ on $\Gamma_n + \Gamma_0$, then $T_u = \lim_{n\to\infty} u^n$ exists and either $T_u \equiv 0$ or $T_u \not\equiv 0$. Moreover T_u coincides with $\mathcal{T}_u$. We have the following facts:

(i) $\mathcal{T}\circ\mathcal{S} = I$ for any $v \in \mathcal{G}_{F-\bar{F}_0}$,

(ii) $\mathcal{S}$ operation preserves the linear independency.

Let $[U]$ be a positively linear subspace of $\mathbb{P}_{F-\bar{F}_0}$ spanned by all the minimals $u_i, i = 1, \dots, n$ such that $\mathcal{T}_{u_i} \not\equiv 0$, $u_i \in \mathbb{P}_{F-\bar{F}_0}$ and let $\mathcal{T}[U]$ be T image of $[U]$. Let $[V]$ be a positively linear subspace of $\mathcal{P}_{F-\bar{F}_0}$ for each element of which $\mathcal{S}$ operation has the sense. Evidently $\mathcal{G}_{F-\bar{F}_0} \subseteq [V] \subset \mathcal{P}_{F-\bar{F}_0}$. Let $\mathcal{S}[V]$ be $\mathcal{S}$ image of $[V]$. Next facts are also easy to verify:

$$\mathcal{S}[V] \subset [U] \text{ and } [V] \subset \mathcal{T}[U].$$

Let u be a minimal in $\mathbb{P}_{F-\bar{F}_0}$ such that $\mathcal{T}_u \not\equiv 0$. Then $\mathcal{S}\mathcal{T}_u = u$ is valid. This shows that $\mathcal{T}[U] \subset [V]$ and $[U] \subset \mathcal{S}[V]$. Hence we see that $[V] = \mathcal{T}[U]$ and $\mathcal{S}[V] = [U]$.

Let u be a minimal belonging to $[U]$, then $\mathcal{T}_u$ is also a minimal in $\mathcal{P}_{F-\bar{F}_0}$. In fact, if we assume that $0 < w \leq \mathcal{T}_u$, then $\mathcal{S}_w$ exists and satisfies $0 < \mathcal{S}_w \leq u$, therefore $\mathcal{S}_w = k u$ holds. This implies the desired fact $w = k\mathcal{T}_u$.

If $F \in O_G^{(k)}$, then $u = \lim_{n\to\infty} g^{(k)}_{F-\bar{F}_0}(z, \zeta_n)$

for a suitable non-compact sequence $\{\zeta_n\}$ for any minimal u in $\mathbb{P}_{F-\bar{F}_0}$.

If $w = \lim_{m\to\infty} G_{F-\bar{F}_0}(z, \zeta_{n_m}) > 0$ on $F - \bar{F}_0$

for a suitable subsequence $\{\zeta_{n_m}\}$ of $\{\zeta_n\}$, then $0 < w \leq \mathcal{T}_u$ which shows that $w = k\mathcal{T}_u$ and w is also a minimal in $\mathcal{P}_{F-\bar{F}_0}$ and $\mathcal{T}_u$ belongs to $\mathcal{G}_{F-\bar{F}_0}$.

Let $\mathcal{Q}_{F-\bar{F}_0}$ be a class of positive solutions v of (A) on $F - \bar{F}_0$ such that $0 < \iint_{F-\bar{F}_0} v(z)P(z)\,d\sigma < \infty$ and $v = 0$ on Γ_0. We shall next prove that S_v has the sense for any $v \in \mathcal{Q}_{F-\bar{F}_0}$. Evidently $v^n \geq v$ on $F_n - \bar{F}_0$ and $v^n > v^m$ if $n > m$. Therefore $\frac{\partial v^n}{\partial \nu} > \frac{\partial v}{\partial \nu} \geq 0$ on Γ_0 and $\frac{\partial v^n}{\partial \nu} \geq \frac{\partial v}{\partial \nu}$ on Γ_n. On the other hand we see

