PROBABILISTIC INVESTIGATIONS ON POPULATION GENETICS

By Yûsaku KOMATU

The present paper reproduces a report just as it was read at the World Population Conference held on 31 August- 10 September, 1954 at Rome; E/CONF. 13/40, Meeting No. 23.

SUMMARY

In a series of successive papers we have developed a unified theory of inheritance from a probabilistic view-point. They have been mainly concerned with the distributions in an ordinary population. In succession, we now deal with analogous problems in a population which consists of families involving several types of consanguineous marriages.

We consider again a single inherited character which consists of any number of multiple alleles at one diploid locus and of which the inheritance is subject to Mendelian law. Our main purpose is then to study the distributions of genotypes in several definite combinations, lineal or collateral, which consist of individuals in families possibly involving consanguineous marriages chosen at random from a population under consideration.

By making use of relative frequencies of genes as basic quantities, we derive systematically the formulas expressing probabilities of several combinations. It is well clarified how the correlations between genotypes of family-members are affected by intervention of consanguineous marriages. On the other hand, many notable phenomena are elucidated. Especially, the distribution of genotypes in a generation immediate after any consanguineous marriage deviates from an equilibrium state in such a manner that the frequency of any homozygous type increases while that of any heterozygous one decreases. This fact will illustrate a cause of a deviation observed on a distribution in an isolated population.

* * *

Introduction. In a series of successive papers[1)] we have developed a unified theory of inheritance from a probabilistic view-point. It has been the purpose of these papers to investigate the phenomenon of inheritance based upon a general mode which is supposed to be subject to Mendelian law and then to apply the results to some practical problems on forensic medicine. One of the main problems has been concerned in determining the distributions of genotypes in several definite combinations of individuals chosen at random from a population of a given equilibrium state. The mother-child combination, among others, has played a distinguished role. This concept is, as its own name shows, a combination consisting of an individual and its immediate descendant. It will be plausible, for instance, to attempt to generalize the concept to a lineal combination consisting of an individual and its distant descendant after any definite number of generations.

The main purpose of the present report[2)] is to discuss further problems of determining the distributions of genotypes in several definite combinations, lineal or collateral, consisting of members in a family possibly involving consanguineous marriages. A population in consideration is again supposed to be in an equilibrium state unless the contrary is stated. It is further supposed that panmixia takes place at any generation except when consanguineous marriages are appointed.

Now we consider again a single inherited character which consists of m multiple alleles at one diploid locus denoted by A_i $(i=1,\cdots,m)$ and of which the inheritance is subject to Mendelian law. According to circumstances, introducing an abbreviated notation $A_{ij}\equiv A_iA_j$, we make an agreement that A_{ji} represents an identical object with A_{ij}. There are therefore $m(m+1)/2$ possible genotypes, m homozygous and $m(m-1)/2$ heterozygous, which are denoted respectively by A_{ii} $(i=1,\dots,m)$ and A_{ij} $(i,j=1,\dots,m;\ i<j)$ and together by A_{ab} $(a,b=1,\dots,m;\ a\leqq b)$.

We now designate by p_i, or often merely by i provided no confusion can arise, the relative frequency of the gene A_i in a population under consideration. The frequency of the genotype A_{ab} is then, in view of the assumption of an equilibrium, represented by

$$\bar{A}_{ii}=p_i^2,\qquad \bar{A}_{ij}=2p_ip_j$$
$$(i,j=1,\dots,m;\ i<j).$$

These basic quantities must, of course, satisfy the fundamental relations $\sum p_b=1, \sum\bar{A}_{ab}=(\sum p_b)^2=1$ where the first and the second summations extend over $b=1,\dots,m$ and $a,b=1,\dots,m;\ a\leqq b$, respectively. We notice here in passing that, also in the following lines, the different Latin suffices except those indicating the running types in summation are supposed, in principle, to designate the different genes.

