ON TRANSFERENCE BETWEEN BOUNDARY VALUE PROBLEMS FOR A SPHERE

By Yûsaku KOMATU and Hisao MIZUMOTO

1. Introduction.

In two-dimensional case it has been shown that two kinds of boundary value problems, Dirichlet and Neumann problems, are readily transferable each other for some domains of simple elementary configuration. Such a method for transference has been availed by Myrberg for a circular disc,[1] and subsequently discussed by one of the present authors for certain simply-connected slit domains[2] and for an annulus[3]

In the present Note we shall show that the method remains valid also for a sphere of any higher dimension. Since the Poisson integral formula for solving Dirichlet problem is classical, our result will offer an elementary construction of an integral formula for solving Neumann problem for a sphere.

In an N-dimensional euclidean space, the rectangular cartesian coordinates $(x_1, \ldots, x_N)$ and the polar coordinates $(r, \vartheta_1, \ldots, \vartheta_{N-1})$ are connected by the relations[4]

$$x_j = r\cos\vartheta_j \prod_{k=1}^{j-1} \sin\vartheta_k \qquad (1 \leq j \leq N-1),$$

$$x_N = r \prod_{k=1}^{N-1} \sin\vartheta_k ;$$

$$r \geqq 0,$$

$$0 \leqq \vartheta_j \leqq \pi \qquad (1 \leqq j \leqq N-2),$$

$$0 \leqq \vartheta_{N-1} < 2\pi ;$$

here and henceforth the empty product is always understood, as usual, to denote unity. The square of the line element is given by

$$ds^2 = \sum_{j=1}^{N} dx_j^2 = dr^2 + r^2 \sum_{j=1}^{N-1} d\vartheta_j^2 \prod_{k=1}^{j-1} \sin^2\vartheta_k$$

and the volume element is expressed in the form

$$d\tau = \prod_{j=1}^{N} dx_j = r^{N-1} dr \prod_{j=1}^{N-1} \sin^{N-j-1}\vartheta_j \, d\vartheta_j .$$

The Laplacian operator is represented by

$$\Delta = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} = \frac{1}{r^{N-1}} \frac{\partial}{\partial r}\left(r^{N-1} \frac{\partial}{\partial r}\right) + \frac{1}{r^2} \Delta^*$$

where Δ^* denotes a self-adjoint differential operator defined by

$$\Delta^* = \sum_{j=1}^{N-1} \left(\prod_{k=1}^{j-1} \operatorname{cosec}^2 \vartheta_k \right) \cdot \operatorname{cosec}^{N-j-1}\vartheta_j \frac{\partial}{\partial \vartheta_j}\left(\sin^{N-j-1}\vartheta_j \frac{\partial}{\partial \vartheta_j} \right).$$

The self-adjoint partial differential equation

$$\Delta^* Y + \lambda Y = 0$$

defines a spherical surface harmonic Y_n of order n as a solution belonging to the eigen-value $n(n+N-2)$.[5] Its general form is a linear combination of $\binom{n+N-1}{N-1} - \binom{n+N-3}{N-1}$ functions which constitute a linearly independent basis.

2. Lemmas.

We begin with a lemma involving a formal identity.

Lemma 1. Let $u(r, \vartheta_1, \cdots, \vartheta_{N-1})$ be a one-valued function with respect to polar coordinates $r, \vartheta_1, \cdots, \vartheta_{N-1}$ which is defined in the unit sphere

$$S: \qquad r<1$$

and is continuously differentiable twice with respect to its arguments and further vanishes at the origin at least of the first order, i. e.

$$u(r,\vartheta_1,\cdots,\vartheta_{N-1})=O(r) \quad \text{for } r\to 0.$$

Then the function defined by

$$v(r,\vartheta_1,\cdots,\vartheta_{N-1})=\int_0^r u(t,\vartheta_1,\cdots,\vartheta_{N-1})\frac{dt}{t},$$
$$v(0,\vartheta_1,\cdots,\vartheta_{N-1})=0,$$

is also one-valued there and satisfies the relation

$$r^2\Delta v(r,\vartheta_1,\cdots,\vartheta_{N-1})=\int_0^r t\Delta u(t,\vartheta_1,\cdots,\vartheta_{N-1})dt,$$

where Δ denotes the Laplacian operator with respect to the respective arguments involved.

