ON AN INEQUALITY CONCERNING THE EIGENVALUE PROBLEM OF MEMBRANE

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Let D be a plane domain with a given area A surrounded by a boundary C. We consider the eigenvalue problem on the equation for a vibrating membrane

$$\Delta u + \lambda u = 0, \qquad \lambda > 0,$$

under the boundary condition

$$u = 0 \quad \text{on } C.$$

The principal frequency of a membrane is diminished by symmetrization with respect to a straight line as well as to a point, and hence, for a given area the principal frequency attains its minimum for a circle. On the other hand, the second frequency does not behave similarly. Pólya and Szegö have pointed out that a rectangle with sides a and $2a$ has the second frequency less than that of a circle with the same area.

In the present paper we shall study the greatest lower bound for the second frequency of all membranes having a given area. Our result may be stated as follows:

Proposition. The second frequency of all membranes, having a given area A and fixed along their boundaries, has the greatest lower bound equal to the principal frequency of a circle having the area $A/2$.

Before giving a proof of our proposition, we start with observing some fundamental properties of the second eigenfunction. Let u_1, u_2 be the first and the second eigenfunctions respectively, and λ_1, λ_2 be the corresponding eigenvalues. It is well known that the second eigenfunction u has necessarily a nodal line in the interior of the domain D. Precisely, the domain should be divided into two parts D' and D'' by a nodal line σ, each of them is a connected domain. u is positive in one of D' and D'' and negative in the other. The boundary C is divided into two parts C' and C'' in such a manner. C' and the nodal line σ surround the domain D' while C'' and σ the domain D''. Then u_2 is a function which satisfies the equation

$$\Delta u + \lambda_2 u = 0 \qquad \text{in } D'$$

and vanishes on $C' + \sigma$, and simultaneously satisfies

$$\Delta u + \lambda_2 u = 0 \qquad \text{in } D''$$

and vanishes on $C'' + \sigma$. Moreover u_2 vanishes neither in the interior of D' nor in that of D''. It is to be noticed that, if a solution of the equation $\Delta u + \lambda u = 0$ under the vanishing boundary condition, never vanishes in the interior of the domain, then it must necessarily be the first eigenfunction. Therefore our second eigenfunction u_2 may be regarded as the first eigenfunction of the domain D' as well as of the domain D'', and accordingly λ_2 can be regarded as the first eigenvalue both of D' and D''.

We are now in the position to give a proof for our proposition. Let μ' be the first eigenvalue of the circle with the same area as D', and μ'' be that of the circle with the same area as D''.

According to the fundamental inequality for the first eigenvalue, we can obtain two inequalities

$$\lambda_2 \geqq \mu', \qquad \lambda_2 \geqq \mu'',$$

which hold simultaneously, hence follows

$$\lambda_2 \geqq \mathrm{Max}(\mu', \mu'').$$

We may suppose that the area of D' is not greater than of D''. Since the first eigenvalue is a monotone decreasing set function, it is then obvious that $\mu' \geqq \mu''$.

Now let us consider a class of domains with the given area A, and find the greatest lower bound for the second eigenvalue of the domain among the above class. According to our last inequality, it is sufficient to minimize μ'. For that purpose, the area of D', that is, the smaller subdomain, must be as large as possible. Hence we have to make the area of D' to be equal to $A/2$. Thus $\inf_D \mu'$ becomes equal to the first eigenvalue of the circle with the area $A/2$. Therefore

$$\lambda_2 \geqq (2\pi/A)j^2,$$

where $j \doteqdot 2.4048$ denotes the first zero of the Bessel function $J_0(x)$, and the right hand member represents, in fact, the fundamental eigenvalue for the circle with $A/2$ as its area. The equality holds only for the extreme case where D degenerates to a set consisting of two circles each of which has $A/2$ as its area and tangenting to each other at a point. However such a degeneration is not permitted, we may take the equality sign off in the above relation.

Furthermore, we can assert that the above lower bound is in fact the greatest one, by using the continuity property of λ_2 with regard to the domain. For instance, we may give a minimizing sequence as follow: Consider a domain which has $A/2$ as its area and bounded by two equal circular arcs having a common chord with length δ. Let δ tend to zero through a sequence. We then obtain a corresponding sequence of domains whose second eigenvalues tend to $(2\pi/A)j^2$. Thus our proposition has been established.

Finally, it would be noticed that our proposition can be extended to the three- dimensional case quite in the same way.

Reference

1) G. Pólya and G. Szegö, Isoperimetric inequalities in mathematical physics, 1951, P. 168.

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