1. Simple lineal combinations. The probability of mother-child combination $\pi\equiv\pi_1$ discussed in a previous paper3) is now generalized to that of a combination consisting of an individual $A_{\alpha\beta}$ and its nth descendant $A_{\xi\eta}$ which will be designated by $\pi_n(\alpha\beta;\ \xi\eta)\equiv\bar{A}_{\alpha\beta}\kappa_n(\alpha\beta;\ \xi\eta)$. The reduced probability κ_n satisfies a recurrence equation

$$\kappa_n(\alpha\beta;\ \xi\eta)=\sum\kappa_{n-1}(\alpha\beta;\ ab)\,\kappa(ab;\ \xi\eta),$$

the running type in summation being A_{ab} $(a,b=1,\dots,m;\ a\leqq b)$. It is shown that the probability is expressed by the formula

$$\kappa_n(\alpha\beta;\ \xi\eta)=\bar{A}_{\xi\eta}+2^{-n+1}Q(\alpha\beta;\ \xi\eta)$$

where the quantity $Q\ (=\kappa_1-\bar{A}_{\xi\eta})$ is independent of the generation-number n and its explicit values are given as follows:

$$Q(ii;\ ii)=i(1-i),\qquad Q(ii;\ ik)=k(1-2i),$$

$$Q(ii;\ kk)=-k^2,\qquad Q(ii;\ hk)=-2hk,$$

$$Q(ij;\ ii)=\tfrac{1}{2}i(1-2i),\quad Q(ij;\ ij)=\tfrac{1}{2}(i+j-4ij),$$

$$Q(ij;\ ik)=\tfrac{1}{2}k(1-4i),\quad Q(ij;\ kk)=-k^2,$$

$$Q(ij;\ hk)=-2hk.$$

The proof of the formula can be performed by induction4) by actually verifying an identity

$$\sum Q(\alpha\beta;\ ab)\,\kappa(ab;\ \xi\eta)$$
$$=\sum Q(\alpha\beta;\ ab)\,Q(ab;\ \xi\eta)$$
$$=\frac{1}{2}Q(\alpha\beta;\ \xi\eta).$$

An asymptotic behavior of κ_n as $n\to\infty$ can be readily deduced from its expression derived above. There holds, in fact, a limit relation $\kappa_n(\alpha\beta;\xi\eta)\to\bar{A}_{\xi\eta}$ $(n\to\infty)$ stating that a consanguineous correlation between the types of an individual and of its nth descendant becomes negligible for a sufficiently large n.

Let next a pair of individuals with assigned types $(A_{\alpha\beta},\ A_{\gamma\delta})$ be given, the order of members being taken into account. We then denote by $\varepsilon_n(\alpha\beta,\gamma\delta;\ \xi\eta)$ the probability that the pair is accompanied by an nth descendant $A_{\xi\eta}$. The values of $\varepsilon\equiv\varepsilon_1$ have been already availed in a previous paper5). The probability in question is given by a recurrence formula

$$\varepsilon_n(\alpha\beta,\ \gamma\delta;\ \xi\eta)$$
$$=\sum\varepsilon(\alpha\beta,\ \gamma\delta;\ ab)\,\kappa_{n-1}(ab;\ \xi\eta).$$

It is shown that the probability is expressible in the form

$$\varepsilon_n(\alpha\beta,\gamma\delta;\ \xi\eta)=\bar{A}_{\xi\eta}+2^{-n+1}E(\alpha\beta,\gamma\delta;\ \xi\eta)$$

valid for $n\geqq 2$. The factor E in the residual term is then defined by $E(\alpha\beta,\gamma\delta;\xi\eta)=2\sum\varepsilon(\alpha\beta,\gamma\delta;ab)Q(ab;\xi\eta)$. Its values can be determined by actual computation while they are omitted here for economy reason of space. An asymptotic behavior of ε_n as $n\to\infty$ is obvious; namely, we get $\varepsilon_n(\alpha\beta,\gamma\delta;\xi\eta)\to\bar{A}_{\xi\eta}\ (n\to\infty)$.