$$\infty \gneqq M_1 > \int_{\Gamma_0} \frac{\partial}{\partial \nu} v\,ds + \iint_{F_n-\bar{F}_0} v P\,d\sigma$$

$$= -\int_{\Gamma_n} \frac{\partial}{\partial \nu} v\,ds > -\int_{\Gamma_n} \frac{\partial}{\partial \nu} v^n ds$$

$$= \int_{\Gamma_0} \frac{\partial}{\partial \nu} v^n ds,$$

which leads to a fact that

$$M_1 \geq \int_{\Gamma_0} \frac{\partial}{\partial \nu} S_v\,ds.$$

and hence we see that

$$S_v \not\equiv \infty$$

This fact can also be verified as follows: We have a decomposition for any $v \in \mathfrak{P}_{F-\bar{F}_0}$ such that

$$v(p) = v^n(p) - \iint_{F_n-\bar{F}_0} g^{(h)}_{F_n-\bar{F}_0}(z,p)\, v(z) P(z)\, d\sigma .$$

If $v \in Q_{F-\bar{F}_0}$, then $\iint_{F-\bar{F}_0} v(z)P(z)\,d\sigma < M$. And we have $g^{(h)}_{F-\bar{F}_0}(z,p) < m$ if $z \in F-\bar{F}_0 - K_p$, where K_p is a parameter disc around p. Let $\bar{g}^{(h)}_{F-\bar{F}_0}(z,p)$ be equal to $g^{(h)}_{F_n-\bar{F}_0}$ (resp. 0) on $F_n - \bar{F}_0$ (resp. $F - \bar{F}_n$), then

$$\iint_{F-\bar{F}_0-K_p} \bar{g}^{(h)}_{F_n-\bar{F}_0}(z,p)\, v(z) P(z)\, d\sigma \leq m M$$

and

$$\iint_{K_p} \bar{g}^{(h)}_{F_n-\bar{F}_0}(z,p)\, v(z) P(z)\, d\sigma \leq R < \infty .$$

Thus $S_v = \lim_{n\to\infty} v^n(p)$ exists and $\neq \infty$.

If $F \in O_G^{(h)}$, then we have an inverse, that is, $v \in Q_{F-\bar{F}_0}$ if $S_v \neq \infty$ for a given $v \in \mathfrak{P}_{F-\bar{F}_0}$. In fact, we shall use again the decomposition for any $v \in \mathfrak{P}_{F-\bar{F}_0}$:

$$v(p) + \iint_{F-\bar{F}_0} \bar{g}^{(h)}_{F_n-\bar{F}_0}(z,p)\, v(z) P(z)\, d\sigma = v^n(p) .$$

By the assumption $S_v \neq \infty$, $v^n(p) < M$ holds uniformly if p belongs to a compact subregion of $F - \bar{F}_0$. Thus we see

$$\iint_{F-\bar{F}_m-K_q} \bar{g}^{(h)}_{F_n-\bar{F}_0}(z,p)\, v(z)\, P(z)\, d\sigma < M_1 ,$$

whence

$$\iint_{F-\bar{F}_m-K_q} g^{(h)}_{F-\bar{F}_0}(z,p)\, v(z)\, P(z)\, d\sigma < M_1 .$$

If $g^{(h)}_{F-\bar{F}_0}(z,p) \geq \delta > 0$ on $F-\bar{F}_m - K_q$, —— this is really valid if $F \in O_G^{(h)}$ ——, then we have

$$\delta \iint_{F-\bar{F}_m-K_q} v(z)\, P(z)\, d\sigma < M_1 ,$$

whence

$$\iint_{F-\bar{F}_0} v(z)\, P(z)\, d\sigma < M_2 .$$

This is the desired fact.

