SOME CLASSES OF POSITIVE SOLUTIONS OF $\Delta u = Pu$ ON RIEMANN SURFACES, I

By Mitsuru OZAWA

§0. Introduction.

In our earlier paper [1] we had introduced a method to classify the Riemann surfaces into the various types making use of the following leading idea: Under what conditions does there exist a solution of a partial differential equation of elliptic type $\Delta u = Pu$ on a given Riemann surface?

Recently Lauri Myrberg [1][2] showed that on every Riemann surface there always exists the Green function of $\Delta u = Pu$. This is an elegant result in this tendency.

Martin topology played an important role in Heins' investigation for the structure of ideal boundary and for a class of positive harmonic functions on an end in Heins' sense. An analogue of Heins' investigation was stated and a maximality of a class of positive harmonic functions on a subregion with non-compact relative boundary was established in our papers [2][3].

In the present paper we shall investigate the structure of the ideal boundary using the positive solutions of $\Delta u = Pu$ instead of positive harmonic functions. In the way of construction of the theory we are obliged to add a more restriction than in the harmonic case in order to establish a full parallelism between the theory in Heins' paper and that in the present paper. In harmonic case, when F belongs to the class O_G, the maximum-minimum principle in the extended sense for any bounded harmonic function holds and we can conclude that each minimal positive harmonic function in Martin's sense on any Heins' end is obtained by a suitable limiting process $m \to \infty$ ($p_m \to$ ideal boundary) for the Green function $G(z, p_m)$ on a given end.

Riemann surfaces F in considerations are the ones in the sense of Weyl-Radó. Differential equation considered here is the following type

$$(A)\quad \frac{\partial^2}{\partial x^2} u(x,y) + \frac{\partial^2}{\partial y^2} u(x,y) = P(x,y)\, u(x,y),$$

where $z = x+iy$ is a local parameter at a point p on F, $P(x,y)$ is a real continuous function of (x,y) being except at most at a countable set of zero-points with accumulation points lying only on the ideal boundary and having continuous partial derivatives. We assume moreover that if we change the local parameter z to z', then $P(z)$ changes as follows:

$$P(z) = P(z')\left|\frac{dz'}{dz}\right|^2.$$

For this type of differential equation (A) we can prove the existence of the Green function and the solvability and uniqueness of the first boundary value problem on compact subsurface. Moreover Harnack's convergence theorem is valid. For the precise formulations one can refer to the well-known classical papers. Recently L. Myrberg [1][2] gave a precise method of formulations.

§1. Generalities and known results.

Let $\{F_n\}$ $n = 0,1,2,\cdots$ be an exhaustion of F in the ordinary sense. Let F_n have a compact analytic curve Γ_n as its relative boundary.

Let $G_n(z,\zeta)$ be the Green function of (A) on F_n satisfying the following conditions:

1. $\Delta G_n(z,\zeta) = P(z)\cdot G_n(z,\zeta)$ on F_n,
2. $G_n(z,\zeta)$ has continuous partial derivatives of second order on $F_n - \zeta$,

3. $G_n(z,\zeta)+\log|z-\zeta|$ is bounded in the neighborhood of ζ,

4. $G_n(z,\zeta)\equiv 0$ on Γ_n.

Let $G(z,\zeta)=\lim_{n\to\infty}G_n(z,\zeta)$ call the Green function of (A) on F. Myrberg [2] showed that for any F there always exists uniquely the Green function of (A) on F. Essential step of his argument is to establish the following identity

$$\iint_{F_n} G_n(z,\zeta)P(z)\,d\sigma = 2\pi - \int_{\Gamma_n}\frac{\partial}{\partial\nu}\hat{\tau}_n(z,\zeta)\,ds,$$

which leads to

$$\iint_F G(z,\zeta)P(z)d\sigma = 2\pi\left(1-\Omega(\zeta,\Gamma)\right),$$

where $\Omega(\zeta,\Gamma)=\lim_{n\to\infty}\Omega_n(\zeta,\Gamma_n)$ such that $\Omega_n(\zeta,\Gamma_n)=1$ on Γ_n and satisfies the differential equation (A). If $\Omega(\zeta,\Gamma)\equiv 0$, then we have