2. Simple collateral combinations. By generalizing the probability of brethren combination $\sigma\equiv\sigma_{11}$ discussed in a previous paper6), we now introduce that of a combination consisting of μth and νth collateral descendants $A_{\xi_1\eta_1}$ and $A_{\xi_2\eta_2}$, respectively, originated from the same couple, which will be designated by $\sigma_{\mu\nu}(\xi_1\eta_1,\ \xi_2\eta_2)$. It can be proved that there holds an identical relation

$$\sigma_{\mu\nu}(\xi_1\eta_1,\ \xi_2\eta_2)=\pi_{\mu+\nu-1}(\xi_1\eta_1;\ \xi_2\eta_2)$$

provided $\mu+\nu>2$, while it should be remembered that σ_{11} is not identical with π_1. The last relation shows, in particular, that the dependence of $\sigma_{\mu\nu}$ with $\mu+\nu>2$ on the generation-numbers is subject merely to their sum $\mu+\nu$.

3. Simple mother-descendants combinations. We designate, in general, by $\pi_{\mu\nu}(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)\equiv\bar{A}_{\alpha\beta}\cdot\kappa_{\mu\nu}(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)$ the probability of a combination consisting of an individual $A_{\alpha\beta}$ and its μth and νth collateral descendants $A_{\xi_1\eta_1}$ and $A_{\xi_2\eta_2}$, respectively, originated from the same spouse of $A_{\alpha\beta}$.

Three systems will be distinguished according to $\mu=\nu=1$, $\mu>1=\nu$ or $\mu=1<\nu$ and $\mu,\nu>1$. The lowest system $\pi\equiv\pi_{11}$ has already been treated in a previous paper7) as the probability of mother-children combination. Based on an evident quasi-symmetry relation $\pi_{\mu\nu}(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)=\pi_{\nu\mu}(\alpha\beta;\xi_2\eta_2,\xi_1\eta_1)$, it suffices to deal with the former of the second system. The reduced probability is then defined by a recurrence equation

$$\kappa_{1\nu}(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)=\sum\kappa(\alpha\beta;\xi_1\eta_1,ab)\,\kappa_{\nu-1}(ab;\xi_2\eta_2)$$

and is expressible in the form

$$\kappa_{1\nu}(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)$$
$$=\kappa(\alpha\beta;\xi_1\eta_1)\bar{A}_{\xi_2\eta_2}+2^{-\nu}W(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2).$$

The quantity W in the residual term is then defined by $W(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)=4\sum\kappa(\alpha\beta;\xi_1\eta_1,ab)Q(ab;\xi_2\eta_2)$. Its values can be determined by actual computation, while they are omitted here. The validity of the formula may be ensured by induction by directly verifying an identity

$$\sum W(\alpha\beta;\xi_1\eta_1,ab)\,\kappa(ab;\xi_2\eta_2)$$
$$=\frac{1}{2}W(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2).$$

The formula for the last generic system is expressed in the form

$$\kappa_{\mu\nu}(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)$$
$$=\bar{A}_{\xi_1\eta_1}\bar{A}_{\xi_2\eta_2}+2^{-\mu+1}\bar{A}_{\xi_2\eta_2}Q(\alpha\beta;\xi_1\eta_1)$$
$$+2^{-\nu+1}\bar{A}_{\xi_1\eta_1}Q(\alpha\beta;\xi_2\eta_2)+2^{-\mu-\nu+1}T(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2).$$

While the explicit values of T are omitted here, the validity of the formula may be proved by means of a recurrence equation

$$\kappa_{\mu\nu}(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)$$
$$=\sum\kappa_{\mu-1,\nu}(\alpha\beta;ab,\xi_2\eta_2)\,\kappa(ab;\xi_1\eta_1),$$

together with the identities

$$\sum W(\alpha\beta;ab,\xi_2\eta_2)\,\kappa(ab;\xi_1\eta_1)$$
$$=\sum T(\alpha\beta;ab,\xi_2\eta_2)\,\kappa(ab;\xi_1\eta_1)$$
$$=\frac{1}{2}T(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2).$$

It should be noted that the formula for generic system does not involve one for the second system as a particular case. In fact, $W(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)$ is not identical with $2\bar{A}_{\xi_1\eta_1}Q(\alpha\beta;\xi_2\eta_2)+T(\alpha\beta;\xi_1\eta_1,\xi_2\eta_2)$.