Proof. The one-valuedness of $v(r,\vartheta_1,\cdots,\vartheta_{N-1})$ is readily evident. Further, direct calculation shows

$$\begin{aligned}
&r^2\Delta v(r,\vartheta_1,\cdots,\vartheta_{N-1})\\
&=\left\{\frac{1}{r^{N-3}}\frac{\partial}{\partial r}\left(r^{N-1}\frac{\partial}{\partial r}\right)+\Delta^*\right\}\int_0^r u(t,\vartheta_1,\cdots,\vartheta_{N-1})\frac{dt}{t}\\
&=\frac{1}{r^{N-3}}\frac{\partial}{\partial r}\left(r^{N-2}u(r,\vartheta_1,\cdots,\vartheta_{N-1})\right)\\
&\qquad+\Delta^*\int_0^r u(t,\vartheta_1,\cdots,\vartheta_{N-1})\frac{dt}{t}\\
&=\int_0^r\left\{\frac{\partial}{\partial t}\left(\frac{1}{t^{N-3}}\frac{\partial}{\partial t}\left(t^{N-2}u(t,\vartheta_1,\cdots,\vartheta_{N-1})\right)\right)\right.\\
&\qquad\left.+\frac{1}{t}\Delta^*u(t,\vartheta_1,\cdots,\vartheta_{N-1})\right\}dt\\
&=\int_0^r\frac{1}{t}\left\{\frac{1}{t^{N-3}}\frac{\partial}{\partial t}\left(t^{N-1}\frac{\partial}{\partial t}\right)+\Delta^*\right\}u(t,\vartheta_1,\cdots,\vartheta_{N-1})dt\\
&=\int_0^r t\Delta u(t,\vartheta_1,\cdots,\vartheta_{N-1})dt,
\end{aligned}$$

what is to be proved. Here an account has been taken into a fact $r^{-(N-3)}\cdot\partial(r^{N-2}u)/\partial r$ vanishes at the origin since $u=O(r)$ for $r\to 0$.

Henceforth we restrict our considerations chiefly to harmonic functions. Lemma 1 then reduces to the following simple form.

Lemma 2. If $u(r,\vartheta_1,\cdots,\vartheta_{N-1})$ is one-valued and harmonic in S and vanishes at the origin, then the function defined in lemma 1 is also one-valued and harmonic there.

Proof. By lemma 1, $\Delta u(r,\vartheta_1,\cdots,\vartheta_{N-1})=0$ implies immediately $\Delta v(r,\vartheta_1,\cdots,\vartheta_{N-1})=0$ provided $r\neq 0$. But, since $u(r,\vartheta_1,\cdots,\vartheta_{N-1})$ is supposed harmonic, the assumption that u vanishes at the origin implies further that there holds $u(r,\vartheta_1,\cdots,\vartheta_{N-1})=O(r)$ near the origin. Thus, $v(r,\vartheta_1,\cdots,\vartheta_{N-1})$ harmonic for $0<r<1$ is bounded near the origin; in fact, moreover $v(r,\vartheta_1,\cdots,\vartheta_{N-1})=O(r)$ for $r\to 0$. Hence, by a theorem due to Picard, it is harmonic throughout S.

Lemma 2 now established can be verified also by means of an expansion into series. In fact, the function $u(r,\vartheta_1,\cdots,\vartheta_{N-1})$ one-valued and harmonic in S and vanishing at the origin can be expanded into a series of the form

$$u(r,\vartheta_1,\cdots,\vartheta_{N-1})=\sum_{n=1}^{\infty}r^n Y_n(\vartheta_1,\cdots,\vartheta_{N-1})$$

uniformly convergent in the wider sense in S, where $Y_n(\vartheta_1,\cdots,\vartheta_{N-1})$ designates a spherical surface harmonic of order n.[4] Consequently, $v(r,\vartheta_1,\cdots,\vartheta_{N-1})$ is represented by the series

$$v(r,\vartheta_1,\cdots,\vartheta_{N-1})=\sum_{n=1}^{\infty}r^n\frac{1}{n}Y_n(\vartheta_1,\cdots,\vartheta_{N-1})$$

which converges also uniformly in the wider sense in S. Since any constant multiple of $Y_n(\vartheta_1,\cdots,\vartheta_{N-1})$ is also a spherical surface harmonic of order n, $v(r,\vartheta_1,\cdots,\vartheta_{N-1})$ is surely harmonic in S, as is desired.