$$\iint_F G(z,\zeta)P(z)\,d\sigma = 2\pi.$$

Let $\omega_n(z,\Gamma_n,F_n-\bar{F}_0)$ be a finite solution of (A) on $F_n-\bar{F}_0$, being identically 1 on Γ_n and 0 on Γ_0. $F\in O_\omega$ (resp. $F\in O_\Omega$) means that $\omega(z,\Gamma,F-\bar{F}_0)=\lim_{n\to\infty}\omega_n(z,\Gamma_n,F_n-\bar{F}_0)$ (resp. $\Omega(z,\Gamma)$) reduces to constant zero. We had already shown that $O_\omega=O_\Omega$. This result remains valid in spite of Myrberg's result. If there is no non-constant bounded solution of (A) on F, then we denote $F\in O_B$. $O_B=O_\omega$ is also valid. Moreover if $F\in O_B$, then on any subregion **D** with compact analytic relative boundary C there is only one linearly independent solution of (A) bounded non-constant in D satisfying $=1$ on C, and vice versa. If $F\in O_B$ and u(z) is a bounded non-negative solution of (A) on D (with compact or non-compact relative boundary C), then the **maximum** principle holds, that is,

$$\sup_D u(z) = \sup_C u(z).$$

Evidently the minimum principle in this case does not hold.

Let $\omega_n'(z,\Gamma_0,F_n-\bar{F}_0)$ and $h_n(z,\Gamma_0+\Gamma_n,F_n-\bar{F}_0)$ be two finite solutions of (A) on $F_n-\bar{F}_0$ such that

$$\omega_n'(z,\Gamma_0,F_n-\bar{F}_0)=\begin{cases}1 & \text{on } \Gamma_0\\ 0 & \text{on } \Gamma_n\end{cases}$$

and

$$h_n(z,\Gamma_0+\Gamma_n,F_n-\bar{F}_0)=1 \quad \text{on } \Gamma_0+\Gamma_n,$$

respectively. Evidently

$$h_n(z,\Gamma_0+\Gamma_n,F_n-\bar{F}_0) = \omega_n(z,\Gamma_n,F_n-\bar{F}_0)+\omega_n'(z,\Gamma_0,F_n-\bar{F}_0)$$

and

$$h_n(z)<h_m(z),\quad \omega_n'(z)>\omega_m'(z)$$

hold for $n>m$. Let

$$h(z,\Gamma_0+\Gamma,F-\bar{F}_0)=\lim_{n\to\infty}h_n(z,\Gamma_0+\Gamma_n,F_n-\bar{F}_0)$$

$$\omega'(z,\Gamma_0,F-\bar{F}_0)=\lim_{n\to\infty}\omega_n'(z,\Gamma_0,F_n-\bar{F}_0).$$

Save when the contrary is explicitly mentioned, an admitted Riemann surface **of** infinite genus in the present paper subjects to the following two conditions:

(i) the surface has precisely one ideal boundary element,

(ii) the surface belongs to the class O_Ω.

An end is a subregion of an admitted surface whose complement is compact. Without loss of generality we may assume that the relative boundary of an end consists of a finite number of compact smooth Jordan curves.

§ 2. Green function and its behavior.

Now we shall classify the ideal boundary Γ of an admitted Riemann surface F into the following three types:

(a) $\lim_{z\to\Gamma}\omega'(z,\Gamma_0,F-\bar{F}_0)>0$,

(b) $\underline{\lim}_{z\to\Gamma}\omega'(z,\Gamma_0,F-\bar{F}_0)=0$ and $\overline{\lim}_{z\to\Gamma}\omega'(z,\Gamma_0,F-\bar{F}_0)>0$,

(c) $\overline{\lim}_{z\to\Gamma}\omega'(z,\Gamma_0,F-\bar{F}_0)=0$.

This classification is evidently a local property and closely depends upon the choice of $P(z)$. We shall state a fact that either case (b) or (c) **occurs, provided that** $P(z)$ satisfies a condition $\iint_F P(z)\,d\sigma=\infty$. Because we have already seen that

$$\iint_F G(z,\zeta)P(z)\,d\sigma = 2\pi$$

If $\varliminf_{z\to\Gamma} G(z,\zeta) \geq \delta > 0$ occurs, then we have

$$\iint_{F_n} G(z,\zeta) P(z)\, d\sigma + \delta \iint_{F-F_n} P(z)\, d\sigma \leq 2\pi ,$$

which is absurd by $\iint_{F-F_n} P(z)\, d\sigma = \infty$.