4. Ancestors-descendant combinations through intermediate marriages. Suppose that two individuals $A_{\alpha\beta}$ and $A_{\gamma\delta}$ are accompanied by their μth and νth descendants A_{ab} and A_{cd}, respectively, and that these descendants are married and originate themselves an nth descendant $A_{\xi\eta}$. Let the probability of a triple consisting of $A_{\alpha\beta}$, $A_{\gamma\delta}$ and $A_{\xi\eta}$ be then designated by $\bar{A}_{\alpha\beta} A_{\gamma\delta} \mathcal{E}_{\mu\nu;n}(\alpha\beta, \gamma\delta; \xi\eta)$. . The probability of parents-descendant combination, treated in **1**, may be regarded to correspond to the lowest case $\mu=\nu=0$: $\mathcal{E}_{00;n} \equiv \mathcal{E}_n$. In case of higher generation-numbers μ, ν we state here merely the final formulas:

$$\mathcal{E}_{\mu0;1}(\alpha\beta, \gamma\delta; \xi\eta)$$
$$= K(\gamma\delta; \xi\eta) + 2^{-\mu} C_0(\alpha\beta, \gamma\delta; \xi\eta),$$
$$\mathcal{E}_{\mu0;n}(\alpha\beta, \gamma\delta; \xi\eta)$$
$$= K_n(\gamma\delta; \xi\eta) + 2^{-\mu-n} C(\alpha\beta, \gamma\delta; \xi\eta) \qquad (n>1),$$
$$\mathcal{E}_{\mu\nu;1}(\alpha\beta, \gamma\delta; \xi\eta)$$
$$= \bar{A}_{\xi\eta} + 2^{-\mu} Q(\alpha\beta; \xi\eta) + 2^{-\nu} Q(\gamma\delta; \xi\eta) + 2^{-\mu-\nu} D_0(\alpha\beta, \gamma\delta; \xi\eta),$$
$$\mathcal{E}_{\mu\nu;n}(\alpha\beta, \gamma\delta; \xi\eta)$$
$$= \bar{A}_{\xi\eta} + 2^{-n+1}\{2^{-\mu} Q(\alpha\beta; \xi\eta) + 2^{-\nu} Q(\gamma\delta; \xi\eta)\} \qquad (n>1).$$

The defining equations for C_0, C and D_0 will be readily written down. Their explicit values can be obtained by actual computation, while they are omitted here.

5. Mother-descendant combinations through a single consanguineous marriage. Up to the last paragraph, any consanguineous marriage has never yet been implicated. We now begin to attack the problems concerning a consanguineous marriage. Let μ th and ν th descendants originated collaterally from a mother $A_{\alpha\beta}$ and her same spouse be married consanguineously and then originate themselves an n th descendant $A_{\xi\eta}$. Our present purpose is to determine the probability of combination $(A_{\alpha\beta}; A_{\xi\eta})$ which will be designated by

$$\pi_{\mu\nu;n}(\alpha\beta; \xi\eta) \equiv \bar{A}_{\alpha\beta} K_{\mu\nu;n}(\alpha\beta; \xi\eta).$$

Distinguishing two cases according to $n=1$ and $n>1$, we get the defining equations

$$K_{\mu\nu;1}(\alpha\beta; \xi\eta)$$
$$= \sum K_{\mu\nu}(\alpha\beta; ab, cd) \mathcal{E}(ab, cd; \xi\eta),$$
$$K_{\mu\nu;n}(\alpha\beta; \xi\eta)$$
$$= \sum K_{\mu\nu;1}(\alpha\beta; ab) K_{n-1}(ab; \xi\eta)$$

which yield the final formulas

$$K_{\mu\nu;1}(\alpha\beta; \xi\eta)$$
$$= \bar{A}_{\xi\eta} + L_{\mu\nu} Q(\alpha\beta; \xi\eta) + 2^{-\lambda} T(\alpha\beta; \xi\eta),$$
$$K_{\mu\nu;n}(\alpha\beta; \xi\eta)$$
$$= \bar{A}_{\xi\eta} + L_{\mu\nu} 2^{-n+1} Q(\alpha\beta; \xi\eta) \quad (n>1)$$