The lemma 2 can be converted as shown in the following lemma.

Lemma 3. If $v(r,\vartheta_1,\cdots,\vartheta_{N-1})$ is one-valued and harmonic in S, then the function defined by

$$u(r,\vartheta_1,\ldots,\vartheta_{N-1})=r\frac{\partial v}{\partial r}(r,\vartheta_1,\ldots,\vartheta_{N-1})$$
is also one-valued and harmonic there.

Proof. The one-valuedness of $u(r, \vartheta_1, \ldots, \vartheta_{N-1})$ is immediate. Since its defining equation implies
$$v(r,\vartheta_1,\ldots,\vartheta_{N-1})-v_0 =\int_0^r u(t,\vartheta_1,\ldots,\vartheta_{N-1})\frac{dt}{t}$$
where v_0 denotes the value of $v(r, \vartheta_1,\ldots,\vartheta_{N-1})$ at the origin, there holds, by lemma 1,
$$\int_0^r t\,\Delta u(t,\vartheta_1,\ldots,\vartheta_{N-1})\,dt = r^2\Delta v(r,\vartheta_1,\ldots,\vartheta_{N-1})=0,$$
whence follows
$$r\,\Delta u(r,\vartheta_1,\ldots,\vartheta_{N-1})=0.$$
$u(r,\vartheta_1,\ldots,\vartheta_{N-1})$ being bounded near the origin, the last equation yields that $u(r,\vartheta_1,\ldots,\vartheta_{N-1})$ is harmonic throughout $\mathfrak{S}$.

Lemma 3 also can be verified again by means of an expansion into the series in quite a similar manner as for lemma 2. On the other hand, corresponding to lemma 1, it is shown that $u=r\partial v/\partial r$ implies, by direct calculation,
$$r\,\Delta u(r,\vartheta_1,\ldots,\vartheta_{N-1}) =\frac{\partial}{\partial r}\left(r^2\Delta v(r,\vartheta_1,\ldots,\vartheta_{N-1})\right)$$
what is also available for proving lemma 3.

3. Theorems.

We are now in position to state our main theorems stating how the transference between both kinds of boundary value problems is to be performed.

Theorem 1. Let a Dirichlet problem for the unit sphere $\mathfrak{S}$ with the boundary condition
$$u(1,\vartheta_1,\ldots,\vartheta_{N-1})=U(\vartheta_1,\ldots,\vartheta_{N-1})$$
be proposed. Solve by $v(r,\vartheta_1,\ldots,\vartheta_{N-1})$ an associated Neumann problem with the boundary condition
$$\frac{\partial v}{\partial \nu}(1,\vartheta_1,\ldots,\vartheta_{N-1}) =-U(\vartheta_1,\ldots,\vartheta_{N-1})+u_0,$$
where $\partial/\partial\nu$ designates the differentiation along inward normal and u_0 denotes the value of u at the origin which can be directly defined by
$$u_0=\frac{1}{\omega_N}\int\cdots\int_\Omega U(\vartheta_1,\ldots,\vartheta_{N-1})\,d\sigma$$
here the integration extends over the whole surface
$$\Omega:\quad \begin{array}{ll} 0\leqq\vartheta_j\leqq\pi & (1\leqq j\leqq N-2), \\ 0\leqq\vartheta_{N-1}<2\pi, & \end{array}$$
and $d\sigma$ and ω_N denote the surface element of Ω and its total area, respectively, i. e.
$$d\sigma=\prod_{j=1}^{N-1}\sin^{N-j-1}\vartheta_j\,d\vartheta_j,$$
$$\omega_N=\int\cdots\int_\Omega d\sigma=\frac{2\pi^{N/2}}{\Gamma(N/2)}.$$
The solution $u(r,\vartheta_1,\ldots,\vartheta_{N-1})$ of the original Dirichlet problem is then given by
$$u(r,\vartheta_1,\ldots,\vartheta_{N-1}) = r\frac{\partial v}{\partial r}(r,\vartheta_1,\ldots,\vartheta_{N-1})+u_0.$$