Lemma 2.1. Let u(z) be any bounded solution of (A) on $F-\bar{F}_0$ being non-negative on Γ_0. Then $u(z) \geq 0$ on $F-\bar{F}_0$.

Lemma 2.2. Let u(z) be any bounded solution of (A) on $F-\bar{F}_0$ being non-negative and $\not\equiv 0$ on Γ_0, then $\varliminf_{z\to\Gamma} u(z) > 0$ (resp. $= 0$) implies that $\varliminf_{z\to\Gamma} \omega'(z,\Gamma,F-\bar{F}_0) > 0$ (resp. $= 0$) and vice versa. The same holds for $\varlimsup_{z\to\Gamma}$.

Proof of Lemma 2.1. Let D_- be a connected component on which $u(z) < 0$. Then on D_- there is at least one positive solution of (A) being bounded and constant zero on the relative boundary of D_-, which leads to a fact $F \notin O_\Omega$ by theorem 4.2 in our previous paper [1]. This is a fact to be rejected by the assumption $F \in O_\Omega$.

Proof of Lemma 2.2. At first we shall assume that u(z) is strictly positive on Γ_0. Then by Lemma 2.1 and the maximum principle we have

$$0 \leq u(z) \leq \max_{\Gamma_0} u(z) = M$$

on $F-\bar{F}_0$. By the assumption $F \in O_\Omega$, we have

$$h(z,\Gamma_0+\Gamma,F-\bar{F}_0) = \omega'(z,\Gamma_0,F-\bar{F}_0)$$

and

$$M h(z) \geq u(z) \geq m h(z),$$

$$M = \max_{\Gamma_0} u(z),\ m = \min_{\Gamma_0} u(z) \gneqq 0 .$$

This implies the desired fact.

Next we shall prove a proposition that we may assume that u(z) is strictly positive on Γ_0. Let K_z be a parameter disc around z on $F-\bar{F}_0$. Let $G_{K_z}(\zeta,z)$ be the Green function of (A) on K_z. Then we have

$$u(z) = \frac{1}{2\pi}\int_{K_z} u(\zeta) \frac{\partial}{\partial\nu} G_{K_z}(\zeta,z)\, ds_\zeta .$$

Moreover we see that

$$\frac{\partial}{\partial\nu} G_{K_z}(\zeta,z) \geq \delta > 0$$

on the periphery of K_z, if z belongs to a closed subdomain D of K_z. Thus $u(\zeta) \geq 0$, $\not\equiv 0$ on K_z implies that $u(z) \geq N > 0$, $z \in D$. This fact implies that a set E on which $u(z) = 0$ is empty on $F-\bar{F}_0$. Therefore we can select a suitable subend D_1 of $F-\bar{F}_0$ such that $u(z) \geq m > 0$ on the relative boundary of D_1.

Let $G(z,\zeta)$ be the Green function of (A) on F (or $F-\bar{F}_0$), then by Lemma 2.2, we have that $\varliminf_{z\to\Gamma} G(z,\zeta) > 0$ (resp. $= 0$) implies to $\varliminf_{z\to\Gamma} \omega'(z,\Gamma_0,F-\bar{F}_0) > 0$ (resp. $= 0$) and that $\varlimsup_{z\to\Gamma} G(z,\zeta) > 0$ (resp. $= 0$) implies to $\varlimsup_{z\to\Gamma} \omega'(z,\Gamma_0,F-\bar{F}_0) > 0$ (resp. $= 0$).

If the case (a) occurs, then $\varliminf_{z\to\Gamma} G(z,\zeta) > 0$ and hence $\varliminf_{z\to\Gamma} G(\zeta,z) > 0$. Therefore we can select a suitable subsequence $\{z_n\}$ such that $\lim_{n\to\infty} G(\zeta,z_n)$ exists with $z_n \to \Gamma$ and is a non-negative unbounded solution of (A) on F (or $F-\bar{F}_0$).