valid for any $\mu \geq 1$, $\nu \geq 1$; here we put

$$L_{\mu\nu} = 2^{-\mu} + 2^{-\nu}, \qquad \lambda = \mu+\nu-1.$$

The values of $T(\alpha\beta; \xi\eta) = 2\{K_{11;1}(\alpha\beta; \xi\eta) - K(\alpha\beta; \xi\eta)\}$ are set out as follows:

$$T(ii; ii) = \tfrac{1}{4}(1-i)(2-i), \quad T(ii; ik) = -\tfrac{1}{2}k(1-2i),$$
$$T(ii; kk) = \tfrac{1}{4}k(1+k), \quad T(ii; hk) = \tfrac{1}{2}hk,$$
$$T(ij; ii) = \tfrac{1}{8}(1-2i+2i^2), \quad T(ij; ij) = \tfrac{1}{4}(1-2i-2j+2ij),$$
$$T(ij; ik) = -\tfrac{1}{2}k(1-i), \quad T(ij; kk) = \tfrac{1}{4}k(1+k),$$
$$T(ij; hk) = \tfrac{1}{2}hk.$$

More generally, the reduced probability defined by

$$K_{l|\mu\nu;n}(\alpha\beta; \xi\eta)$$
$$= \sum K_l(\alpha\beta; ab) K_{\mu\nu;n}(ab; \xi\eta)$$

can be determined in the following forms:

$$K_{l|\mu\nu;1}(\alpha\beta; \xi\eta)$$
$$= \bar{A}_{\xi\eta} + 2^{-l} L_{\mu\nu} Q(\alpha\beta; \xi\eta) + 2^{-\lambda} R(\xi\eta) + 2^{-l-\lambda} S(\alpha\beta; \xi\eta),$$

$$K_{l|\mu\nu;n}(\alpha\beta;\xi\eta)$$
$$=\bar{A}_{\xi\eta}+2^{-l-n+1}L_{\mu\nu}Q(\alpha\beta;\xi\eta)\quad (n>1).$$

The values of the quantities defined by $R(\xi\eta)=\sum\bar{A}_{ab}T(ab;\xi\eta)$ and $S(\alpha\beta;\xi\eta)=2\sum Q(\alpha\beta;ab)T(ab;\xi\eta)$ are set out as follows:

$$R(ii)=\frac{1}{2}i(1-i),\qquad R(ij)=-ij;$$
$$S(ii;ii)=\frac{1}{4}(1-i)(2-i),\quad S(ii;iK)=-\frac{1}{2}K(1-2i),$$
$$S(ii;KK)=-\frac{1}{4}K(1-2K),\quad S(ii;hK)=hK,$$
$$S(ij;ii)=\frac{1}{8}(1-2i)^2,\quad S(ij;ij)=-\frac{1}{4}(i+j-4ij),$$
$$S(ij;iK)=-\frac{1}{4}K(1-4i),\quad S(ij;KK)=\frac{1}{4}K(1-2K),$$
$$S(ij;hK)=hK.$$

We now propose to explain a meaning of the quantity

$$L_{\mu\nu}\equiv 2^{-\mu}+2^{-\nu}$$

introduced above, from a view-point of genetics. As shown, the probability $K_{\mu\nu;n}$ $(n>1)$ of mother-descendant combination distant after a consanguineous marriage is expressed in the form

$$K_{\mu\nu;n}(\alpha\beta;\xi\eta)=\bar{A}_{\xi\eta}+L_{\mu\nu}2^{-n+1}Q(\alpha\beta;\xi\eta).$$

On the other hand, the probability K_{n^*} of mother-descendant combination without any consanguineous marriage has been established in the form

$$K_{n^*}(\alpha\beta;\xi\eta)=\bar{A}_{\xi\eta}+2^{-n^*+1}Q(\alpha\beta;\xi\eta).$$