Next we shall prove that $\mathfrak{G}_{F-\bar{F}_0} \subsetneqq Q_{F-\bar{F}_0}$. Evidently we see

$$\iint_{F-\bar{F}_0} G_{F-\bar{F}_0}(z,\zeta_m) P(z)\, d\sigma = 2\pi - \int_{\Gamma_0} \frac{\partial}{\partial\nu} G_{F-\bar{F}_0}(z,\zeta_m)\, ds \leq 2\pi .$$

Let m tend to infinity for which $\lim_{m\to\infty} G_{F-\bar{F}_0}(z,\zeta_m) > 0$, then

$$0 < \iint_{F-\bar{F}_0} \lim_{m\to\infty} G_{F-\bar{F}_0}(z,\zeta_m) P(z)\, d\sigma \leq 2\pi .$$

Therefore the desired fact is valid.

<u>Theorem 2</u>. Let $F \in O_\Omega$ and $\Gamma \in (b)$, then we have

$$\dim_2(\Gamma) \leq \dim_0(\Gamma) .$$

<u>Theorem 3</u>. $\mathfrak{G}_{F-\bar{F}_0} \subsetneqq Q_{F-\bar{F}_0} \subsetneqq [V] = \mathfrak{T}[U]$.

Especially if $F \in O_G^{(h)}$, we have $Q_{F-\bar{F}_0} = [V] = \mathfrak{T}[U]$.

We shall consider the surface F of finite genus. The ideal boundary is a point p of F, hence $F - \bar{F}_0$ may be considered as $|q - p| < 1$. Let Γ_n be a circumference $|q - p| = r_n$ with $r_n \downarrow 0$. If P(q) satisfied the required conditions at p, then there is only one Green function $G_{F-\bar{F}_0}(q,p)$ with pole at p. Thus $\dim_2(\Gamma) = 1$.

If $\iint_{F-\bar{F}_0} P(q)\, d\sigma_q = \infty$, then $\Gamma \in (b)$ or $\Gamma \in (c)$. However in this case $\Gamma \in (b)$ does not occur. In fact we see easily that

$$\frac{1}{k} G_{F-\bar{F}_0}(\zeta_n, z) \leq G_{F-\bar{F}_0}(\zeta'_n, z) \leq k\, G_{F-\bar{F}_0}(\zeta_n, z)$$

where k is independent of n and $z \in F_m - \bar{F}_0$ and $\zeta_n, \zeta'_n \in F_n - \bar{F}_{n-1}$. This is the same as Harnack's inequality. Thus for any $\{\zeta_n\}$ we can conclude that

$$\lim_{n\to\infty} G_{F-\bar{F}_0}(\zeta_n, z) = 0 .$$

§5. Subregion and its dimensions.

Let D be a subregion with non-compact analytic relative boundary C imbedded in $F (\in O_\Omega)$. Let $D_n = F_n \cap D$, $C_n = F_n \cap C$ and $\gamma_n = \bar{D} \cap \Gamma_n$.

$\mathfrak{G}_D$, $\mathbb{P}_D$, $\mathfrak{P}_D$ are similarly defined as in § 3. Let $\mathbb{Q}_D$ be a family of positive harmonic functions on D such that $\int_C \frac{\partial}{\partial\nu} u\, ds < \infty$. Let $\mathbb{G}_D$ be a class spanned by a positively linear combination of limit functions $\lim\limits_{n\to\infty}$ $g_D^{(k)}(z, \zeta_n^{(i)})$ each of which is minimal in Martin's sense on D. Then we see evidently $\mathfrak{G}_D \subset \mathfrak{P}_D$, $\mathbb{G}_D \subset \mathbb{P}_D$ and $\mathbb{Q}_D \subset \mathbb{P}_D$. Let $\mathfrak{O}_D$ be a class of positive solutions of (A) on D such that $v \equiv 0$ on C and

$$\int_C \frac{\partial}{\partial\nu} v\, ds < \infty$$

and

$$\iint_D v P\, d\sigma < \infty .$$

We shall first prove that $\mathfrak{G}_D \subseteqq \mathfrak{O}_D$. For any point ζ_m on D we see that