§ 3. Dimensions of ideal boundary.

Let $\mathfrak{P}_\square$ be a class of positive solutions u of (A) on a domain $\square$. If $\square$ has a relative boundary, then we shall add a further restriction such that $u = 0$ on the relative boundary of $\square$. In this section we shall restrict to the case that $\square$ is an end or an original admitted Riemann surface.

Between $\mathfrak{P}_{F-\bar{F}_0}$ and $\mathfrak{P}_{F-\bar{F}_1}$ there exists a one-to-one and onto mapping. Let u(z) be a member of $\mathfrak{P}_{F-\bar{F}_0}$, then we put that $T_u(z)$ is the largest minorant of u(z) among $\mathfrak{P}_{F-\bar{F}_1}$. Let v(z) be a member of $\mathfrak{P}_{F-\bar{F}_1}$, then we put that $S_v(z)$ is the least majorant of v(z) among $\mathfrak{P}_{F-\bar{F}_0}$. Evidently these operations have their senses. And moreover we can conclude that

$$ST_u = u \quad\text{and}\quad TS_v = v$$

and S and T operations are one-to-one, onto and positively linear mappings between $\mathfrak{P}_{F-\bar{F}_0}$ and $\mathfrak{P}_{F-\bar{F}_1}$.

Thus we can define a sort of dimension $\mathrm{dem}_1(\Gamma)$ as the maximal cardinal number of linearly independent solutions of (A) belonging to $\mathfrak{P}_{F-\bar{F}_0}$.

Let us now define dimension $\dim_2(\Gamma)$ of the second kind of Γ as follows: Let $G(z,\zeta)$ be the Green function of (A) on $F-\bar{F}_0$. By the result stated in §2, three cases can happens, that is, either (a) $\varliminf_{\zeta\to\Gamma} G(z,\zeta)>0$ or (b) $\varliminf_{\zeta\to\Gamma} G(z,\zeta)=0$ but $\varlimsup_{\zeta\to\Gamma} G(z,\zeta)>0$ or (c) $\varlimsup_{\zeta\to\Gamma} G(z,\zeta)=0$. And we denote these three cases, for convenience, by $\Gamma\in(a)$, $\Gamma\in(b)$ and $\Gamma\in(c)$, respectively.

If $\Gamma\in(c)$ happens, then we put $\dim_2(\Gamma)=0$. If either (a) or (b) happen, then we put that $\dim_2(\Gamma)$ is maximal cardinal number of linearly independent limit functions of $G(z,\zeta_n)$ with $\zeta_n\to\Gamma$ belonging to $\mathfrak{P}_{F-\bar{F}_0}$. This cardinal number $\dim_2(\Gamma)$ is also a local invariant. Evidently $\dim_2(\Gamma) \leqq \dim_1(\Gamma)$ holds. In cases (b) and (c) we see $\dim_2(\Gamma) \lneqq \dim_1(\Gamma)$. For the case (a) we now offer an unsolved problem:

Does there hold a relation $\dim_1(\Gamma) = \dim_2(\Gamma)$?
This problem relates closely to the following problem which is wider than the above one.

Can any minimal positive solution of (A) be expressed as a suitable limit function such that

$$\lim_{n\to\infty} \frac{G_{F-\bar{F}_0}(z,\zeta_n)}{G_{F-\bar{F}_0}(z_0,\zeta_n)} \quad ?$$

In this problem we donot require that F is of null-boundary of any sort. Here minimal positive solution u(z) of (A) on $F-\bar{F}_0$ means the one belonging to $\mathfrak{P}_{F-\bar{F}_0}$ with a postulate for minimality: If $v(z) \leqq u(z)$ remains valid on $F-\bar{F}_0$, then $v(z) = k\,u(z)$, $k>0$, unless $v\equiv 0$.