The comparison of these formulas will well interpret a meaning of the factor $L_{\mu\nu}$. In fact, we introduce a positive number $\varrho=\varrho_{\mu\nu}$ by

$$2^{-\rho}=L_{\mu\nu}$$

or

$$\rho=\log L_{\mu\nu}/\log 2$$
$$=\mu+\nu-\log(2^{\mu}+2^{\nu})/\log 2.$$

The probability $K_{\mu\nu;n}$ is then brought into the form $K_{\mu\nu;n}(\alpha\beta;\xi\eta)=\bar{A}_{\xi\eta}+2^{-(\rho+n)+1}Q(\alpha\beta;\xi\eta)$ which coincides formally with $K_{\rho+n}(\alpha\beta;\xi\eta)$, though the number ρ is, in general, i. e. unless $\mu=\nu$, not equal to an integer.

By reason of their own meanings explained just above, we may call the number $\rho_{\mu\nu}$ an equivalent generation-number and the factor $L_{\mu\nu}$ a contracting factor. As readily shown, there holds

$$\mathrm{Min}(\mu,\nu)-1\leqq\rho_{\mu\nu}\leqq\mathrm{Max}(\mu,\nu)-1.$$

6. Mother-descendant combinations through several consanguineous marriages. We have discussed in the preceding paragraph the mother-descendant combinations through a single consanguineous marriage. We now attempt to generalize the problem to the case where several consanguineous marriages intervene between a mother and her descendant. Our present purpose is thus to determine the probability of a combination which consists of an original individual $A_{\alpha\beta}$ and of its descendant $A_{\xi\eta}$, consanguineous marriages interjacent between them occurring t times, designated, with an easily comprehensible routine notation, by

$$\pi_{\mu_1\nu_1;n_1|\cdots|\mu_t\nu_t;n_t}(\alpha\beta;\xi\eta)$$
$$\equiv\bar{A}_{\alpha\beta}K_{\mu_1\nu_1;n_1|\cdots|\mu_t\nu_t;n_t}(\alpha\beta;\xi\eta)$$

or briefly, provided no confusion can arise, by

$$\pi_{(\mu\nu;n)_t}(\alpha\beta;\xi\eta)\equiv\bar{A}_{\alpha\beta}K_{(\mu\nu;n)_t}(\alpha\beta;\xi\eta).$$

By definition, the reduced probability is given by

$$K_{(\mu\nu;n)_t}(\alpha\beta;\xi\eta)$$
$$=\sum\prod_{x=1}^{t}K_{\mu_x\nu_x;n_x}(a_{x-1}b_{x-1};a_xb_x)$$

with $a_0b_0\equiv\alpha\beta$ and $a_tb_t\equiv\xi\eta$, where the summation extends over all the possible sets of $t-1$ genotypes $A_{a_xb_x}$ $(x=1,\cdots,t-1)$. It satisfies therefore a recurrence equation

$$K_{(\mu\nu;n)_t}(\alpha\beta;\xi\eta)$$
$$=\sum K_{(\mu\nu;n)_{t-1}}(\alpha\beta;ab)K_{\mu_t\nu_t;n_t}(ab;\xi\eta).$$

In a special case where the n's are all equal to unity, it can be proved by induction that there holds the formula

$$K_{(\mu\nu;1)_t}(\alpha\beta;\xi\eta)$$

$$=\bar{A}_{\xi\eta}+2^{-t+1}\Lambda_t Q(\alpha\beta;\xi\eta)$$

$$+4u_t R(\xi\eta)+2v_t S(\alpha\beta;\xi\eta)+2w_t T(\alpha\beta;\xi\eta);$$

$$\lambda_r=\mu_r+\nu_r-1,$$

$$\Lambda_t=\prod_{r=1}^{t}(2^{-\mu_r}+2^{-\nu_r}),$$

$$u_t=\sum_{r=1}^{t-1}\prod_{s=r+1}^{t}2^{-\lambda_s-2},$$

$$v_t=\sum_{r=1}^{t-1}2^{-r+1}\Lambda_r\prod_{s=r+1}^{t}2^{-\lambda_s-2},$$

$$w_t=\prod_{s=1}^{t}2^{-\lambda_s-2};$$

the formula remains valid even for $t=1$ provided an empty sum is understood to be equal to zero. On the other hand, we get the formula