$$2\pi - \int_{C_n+\gamma_n} \frac{\partial}{\partial\nu} G_{D_n}(z, \zeta_m)\, ds = \iint_{D_n} G_{D_n}(z, \zeta_m) P(z)\, d\sigma$$

for a sufficiently large n. And moreover $F \in O_\Omega$ implies that

$$\int_{\gamma_n} \frac{\partial}{\partial\nu} G_{D_n}(z, \zeta_m)\, ds = 2\pi\, \omega(\zeta_m, \gamma_n, D_n)$$

tends to zero as $n \to \infty$. Thus

$$2\pi - \int_C \frac{\partial}{\partial\nu} G_D(z, \zeta_m)\, ds = \iint_D G_D(z, \zeta_m) P(z)\, d\sigma$$

holds, since $G_{D_n}(z, \zeta_m) > G_{D_p}(z, \zeta_m)$ for $n > p$. On the other hand

$$\int_{C_n} \frac{\partial}{\partial\nu} G_{D_n}(z, \zeta_m)\, ds = 2\pi\, \omega'(\zeta_m, C_n, D_n)$$

holds and this implies a fact

$$\iint_D G_D(z, \zeta_m) P(z)\, d\sigma \leqq 2\pi$$

Let m tend to infinity with $\lim\limits_{m\to\infty} G_D(z, \zeta_m) > 0$ on D, then

$$0 < \iint_D \lim_{m\to\infty} G_D(z, \zeta_m) P(z)\, d\sigma \leqq 2\pi .$$

Thus we have the desired result:

$$\mathfrak{G}_D \subseteqq \mathfrak{O}_D .$$

Next fact is easy to verify:

$$\mathbb{G}_D \subseteqq \mathbb{Q}_D .$$

<u>Theorem 4.</u> $\mathfrak{G}_D \subseteqq \mathfrak{O}_D$, $\mathbb{G}_D \subseteqq \mathbb{Q}_D$. There is a one-to-one positively linear mapping S which carries $\mathfrak{G}_D$ and $\mathfrak{O}_D$ into $\mathbb{G}_D$ and $\mathbb{Q}_D$, respectively. S operation preserves the minimality and has its left inverse operation T. Thus there holds

$$\dim \mathfrak{G}_D \begin{matrix} \leqq & \dim \mathfrak{O}_D & \leqq \\ \leqq & \dim \mathbb{G}_D & \leqq \end{matrix} \dim \mathbb{Q}_D .$$

Especially $F \in O_G$ implies that $\mathbb{G}_D = \mathbb{Q}_D$.

Proof. Let $v \in \mathfrak{O}_D$, then $\mathcal{S}_v$ is defined as the least harmonic majorant of v in $\mathbb{P}_D$. Then $\mathcal{S}_v$ is not constant ∞. In fact, we see that

$$\infty > M_1 > \int_{C_n} \frac{\partial}{\partial\nu} v\, ds + \iint_{D_n} v P\, d\sigma$$

$$= -\int_{\gamma_n} \frac{\partial}{\partial\nu} v\, ds > -\int_{\gamma_n} \frac{\partial}{\partial\nu} v^n\, ds$$

$$= \int_{C_n} \frac{\partial}{\partial\nu} v^n\, ds ,$$

where v^n is a harmonic function on D_n such that $v^n = v$ on $\gamma_n + C_n$. Since $v^n > v^m$ is valid for $n > m$, $S_v = \lim\limits_{n\to\infty} v^n \not\equiv \infty$. Evidently $S_v \geqq \mathcal{S}_v$ and hence $\mathcal{S}_v$ surely exists and $\not\equiv \infty$. Since $\mathcal{S}_v \geqq v$ on D, we see $\mathcal{S}_v \geqq v^n$ on D_n, whence $\mathcal{S}_v \geqq S_v$. Thus $\mathcal{S}_v$ coincides with S_v which belongs to the class $\mathbb{Q}_D$, that is,

$$0 < \int_C \frac{\partial}{\partial\nu} S_v\, ds < \infty .$$

$\mathfrak{T}_u$ is defined for any $u \in \mathbb{P}_D$ as the largest minorant of u in $\mathfrak{P}_D$. The following three facts are similarly verified as in §§ 3, 4.