If $F\in O_\Omega$ and $\Gamma\in(a)$, then $F\in O_G^{(h)}$, that is, F is of null-boundary in Nevanlinna's sense. In fact, let $F\notin O_G^{(h)}$, then the harmonic measure of $F-\bar{F}_0$ with regard to Γ_2 : $\omega^{(h)}(z,\Gamma_2,F-\bar{F}_2)$ satisfies $\varliminf_{z\to\Gamma}\omega^{(h)}(z)=1$ and hence $\varliminf_{z\to\Gamma}(1-\omega^{(h)}(z))=0$ is valid. On the other hand $(1-\omega^{(h)}(z))\geqq k\,g^{(h)}_{F-\bar{F}_0}(z,\zeta)\geqq 0$ holds on $F-\bar{F}_0$ for $\zeta\in F-\bar{F}_0$, where $g^{(h)}_{F-\bar{F}_0}(z,\zeta)$ is the harmonic Green function on $F-\bar{F}_0$ with pole at ζ. This leads to a fact $\varliminf_{z\to\Gamma} g^{(h)}_{F-\bar{F}_0}(z,\zeta)=0$, which implies that $\varliminf_{z\to\Gamma} G_{F-\bar{F}_0}(z,\zeta)=0$. This is absurd.

Let $\mathbb{P}_{F-\bar{F}_0}$ be a class of positive harmonic functions on $F-\bar{F}_0$ with vanishing boundary value on Γ_0. We shall now denote its dimension in Heins' sense by $\dim_0(\Gamma)$.

In this section we shall restrict to the case that $F\in O_\Omega$ and $\Gamma\in(a)$.

Next we shall construct a positively linear mapping $\mathfrak{T}$ and its inverse mapping $\mathfrak{S}$ from $\mathbb{P}_{F-\bar{F}_0}$ into $\mathfrak{P}_{F-\bar{F}_0}$.

Let u be a minimal positive harmonic function of $\mathbb{P}_{F-\bar{F}_0}$, then $u=\lim_{n\to\infty} g^{(h)}_{F-\bar{F}_0}(z,\zeta_n)$ for a suitable non-compact sequence $\{\zeta_n\}$. Evidently $G_{F-\bar{F}_0}(z,\zeta_n)\leqq g^{(h)}_{F-\bar{F}_0}(z,\zeta_n)$. And hence we can extract a suitable subsequence $\{\zeta_{n_m}\}$ such that $v=\lim_{m\to\infty} G_{F-\bar{F}_0}(z,\zeta_{n_m})$ exists and is not constant zero on $F-\bar{F}_0$, for $\lim_{m\to\infty} G_{F-\bar{F}_0}(\zeta_{n_m},z)>0$ by $\Gamma\in(a)$. And we shall put that $\mathfrak{T}_u$ is the largest minorant of u belonging to $\mathfrak{P}_{F-\bar{F}_0}$, then $\mathfrak{T}_u>0$ for $0<v\leqq\mathfrak{T}_u<u$, $v\in\mathfrak{P}_{F-\bar{F}_0}$. Evidently $\mathfrak{T}_u$ is a positively linear mapping from $\mathbb{P}_{F-\bar{F}_0}$ into $\mathfrak{P}_{F-\bar{F}_0}$.

Let v be a limit function $\lim_{n\to\infty} G_{F-\bar{F}_0}(z,\zeta_n)$ along a suitable non-compact sequence $\{\zeta_n\}$. Then we shall construct a sequence of the harmonic Green functions $g^{(h)}_{F-\bar{F}_0}(z,\zeta_n)$. Since $F\in O_G^{(h)}$, we can extract a subsequence such that $u=\lim_{m\to\infty} g^{(h)}_{F-\bar{F}_0}(z,\zeta_{n_m})$ exists on $F-\bar{F}_0$ in the wider sense and $\not\equiv\infty$. Let $\mathfrak{S}_v$ be the least harmonic majorant of v, then $\mathfrak{S}_v$ surely exists and satisfies $v<\mathfrak{S}_v \leqq u \not\equiv \infty$. $\mathfrak{S}$ operation is evidently positively linear on $\mathfrak{G}_{F-\bar{F}_0}$, where $\mathfrak{G}_{F-\bar{F}_0}$ is a class of positive solutions of (A) belonging to $\mathfrak{P}_{F-\bar{F}_0}$ and being expressible as a suitable positively linear combination of any linearly independent positive solutions of (A) each of which is obtained as a limit function $\lim_{n\to\infty} G_{F-\bar{F}_0}(z,\zeta_n)$ along a suitable non-compact sequence $\{\zeta_n\}$.