$$K_{(\mu\nu;n)_t}(\alpha\beta;\xi\eta)=\bar{A}_{\xi\eta}+2^{-N_t+1}\Lambda_t Q(\alpha\beta;\xi\eta);$$

$$N_t=\sum_{r=1}^{t}n_r,$$

which remains valid regardless of the values of the n_r's with $1\leqq r<t$ provided merely $n_t>1$.8)

The probability of the form $\pi_{\ell|(\mu\nu;n)_t|(\mu'\nu';1)_{t'}}(\alpha\beta;\xi\eta)$ with $n_r\geqq 1$ for $1\leqq r<t$ and $n_t>1$ can also be treated in a similar manner.

We now proceed to illustrate a very notable phenomenon concerning the distribution of genotypes in a generation after consanguineous marriages. Up to now we have dealt with several mother-descendant combinations $(A_{\alpha\beta};A_{\xi\eta})\dots$ of which the probabilities are designated by

$$\pi_{\dots}(\alpha\beta;\xi\eta)=\bar{A}_{\alpha\beta}K_{\dots}(\alpha\beta;\xi\eta),$$

respectively. For each combination, if we eliminate mother's type by summing up over whole possible range, then we obtain the probability of descendant $A_{\xi\eta}$ alone, i. e. the relative frequency of genotype $A_{\xi\eta}$ in the generation of descendant, which will be designated by

$$\bar{A}_{\dots}(\xi\eta)=\sum\pi_{\dots}(ab;\xi\eta)$$

$$\equiv\sum\bar{A}_{ab}K_{\dots}(ab;\xi\eta).$$

In case of a simple mother-descendant combination, with probability π_n, without intervention of any consanguineous marriage, it is given by

$$\bar{A}_n(\xi\eta)=\sum\pi_n(ab;\xi\eta)=\bar{A}_{\xi\eta}.$$

This shows that the distribution of $A_{\xi\eta}$ in the generation of descendant then coincides just with one in the original generation.

We next consider the mother-descendant combination $(A_{\alpha\beta};A_{\xi\eta})_{(\mu\nu;1)_t}$. The distribution in the generation of descendant is then given by

$$\bar{A}_{(\mu\nu;1)_t}(\xi\eta)$$

$$=\bar{A}_{\xi\eta}+4(u_t+w_t)R(\xi\eta);$$

$$u_t+w_t=\sum_{r=0}^{t-1}\prod_{s=r+1}^{t}2^{-\lambda_s+2}.$$

In case of mother-descendant combination $(A_{\alpha\beta};A_{\xi\eta})_{(\mu\nu;n)_t}$ with $n_t>1$, we get simply

$$\bar{A}_{(\mu\nu;n)_t}(\xi\eta)=\bar{A}_{\xi\eta}.$$

The probability of the form $\bar{A}_{\ell|(\mu\nu;n)_t|(\mu'\nu';1)_{t'}}(\xi\eta)$ with $n_t>1$ can also be treated similarly.

We can thus assert the following proposition: The distribution of genotypes deviates in a generation immediate after any consanguineous marriage, while the deviation disappears in a generation distant after a consanguineous marriage. The deviation of frequency of genotype $A_{\xi\eta}$ in the former case amounts to $R(\xi\eta)$ multiplied by a definite positive factor $4(u_t+w_t)$ depending only on the generation-numbers. Consequently, the frequency of $A_{\xi\eta}$ in the generation of descendant increases or decreases according to whether it is homozygous or heterozygous. Thus, we are led to a conjecture concerning practical observations. Suppose, in fact, there exists a population which is isolated hereditarily from its neighboring populations and in which consanguineous marriages take place to a considerable extent. If one observes frequencies on an inherited character in the population, it will then be found out that the frequencies of homozygous and heterozygous types are more than and less than those expected from the supposition of an equilibrium state, respectively. Moreover, the deviation is probabilistically given quantitatively as above.