Proof. The solvability of the associated Neumann problem is surely guaranteed by the relation
$$\int\cdots\int_\Omega\left(-U(\vartheta_1,\ldots,\vartheta_{N-1})+u_0\right)d\sigma=0$$
By lemma 3, the function u defined by the equation in the theorem is one-valued and harmonic in $\mathfrak{S}$. Further, it satisfies the assigned boundary condition, since there holds
$$u(1,\vartheta_1,\ldots,\vartheta_{N-1}) =-\frac{\partial v}{\partial \nu}(1,\vartheta_1,\ldots,\vartheta_{N-1})+u_0$$
$$=U(\vartheta_1,\ldots,\vartheta_{N-1}).$$

Theorem 2. Let a Neumann problem for the unit sphere with the boundary be proposed. Solve by $u(r,\vartheta_1,\dots,\vartheta_{N-1})$ an associated Dirichlet problem with the boundary condition

$$u(1,\vartheta_1,\dots,\vartheta_{N-1})=-V(\vartheta_1,\dots,\vartheta_{N-1}).$$

The solution $v(r,\vartheta_1,\dots,\vartheta_{N-1})$ of the original Neumann problem is then given by

$$v(r,\vartheta_1,\dots,\vartheta_{N-1}) = c+\int_0^r u(t,\vartheta_1,\dots,\vartheta_{N-1})\frac{dt}{t},$$

where c is an arbitrary constant.

Proof. Since $V(\vartheta_1,\dots,\vartheta_{N-1})$ possesses the vanishing mean, the value u_0 of $u(r,\vartheta_1,\dots,\vartheta_{N-1})$ at the center also vanishes:

$$u_0=\int_\Omega\cdots\int(-V(\vartheta_1,\dots,\vartheta_{N-1}))\,d\sigma=0.$$

Hence, by lemma 2, the function $v(r,\vartheta_1,\dots,\vartheta_{N-1})$ defined in the present theorem is harmonic throughout S. Further, it satisfies the assigned boundary condition:

$$\frac{\partial v}{\partial \nu}(1,\vartheta_1,\dots,\vartheta_{N-1}) = -\frac{\partial v}{\partial r}(1,\vartheta_1,\dots,\vartheta_{N-1}) = V(\vartheta_1,\dots,\vartheta_{N-1}).$$

4. Application.

It is well known that Dirichlet problem for a sphere is solved by means of the classical Poisson integral formula. We are now in position to show that, based on theorem 2, this formula can be transferred into one for solving Neumann problem for a sphere. Actual procedure of calculation will be performed in the following lines.

Let the basic domain be again, for the sake of simplicity, the unit sphere S and let a Neumann problem with the boundary condition

$$\frac{\partial v}{\partial \nu}(1,\vartheta_1,\dots,\vartheta_{N-1})=V(\vartheta_1,\dots,\vartheta_{N-1}),$$

$$\int_\Omega\cdots\int V(\vartheta_1,\dots,\vartheta_{N-1})\,d\sigma=0,$$

be proposed, Ω denoting again the surface of S. We now suppose that the Poisson formula for Dirichlet problem has been established. An associated Dirichlet problem with the boundary condition

$$u(1,\vartheta_1,\dots,\vartheta_{N-1})=-V(\vartheta_1,\dots,\vartheta_{N-1})$$

is accordingly solved by

$$u(r,\vartheta_1,\dots,\vartheta_{N-1}) = -\frac{\Gamma(N/2)}{2\pi^{N/2}}\int_\Omega\cdots\int V(\vartheta'_1,\dots,\vartheta'_{N-1})\frac{1-r^2}{(1-2r\cos\psi+r^2)^{N/2}}\,d\sigma'$$

where $d\sigma'$ denotes the surface element of Ω with respect to integration variable $(\vartheta'_1,\dots,\vartheta'_{N-1})$, i. e.