Next we shall prove that $\mathfrak{T}$ and $\mathfrak{S}$ operations are both one-to-one and onto mapping between $\mathbb{P}_{F-\bar{F}_0}$ and $\mathfrak{G}_{F-\bar{F}_0}$.

$\mathcal{S}$ operation determines uniquely an element of $\mathbb{P}_{F-\bar{F}_0}$ for any given element of $\mathfrak{G}_{F-\bar{F}_0}$. Let $\mathcal{S}_v^{(1)}$ and $\mathcal{S}_v^{(2)}$ be two elements of $\mathbb{P}_{F-\bar{F}_0}$ corresponding to a given element v of $\mathfrak{G}_{F-\bar{F}_0}$ and there be no majorant relation between $\mathcal{S}_v^{(1)}$ and $\mathcal{S}_v^{(2)}$. Let $U=\min(\mathcal{S}_v^{(1)},\mathcal{S}_v^{(2)})$ on $F-\bar{F}_0$, then U is a superharmonic function such that $U \lneqq \mathcal{S}_v^{(1)}$ and $U \lneqq \mathcal{S}_v^{(2)}$. Let U_n be a harmonic function such that $U_n = U$ on $\Gamma_n+\Gamma_0$, then $U_n \geqq U$ and $U_n \leqq \mathcal{S}_v^{(1)}, \mathcal{S}_v^{(2)}$ on $F_n-\bar{F}_0$. And moreover U_n monotonically increases with increasing n. Thus $\lim_{n\to\infty} U_n$ exists and we shall denote this by $U^{(k)}$. Then $U^{(k)} > v$ and $U^{(k)} < \mathcal{S}_v^{(1)}$ and $U^{(k)} < \mathcal{S}_v^{(2)}$ hold on $F-\bar{F}_0$. This is impossible by the definitions of $\mathcal{S}_v^{(1)}$ and $\mathcal{S}_v^{(2)}$. Thus we see that $\mathcal{S}_v^{(1)} \equiv \mathcal{S}_v^{(2)}$ on $F-\bar{F}_0$.

$\mathfrak{T}$ operation determines uniquely an element of $\mathfrak{P}_{F-\bar{F}_0}$ for any given element of $\mathbb{P}_{F-\bar{F}_0}$. Let $\mathfrak{T}_u^{(1)}$ and $\mathfrak{T}_u^{(2)}$ be two elements of $\mathfrak{P}_{F-\bar{F}_0}$ corresponding to a given element u of $\mathbb{P}_{F-\bar{F}_0}$. Let V^n be a positive solution of (A) on $F-\bar{F}_0$ such that $V^n = \max(\mathfrak{T}_u^{(1)}, \mathfrak{T}_u^{(2)})$ on $\Gamma_n+\Gamma_0$. Then evidently $V^n \geqq \max(\mathfrak{T}_u^{(1)}, \mathfrak{T}_u^{(2)})$ and $V^n < V^m$ if $n < m$ and $V^n \leqq u$ for any n. And hence $V = \lim_{n\to\infty} V^n$ exists and is a solution of (A) on $F-\bar{F}_0$. If $\mathfrak{T}_u^{(1)} \not\equiv \mathfrak{T}_u^{(2)}$ and there is no majorant relation between $\mathfrak{T}_u^{(1)}$ and $\mathfrak{T}_u^{(2)}$, then $V > \mathfrak{T}_u^{(1)}$ and $V > \mathfrak{T}_u^{(2)}$ remain valid on $F-\bar{F}_0$. This is impossible by definitions of $\mathfrak{T}_u^{(1)}$ and $\mathfrak{T}_u^{(2)}$.