Asymptotic behaviors of $K_{(\mu\nu;n)_t}$ as each among μ_r, ν_r and n_r tends to infinity, can be readily deduced from its expression derived above. We here supplement an asymptotic behavior of $K_{(\mu\nu;1)_t}$ as $t\to\infty$, which depends on the choice of a sequence $\{\mu\nu\}\equiv\{\mu_r,\nu_r\}_{r=1}^{\infty}$. The following result can be deduced: Introducing the notations defined by

$$\bar{\lambda}=\overline{\lim_{r\to\infty}}\,\lambda_r,\qquad \underline{\lambda}=\underline{\lim_{r\to\infty}}\,\lambda_r,$$

$$\underline{\upsilon}=\frac{4}{2^{\bar{\lambda}+2}-1},\qquad \bar{\upsilon}=\frac{4}{2^{\underline{\lambda}+2}-1},$$

for which there hold evidently the inequalities $1\leqq\underline{\lambda}\leqq\bar{\lambda}\leqq\infty$, $0\leqq\underline{\upsilon}\leqq\bar{\upsilon}\leqq 4/7$, we can assert that there hold the estimations

$$\begin{aligned}\bar{A}_{ii}+\underline{\upsilon}\,R(ii)&\leqq\underline{\lim_{t\to\infty}}\,K_{(\mu\nu;1)_t}(\alpha\beta;ii)\\&\leqq\overline{\lim_{t\to\infty}}\,K_{(\mu\nu;1)_t}(\alpha\beta;ii)\\&\leqq\bar{A}_{ii}+\bar{\upsilon}\,R(ii),\end{aligned}$$

$$\begin{aligned}\bar{A}_{ij}+\bar{\upsilon}\,R(ij)&\leqq\underline{\lim_{t\to\infty}}\,K_{(\mu\nu;1)_t}(\alpha\beta;ij)\\&\leqq\overline{\lim_{t\to\infty}}\,K_{(\mu\nu;1)_t}(\alpha\beta;ij)\\&\leqq\bar{A}_{ij}+\underline{\upsilon}\,R(ij).\end{aligned}$$

REFERENCES

1) Y. Komatu, Probability-theoretic investigations on inheritance. I-XVI. Proc. Japan Acad. **27-29** (1951-1953).

2) Here we confine ourselves to a brief announcement of the results. The details on the formulas as well as their consequences will be fully discussed in our subsequent paper which will appear shortly: Y. Komatu and H. Nishimiya, Probabilistic investigations on inheritance in consanguineous families. Bull. Tokyo Inst. Tech. (1954), 1-66, 67-152, 153-222 et seq.

3) Cf. a paper in 1): IV. Mother-child combinations. **27** (1951), 587-620.

4) Another deductive derivation of the formula will be found in Y. Komatu and H. Nishimiya, Lineal combinations on a Mendelian inherited character. Rep. Stat. Appl. Res., JUSE **3** (1953), 13-22. On the other hand, a stochastic treatment of the problems in case of $n=1$ will be made in Y. Komatu, Mother-child combinations concerning an inherited character after a panmixia. Journ. Math. Soc. Japan **6** (1954), 283-302, and Further discussions on mother-child combinations concerning an inherited character after a panmixia. Rep. Stat. Appl. Res., JUSE **3** (1954), 42-53.

5) Cf. a paper in 1): I. Distribution of genes. **27** (1951), 371-377.

6) Cf. a paper in 1): V. Brethren combinations. **27** (1951), 689-699.

7) Cf. a paper cited in 3).

8) We have implicitly supposed that none among the n_r's with $1\leqq r\leqq t$ vanishes. However, we could really permit degenerate cases where some of the n_r's with $1\leqq r<t$ vanish. The discussions will then become too complicated to perform in a restricted space. These problems, together with various related ones, will be fully developed in the paper cited in 2).

Department of Mathematics,
Tokyo Institute of Technology.

(*) Received October 30, 1954.