$$d\sigma'=\prod_{j=1}^{N-1}\sin^{N-j-1}\vartheta'_j\,d\vartheta'_j$$

and ψ denotes the angle between the two directions represented by $(\vartheta_1,\dots,\vartheta_{N-1})$ and $(\vartheta'_1,\dots,\vartheta'_{N-1})$ and hence

$$\cos\psi=\sum_{j=1}^{N-2}\cos\vartheta_j\cos\vartheta'_j\prod_{k=1}^{j-1}\sin\vartheta_k\sin\vartheta'_k + \cos(\vartheta_{N-1}-\vartheta'_{N-1})\prod_{k=1}^{N-2}\sin\vartheta_k\sin\vartheta'_k.$$

Remembering the auxiliary condition for solvability of Neumann problem, the above formula for u can be brought into the form

$$u(r,\vartheta_1,\dots,\vartheta_{N-1}) = -\frac{\Gamma(N/2)}{2\pi^{N/2}}\int_\Omega\cdots\int V(\vartheta'_1,\dots,\vartheta'_{N-1}) \cdot\left(\frac{1-r^2}{(1-2r\cos\psi+r^2)^{N/2}}-1\right)d\sigma',$$

in which the kernel $(1-r^2)/(1-2r\cos\psi+r^2)^{N/2}-1$ contained in the integrand vanishes at the origin at least of the same order as r itself.

Based on theorem 2, the solution of the original Neumann problem is then given by

$$v(r,\vartheta_1,\dots,\vartheta_{N-1}) = c+\int_0^r u(t,\vartheta_1,\dots,\vartheta_{N-1})\frac{dt}{t}$$

$$= c - \frac{\Gamma(N/2)}{2\pi^{N/2}} \int_{\Omega} \cdots \int V(\vartheta_1', \ldots, \vartheta_{N-1}')$$
$$\times \int_0^r \left(\frac{1-t^2}{(1-2t\cos\psi + t^2)^{N/2}} - 1 \right) \frac{dt}{t} \, d\sigma',$$

c being an arbitrary constant. This is the desired formula which remains valid regardless of the value $N \geqq 2$.

The inward integral with respect to t can be evaluated by means of an elementary functions of r and ψ, though the actual procedure will be somewhat troublesome for a higher N. For lowest cases $N=2$ and $N=3$, the formula reduces after a simple calculation into

$$v(r,\vartheta) = c - \frac{1}{2\pi} \int_0^{2\pi} V(\vartheta') \lg \frac{1}{1-2r\cos(\vartheta'-\vartheta)+r^2} d\vartheta'$$

and

$$v(r,\vartheta,\varphi) = c - \frac{1}{4\pi} \int_{\vartheta'=0}^{\pi} \int_{\varphi'=0}^{2\pi} V(\vartheta',\varphi') \left\{ \frac{2}{\sqrt{1-2r\cos\psi+r^2}} + \lg \frac{1}{1-r\cos\psi+\sqrt{1-2r\cos\psi+r^2}} \right\} \sin\vartheta' d\vartheta' d\varphi',$$

$$\cos\psi = \cos\vartheta\cos\vartheta' + \sin\vartheta\sin\vartheta'\cos(\varphi-\varphi'),$$

respectively; here an account is taken into the condition for solvability of the Neumann problem.

On the other hand, the above general formula can be brought into an alternative form[6)]

$$v(r,\vartheta_1,\ldots,\vartheta_{N-1})$$
$$= c - \frac{\Gamma(N/2)}{2\pi^{N/2}} \int_{\Omega} \cdots \int V(\vartheta_1',\ldots,\vartheta_{N-1}')$$
$$\times \left\{ \frac{2}{N-2} \frac{1}{(1-2r\cos\psi+r^2)^{N/2-1}} + \int_0^r \left(\frac{1}{(1-2t\cos\psi+t^2)^{N/2-1}} - 1 \right) \frac{dt}{t} \right\} d\sigma'$$

provided $N \geqq 3$, while in logarithmic case $N=2$ the expression within the braces must be replaced by

$$\lim_{N \to 2} \frac{2}{N-2} \left(\frac{1}{(1-2r\cos\psi+r^2)^{N/2-1}} - 1 \right)$$
$$= \lg \frac{1}{1-2r\cos\psi+r^2},$$