$\mathfrak{T}_{u_1} \not\equiv \mathfrak{T}_{u_2}$ holds unless $u_1 = u_2$. Since $\mathfrak{T}_u$ has the sense for any $u \in \mathbb{P}_{F-\bar{F}_0}$ and $u > \mathfrak{T}_u$, $\mathcal{S}\mathfrak{T}_u$ can be defined and $\mathcal{S}\mathfrak{T}_u \leqq u$ holds even if $\mathfrak{T}_u \in \mathfrak{G}_{F-\bar{F}_0}$ does not yet guaranteed. Conversely, if $\mathcal{S}_v$ has the sense for an element $v \in \mathfrak{P}_{F-\bar{F}_0}$, $\mathfrak{T}\mathcal{S}_v$ can be defined and $\mathcal{S}_v \in \mathbb{P}_{F-\bar{F}_0}$ and $\mathfrak{T}\mathcal{S}_v \geqq v$ holds on $F-\bar{F}_0$. Let u be $\mathcal{S}_v$, then $\mathcal{S}\mathfrak{T}\mathcal{S}_v \leqq \mathcal{S}_v$ and $\mathcal{S}\mathfrak{T}\mathcal{S}_v \geqq \mathcal{S}_v$ on $F-\bar{F}_0$. This leads to a relation $\mathcal{S}\mathfrak{T}\mathcal{S}_v = \mathcal{S}_v$ on $F-\bar{F}_0$. Let v be $\mathfrak{T}_u$, then $\mathfrak{T}\mathcal{S}\mathfrak{T}_u \geqq \mathfrak{T}_u$ and $\mathfrak{T}\mathcal{S}\mathfrak{T}_u \leqq \mathfrak{T}_u$, and hence $\mathfrak{T}\mathcal{S}\mathfrak{T}_u = \mathfrak{T}_u$ on $F-\bar{F}_0$. If u is minimal in Martin's sense, then $ku = \mathcal{S}\mathfrak{T}_u$ with $0 < k \leqq 1$. Hence we have $k\mathfrak{T}_u = \mathfrak{T}\mathcal{S}\mathfrak{T}_u = \mathfrak{T}_u$ which leads to a fact that $k = 1$, $\mathcal{S}\mathfrak{T}_u = u$. For any minimals and hence their positively linear combination, that is, any elements of $\mathbb{P}_{F-\bar{F}_0}$, $\mathcal{S}\circ\mathfrak{T} = \mathbf{I}$ holds. Therefore, $\mathfrak{T}_{u_1} \equiv \mathfrak{T}_{u_2}$ implies that $u_1 \equiv u_2$.

We shall give another exposition for $\mathcal{S}$ and $\mathfrak{T}$ operations. If $\mathcal{S}_v$ has its sense for an element $v \in \mathfrak{P}_{F-\bar{F}_0}$, then we can define a relating harmonic function S_v by $\lim_{n\to\infty} v^n$, where v^n is a harmonic function on $F_n-\bar{F}_0$ such that $v^n = v$ on $\Gamma_n+\Gamma_0$. Then $v^n \leqq \mathcal{S}_v$ for any n and $v^n \leqq v^m$ if $n < m$ and hence $\lim_{n\to\infty} v^n$ exists and $\not\equiv \infty$ and hence $S_v \in \mathbb{P}_{F-\bar{F}_0}$. Evidently $S_v \leqq \mathcal{S}_v$ and $S_v \geqq \mathcal{S}_v$, since S_v is a harmonic majorant of v. Therefore $S_v \equiv \mathcal{S}_v$. Let u belong to $\mathbb{P}_{F-\bar{F}_0}$, then $\mathfrak{T}_u$ (>0) exists. Let u^n be a positive solution of (A) on $F_n-\bar{F}_0$ such that $u^n = u$ on $\Gamma_n+\Gamma_0$, then $\mathfrak{T}_u \leqq u^n < u$ on $F_n-\bar{F}_0$ which leads to a fact that $T_u = \lim_{n\to\infty} u^n$ exists by monotoneity, that is, if $n < m$, $u^n > u^m$ holds, and $\mathfrak{T}_u \leqq T_u < u$ is valid on $F-\bar{F}_0$. By definition $\mathfrak{T}_u \geqq T_u$ and hence $\mathfrak{T}_u \equiv T_u$.