(i) $\mathfrak{T} \circ \mathcal{S} = I$ for any $\mathfrak{O}_D$.
(ii) $\mathcal{S}$ operation preserves the minimality.
(iii) $\mathcal{S}_v \in \mathbb{G}_D$ if $v \in \mathfrak{G}_D$.

The final statement of Theorem 4 was already proved. (Cf. Ozawa [3].)

Next we shall investigate the relation between $\mathfrak{H}_D$ and $Q_{F-\bar{F}_0}$. Without loss of generality we may assume that $D \subset F-\bar{F}_0$.

Let $v \in \mathfrak{H}_D$, then S_v is defined as a limit function $\lim_{n\to\infty} v^n$ such that v^n is a solution of (A) on $F_n-\bar{F}_0$ with boundary values v on γ_n and 0 on $(\Gamma_n-\gamma_n)+\Gamma_0$. We see that

$$\begin{aligned}\infty > M_1 &> \int_{C_n} \frac{\partial}{\partial\nu} v\,ds + \iint_{D_n} vP\,d\sigma \\ &= -\int_{\gamma_n}\frac{\partial}{\partial\nu} v\,ds \geqq -\int_{\gamma_n}\frac{\partial}{\partial\nu} v^n\,ds \\ &= \int_{\Gamma_n-\gamma_n}\frac{\partial}{\partial\nu}v^n ds + \int_{\Gamma_0}\frac{\partial}{\partial\nu}v^n ds + \iint_{F_n-\bar{F}_0} v^n P\,d\sigma \\ &\geqq \iint_{F_n-\bar{F}_0} v^n P\,d\sigma + \int_{\Gamma_0}\frac{\partial}{\partial\nu}v^n ds.\end{aligned}$$

Let n tend to infinity, then we have

$$M_1 \geqq \iint_{F-\bar{F}_0} S_v P\,d\sigma + \int_{\Gamma_0}\frac{\partial}{\partial\nu}S_v\,ds,$$

which shows that $S_v \not\equiv \infty$ and belongs to the class $Q_{F-\bar{F}_0}$. T operation is defined in an inverse manner as that of S operation. Then we have again

(i) $T\circ S = I$ holds for any $v \in \mathfrak{H}_D$,

(ii) S operation preserves the minimality, if it has the sense,

(iii) S_v for any $v \in \mathfrak{G}_D$ belongs to $\mathfrak{G}_{F-\bar{F}_0}$.

Thus we have the following theorem:

Theorem 5. There is a one-to-one and positively linear mapping S which carries $\mathfrak{G}_D$ and $\mathfrak{H}_D$ into $\mathfrak{G}_{F-\bar{F}_0}$ and $Q_{F-\bar{F}_0}$, respectively. S operation preserves the minimality if it has the sense and has its left inverse operation T. Hence

$$\dim \mathfrak{G}_D \begin{smallmatrix}\leqq\\ \leqq\end{smallmatrix} \begin{matrix}\dim \mathfrak{H}_D \\ \dim \mathfrak{G}_{F-\bar{F}_0}\end{matrix} \begin{smallmatrix}\leqq\\ \leqq\end{smallmatrix} \dim Q_{F-\bar{F}_0}.$$

Let D_i, $i=1,2,3$ be a triple of domains such that $D_1\cap D_2 = 0$, $D_1\cup D_2 \subseteqq D_3$. Then we have that

$$\dim \frac{\mathfrak{G}_{D_1}}{\mathfrak{H}_{D_1}} + \dim \frac{\mathfrak{G}_{D_2}}{\mathfrak{H}_{D_2}} = \dim \frac{\mathfrak{G}_{D_3}}{\mathfrak{H}_{D_3}}.$$

§ 6. Existence proof of the Green function.