Finally, it would be noted that, based on theorem 1, we can also proceed conversely in order to derive the formula on Dirichlet problem from that on Neumann problem provided the latter is supposed to known. In fact, suppose now that a Neumann problem for S with the boundary condition

$$\frac{\partial v}{\partial \nu}(1,\vartheta_1,\ldots,\vartheta_{N-1})$$
$$= -U(\vartheta_1,\ldots,\vartheta_{N-1}) + u_0,$$
$$u_0 = \frac{\Gamma(N/2)}{2\pi^{N/2}} \int_{\Omega} \cdots \int U(\vartheta_1',\ldots,\vartheta_{N-1}') \, d\sigma',$$

is known to be solved by the integral formula

$$v(r,\vartheta_1,\ldots,\vartheta_{N-1})$$
$$= c - \frac{\Gamma(N/2)}{2\pi^{N/2}} \int_{\Omega} \cdots \int (-U(\vartheta_1',\ldots,\vartheta_{N-1}') + u_0)$$
$$\times \int_0^r \left(\frac{1-t^2}{(1-2t\cos\psi+t^2)^{N/2}} - 1 \right) \frac{dt}{t} \, d\sigma',$$

c being any constant. Theorem 1 implies then a formula for solving the Dirichlet problem with the boundary condition

$$u(1,\vartheta_1,\ldots,\vartheta_{N-1}) = U(\vartheta_1,\ldots,\vartheta_{N-1})$$

in the form

$$u(r,\vartheta_1,\ldots,\vartheta_{N-1})$$
$$= r\frac{\partial v}{\partial r}(r,\vartheta_1,\ldots,\vartheta_{N-1}) + u_0$$
$$= -\frac{\Gamma(N/2)}{2\pi^{N/2}} \int_{\Omega} \cdots \int (-U(\vartheta_1',\ldots,\vartheta_{N-1}') + u_0)$$
$$\times \left(\frac{1-r^2}{(1-2r\cos\psi+r^2)^{N/2}} - 1 \right) d\sigma' + u_0$$
$$= \frac{\Gamma(N/2)}{2\pi^{N/2}} \int_{\Omega} \cdots \int U(\vartheta_1',\ldots,\vartheta_{N-1}')$$
$$\times \frac{1-r^2}{(1-2r\cos\psi+r^2)^{N/2}} \, d\sigma',$$

since there holds an elementary identity

$$\frac{\Gamma(N/2)}{2\pi^{N/2}} \int_{\Omega} \cdots \int \frac{1-r^2}{(1-2r\cos\psi+r^2)^{N/2}} \, d\sigma' = 1.$$

We thus have returned to the Poisson formula, which has been taken as a starting tool for the former inverse step for deriving the integral representation on Neumann problem.

REFERENCES

1) L. Myrberg, Über die vermischte Randwetaufgabe der harmonischen Funktionen. Ann. Acad. Sci. Fenn. A, I. 103 (1951), 8 pp.

2) Y. Komatu, On transference of boundary value problems. Kōdai Math. Sem. Rep. (1954), 71-80; A supplement to "On transference of boundary value problems". Ibid. (1954), 97-100.

3) Y. Komatu, Über eine Übertragung zwischen Randwertaufgaben für einen Kreisring. Kōdai Math. Sem. Rep. (1954), 101-108.

4) A. Dinghas, Geometrische Anwendungen der Kugelfunktionen. Göttinger Nachr. Neue Folge 1, No,18 (1940), 213-235.

5) Cf. R. Courant-D. Hilbert, Methoden der mathematischen Physik, I. (1931), pp. 270 and 441, where the case $N=3$ is dealt with.

6) Cf. S. Hitotumatu, On the Neumann function of a sphere. Comm. Math. Univ. Sancti Pauli 3 (1954), 1-5, in which a formula is reported in this form.

Department of Mathematics,
Tokyo Institute of Technology.

(*) Received November 4, 1954.