Let $u = S_v$, then we see that $S_v - u^n = 0$ on $\Gamma_n+\Gamma_0$ and $v^n - v = 0$ on $\Gamma_n+\Gamma_0$ and $S_v - u^n - v^n + v \equiv \varphi$ satisfies a relation $\Delta\varphi = -P(u^n - v)$. On the other hand we have $u^n \geqq v$ holds for any n. Thus $\Delta\varphi \leqq 0$ and hence $\varphi \geqq 0$ on $F_n-\bar{F}_0$, that is, $S_v - u^n \geqq v^n - v$ on $F_n-\bar{F}_0$. Let n tend to infinity, then $S_v - TS_v \geqq S_v - v$ and $TS_v \leqq v$ on $F-\bar{F}_0$. Thus $TS_v = v$ for any v for which S_v has the sense. Especially $\mathfrak{G}_{F-\bar{F}_0}$ belongs to a family for each element of which S operation has the sense. Therefore we see that $\mathfrak{T}\circ\mathcal{S} = \mathbf{I}$ for $\mathfrak{G}_{F-\bar{F}_0}$. Now we may conclude that $\mathcal{S}_{v_1} \equiv \mathcal{S}_{v_2}$ implies that $v_1 \equiv v_2$, if $\mathcal{S}$ operation has the sense for v_1 and v_2.

Let u be minimal in Martin's sense, then $\mathfrak{T}_u$ is also minimal in $\mathfrak{P}_{F-\bar{F}_0}$. Let $\mathfrak{T}_u \geqq v$ hold, then $u = \mathcal{S}\mathfrak{T}_u \geqq \mathcal{S}_v$ which leads to a fact that $ku = \mathcal{S}_v$ for $0 < k \leqq 1$, provided $\mathcal{S}_v \not\equiv 0$. Hence $k\mathfrak{T}_u = \mathfrak{T}\mathcal{S}_v = v$, $0 < k \leqq 1$, unless $v \equiv 0$. Therefore, $\mathfrak{T}_u$ is also a minimal in $\mathfrak{P}_{F-\bar{F}_0}$. Moreover u can be expressed as a limit function $\lim_{n\to\infty} g^{(k)}_{F-\bar{F}_0}(z, \zeta_n)$, hence we can extract a suitable subsequence $\{\zeta_{n_m}\}$ such that $V_u = \lim_{m\to\infty} G_{F-\bar{F}_0}(z, \zeta_{n_m}) \in \mathfrak{G}_{F-\bar{F}_0}$. Evidently $V_u \leqq \mathfrak{T}_u$

and hence $V_u = k\mathfrak{T}_u$ which shows that V_u is also a minimal in $\mathfrak{G}_{F-\bar{F}_0}$ and $\mathfrak{T}_u$ belongs to $\mathfrak{G}_{F-\bar{F}_0}$. Therefore we can conclude that the image of $\mathbb{P}_{F-\bar{F}_0}$ by $\mathfrak{T}$ operation belongs to a class $\mathfrak{G}_{F-\bar{F}_0}$.

Now we may state a result:

Theorem 1. Let $F \in O_\Omega$ and $\Gamma \in$ **(a)**. Between $\mathbb{P}_{F-\bar{F}_0}$ and $\mathfrak{G}_{F-\bar{F}_0}$ there exists a one-to-one and onto mapping $\mathfrak{T}$ such that $\mathfrak{T}$ operation preserves minimality. We have $\dim_2(\Gamma) = \dim_0(\Gamma) \leqq \dim_1(\Gamma)$. The maximal range set of $\mathcal{S}$ operation coincides with $\mathfrak{G}_{F-\bar{F}_0}$.

References

M.Heins. Riemann surfaces of infinite genus. Ann. of Math. 55(1952).

Z.Kuramochi. Relations between harmonic dimensions. Proc. Japan Acad. 30(1954).

R.Martin. Minimal positive harmonic functions. Trans. Amer. Math. Soc. 49(1941).

L.Myrberg. [1]. Über die Integration der Differentialgleichung $\Delta u = c(P)u$ auf offenen Riemannschen Flächen. Math. Acad. 2(1954).
[2]. Über die Existenz der Greenschen Funktion der Gleichung $\Delta u = c(P)u$ auf Riemannschen Flächen. Ann. Acad. Sci. Fenn. AI. 170(1954).

M.Ozawa. [1]. Classification of Riemann surfaces. These Reports. 1952 No. 3.
[2]. On harmonic dimension, I; II. ibid. 1954, No. 2.
[3]. On a maximality of a class of positive harmonic functions. Ibid. 1954. No. 3.

Department of Mathematics,
Tokyo Institute of Technology.

(*) Received October 30, 1954.