In this section we shall give another proof of the existence of the Green function of (A) on any Riemann surface. Original proof for this fact is due to Myrberg.

In this section we donot assume that the Riemann surface F in consideration is of null boundary of any sort. We shall proceed to our existence proof of the Green function of (A) on any Riemann surface F under two logical assumptions listed below:

(1) On any parameter disc there always exists the Green function of (A).

(2) The first boundary value problem on any compact analytic subregion is always solvable.

By the first assumption (1) we can imply that the Harnack's principle for positive solutions of (A) and a theorem on normal family for uniformly bounded solutions are valid.

Let $u_n(z)$ be a bounded solution of (A) on $F_n-\bar{F}_0$ such that $u_n(z)=u(z)$ on Γ_0 and $=0$ on Γ_n. Here $M \geqq u(z) \geqq 0$ on Γ_0. Then $u_n(z) \geqq u_m(z)$ if $n>m$. Thus $N_u = \lim_{n\to\infty} u_n(z)$ exists and $\leqq M = \max_{\Gamma_0} u(z) < \infty$. We shall call N_u the normal solution of the first boundary value problem on $F-\bar{F}_0$ with the given boundary value u on Γ_0. For any normal solution we have

$$\iint_{F-\bar{F}_0} N_u P\,d\sigma = -\int_{\Gamma_0}\frac{\partial}{\partial\nu}N_u\,ds - \int_{\Gamma_0} N_u\frac{\partial}{\partial\nu}\omega\,ds,$$

$$\omega = \omega(p,\Gamma,F-\bar{F}_0).$$

In fact, we see

$$\begin{aligned}\iint_{F_n-\bar{F}_0} u_n P\,d\sigma &= -\int_{\Gamma_0}\frac{\partial}{\partial\nu}u_n ds - \int_{\Gamma_n}\frac{\partial}{\partial\nu}u_n ds \\ &= -\int_{\Gamma_0}\frac{\partial}{\partial\nu}u_n ds - \int_{\Gamma_n+\Gamma_0}\omega_n(p,\Gamma_n,F_n-\bar{F}_0)\frac{\partial}{\partial\nu}u_n ds \\ &= -\int_{\Gamma_0}\frac{\partial}{\partial\nu}u_n ds - \int_{\Gamma_0} u\frac{\partial}{\partial\nu}\omega_n ds\end{aligned}$$

and $0 \leqq \omega_n \leqq \omega_m$, $u_n \geqq u_m$ if $n>m$.

Thus the right hand side term tends to

$$-\int_{\Gamma_0}\frac{\partial}{\partial\nu}N_u\,ds-\int_{\Gamma_0}u\frac{\partial}{\partial\nu}\omega\,ds$$

with increasing n. Thus the left hand term converges to

$$\iint_{F-\bar{F}_0}N_u P\,d\sigma.$$

Let K_q^r be a circular disc around a given point q with radius r which belongs to a parameter disc around q. Let G(p,q) be the Green function of (A) on K_q^1 and let $N^{(r)}\equiv N_G^{(r)}$ be the normal solution of the first boundary value problem on $F-K_q^r$ with boundary value G on $|p-q|=r$. Then $N^{(r)}\geq N^{(r')}$ if $r\leqq r'$. $V=\lim_{r\to 0}N^{(r)}$ surely exists and $\neq\infty$. In fact, we have

$$\infty>K>\int_{|p-q|=1}\frac{\partial}{\partial\nu}G\,ds+\iint_{1\geq|p-q|\geq r}GP\,d\sigma$$

$$=-\int_{|p-q|=r}\frac{\partial}{\partial\nu}G\,ds\geqq-\int_{|p-q|=r}\frac{\partial}{\partial\nu}N^{(r)}ds$$

$$=\iint_{F-K_q^r}N^{(r)}P\,d\sigma+\int_{|p-q|=r}N^{(r)}\frac{\partial}{\partial\nu}\omega^{(r)}ds,$$

$$\omega^{(r)}=\omega(p,\Gamma,F-K_q^r).$$

Noting that $N^{(r)}>0$ and $\frac{\partial}{\partial\nu}\omega^{(r)}>0$ on $\{|p-q|=r\}$, we have

$$K>\iint_{F-K_q^r}N^{(r)}P\,d\sigma>0.$$

Let $M_1=\max_{|p-q|=1}V(p)$, then $M_1<\infty$ and hence

$$M_1\Omega(p)+G(p,q)\geqq N^{(r)}\geqq G(p,q)$$

holds on $K_q^1-K_q^r$, where $\Omega(p)$ is a bounded solution of (A) on K_q^1 with constant boundary value 1 on $\{|p-q|=1\}$. Thus we see that, if r tends to zero,

$$M_1\Omega(p)\geqq V(p)-G(p,q)\geqq 0.$$

Hence $V(p)-G(p,q)$ is a bounded positive solution of (A) on K_q^1 without exception. Thus $V(p)+\log|p-q|$ is bounded around q.

Let $V_1(p)$ be a positive solution of (A) on F - q such that $V_1(p)+\log|p-q|$ is bounded around q and $V_1(p)$ is not a majorant of V(p) on F - q. Then min $(V(p), V_1(p))$ determines an associate solution U(p) which is defined as a double limit $\lim_{n\to\infty}\lim_{r\to 0}U_n^r(p)$, where U_n^r is a finite solution of (A) on $F_n-K_q^r$ with boundary value min $(V(p), V_1(p))$ on $\Gamma_n+\{|p-q|=r\}$. By the well-known minimum property of $G(p,q)$ we have $V_1(p)>G(p,q)$ on K_q^1. Thus min $(V(p),V_1(p))>G(p,q)$ on $\{|p-q|=r\}$. Since $\min(V(p),V_1(p))>0$ on Γ_n, $U_n^r(p)>0$ holds on $\{|p-q|=1\}$. Hence we see that

$$U_n^r(p)>G(p,q)$$

on $K_q^1-K_q^r$. Since $U_n^r(p)$ is a monotone decreasing (non-increasing) sequence with $r\to 0$ at first and next $n\to\infty$, $U(p)=\lim_{n\to\infty}\lim_{r\to 0}U_n^r(p)$ surely exists and $U(p)>G(p,q)$. Evidently $U(p)\leqq V(p)$ on F. Hence $U(p)+\log|p-q|$ is also bounded around q.

If $V(p)\gneqq V_1(p)$ happens really at an inner point $p\in F$, then there is at least a true minorant U(p) of V(p) such that $U(p)+\log|p-q|$ is bounded around q and U(p) is a positive solution of (A).

Next we shall define S and T operations as follows:

$S_G=V$ is an operation carrying G to V. In a general case we shall define the S operation similarly and the result coincides with the one extended in a positively linear manner from the basic one. Let $T_V=\lim_{r\to 0}V^r(p)$ such that $V^r(p)$ is a positive solution of (A) on $K_q^1-K_q^r$ with boundary values 0 on $\{|p-q|=1\}$ and V(p) on $\{|p-q|=r\}$.

Then we have the following facts successively:

(1) $S\circ T_V=V$,
(2) $T\circ S_G=G$,
(3) $T_U=G$,
(4) $U=V$.

Verifications of these facts are similar as in the preceding sections. Then we have a contradiction, because $U\lneqq V$. Thus $V_1\geqq V$ on F. Therefore V is a positive minimum solution of

(A) such that

$$V(p) - \log \frac{1}{|p-q|}$$

is bounded around q.

A characteristic property for the Green function of (A) on F with pole at q is now satisfied by V.

M. Heins. Studies in the conformal mapping of Riemann surfaces, I. Proc. Nat. Acad. Sci. 39 (1953); II. Ibid. 40 (1954).

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