NOTE ON SIGULARITIES OF DIFFERENTIAL EQUATIONS

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In this note, Part I is devoted to study the general theory for critical points at the origin and in Part II special cases are treated. In both cases we shall stand on the view point of complex variables.

Part I

1. Let $f(z, w)$ be an analytic function of z and w in certain neighbourhood of the origin. Instead of analyticity we may suppose that $f(z, w)$ has the first continuous derivatives. Further we suppose that the Lipschitz condition holds good and $f(0, 0) = 0$. Then, we consider the equation

$$(1.1) \qquad dz/dt = f(z, \bar{z}).$$

In virtue of the hypotheses the characteristics passing through $z = z_0$ ($\neq 0$) is uniquely determined.

Critical points of the equation (1.1) are the points defined as $f(z, \bar{z}) = 0$. In the sequel we consider the properties of the characteristics of (1.1) in the neighbourhood of the origin $z = 0$.

By means of the hypotheses, in the neighborhood of the origin the equation (1.1) leads to the equation

$$\frac{dz}{dt} = f_n(z, \bar{z}) + f_{n+1}(z, \bar{z}),$$

where

$$f_n(z, \bar{z}) = a_0 z^n + a_1 z^{n-1}\bar{z} + \cdots + a_n \bar{z}^n$$

and $f_{n+1}(z, \bar{z})$ is the function with order at least $n+1$. Then, the properties of critical point at the origin is identical with the equation

$$(1.2) \qquad dz/dt = f_n(z, \bar{z}).$$

2. <u>Indices</u>. Suppose that there exists no zero points of $f_n(z, 1)$ on the unit circle $|z| = 1$. Then, the origin is the only zero point of $f_n(z, \bar{z})$ and of course it is an isolated singularity. Hence, we can calculate its index.

Describe a circle of sufficiently small radius r with center at the origin. Generally let $f(z, \bar{z})$ be a function having the origin as an isolated zero point. Then, if we consider $f(z, \bar{z})$ as a vector defined in the neighborhood of the origin, the index of $z = 0$ for the function $f(z, \bar{z})$ is defined as

$$\text{Index of } z = 0 = I(0) = \frac{1}{2\pi}\oint_{|z|=r} d \arg f(z, \bar{z}).$$

Hence, the index for the function $f_n(z, \bar{z})$ is

$$(1.3) \qquad I(0) = \frac{1}{2\pi}\oint_{|z|=r} d \arg f_n(z, \bar{z}) = n + \frac{1}{2\pi}\oint_{|z|=r} d \arg f_n(1, e^{-2i\theta}).$$

Let k be the number of zero points of $f_n(1, z)$ in the unit circle. Since $f_n(1, z)$ has no zero points on $|z| = 1$, we have by (1.3)

$$I(0) = n - 2k.$$

Indices are invariant under any regular transformation. Therefore, if we consider the equation only in the neighborhood of the origin, the equation (1.2) is topologically equivalent to the equation

$$(1.4) \qquad \frac{dz}{dt} = z^{n-2k}$$

if $n \geqq 2k$, and

(1.5) $$\frac{dz}{dt} = \bar{z}^{2k-n}$$

if $n < 2k$. Further, $I(0) = n$ if $|1 - f_n(1, e^{-2i\theta})/a| < 1$ and $I(0) = -n$ if $|1 - f_n(1, e^{-2i\theta})/a| > 1$, where a is the first nonvanishing coefficient of $f_n(z, \bar{z})$.

3. Stereographical projection. In order to study the point at infinity, we make use of stereographical projection of the complex plane $z = x + iy$ on the Riemann sphere $\xi^2 + \eta^2 + \zeta^2 - \zeta = 0$. The relation between the coordinates (x, y) and (ξ, η, ζ) are

$$\begin{cases} \xi = \dfrac{x}{x^2+y^2+1} \\ \eta = \dfrac{y}{x^2+y^2+1} \\ \zeta = \dfrac{x^2+y^2}{x^2+y^2+1} \end{cases} \qquad \begin{cases} x = \dfrac{\xi}{1-\zeta} \\ y = \dfrac{\eta}{1-\zeta} \\ \xi^2+\eta^2+\zeta^2-\zeta = 0 \end{cases}.$$

Rotate the sphere by the relation

$$\xi_1 = -\xi, \quad \eta_1 = \eta, \quad \zeta_1 = 1 - \zeta.$$

Then, projecting the sphere $\xi_1^2 + \eta_1^2 + \zeta_1^2 - \zeta_1 = 0$ to the complex plane $z_1 = x_1 + iy_1$ from the new north pole, we obtain the relation

(1.6) $$z = -1/z_1.$$

Hence, the point at infinity $z_1 = \infty$ corresponds to the origin $z = 0$. If we round $z = 0$ counterclockwise, we round $z_1 = \infty$ clockwise.

Given a equation such that

(1.7) $$dz/dt = f_n(z, \bar{z}).$$

By means of (1.6), we have

(1.8) $$dz/dt = z^2 f_n(\bar{z}, z).$$

Hence,

$$\frac{1}{2\pi}\oint_{|z|=r} d \arg z^2 f_n(\bar{z}, z) = 2 + \frac{1}{2\pi}\oint_{|z|=r} d \arg f_n(\bar{z}, z) = 2 - \frac{1}{2\pi}\oint_{|z|=r} d \arg f_n(z, \bar{z}).$$

This shows that it is essential that the index of the sphere is equal to $+2$.

4. Seperatrices. In the equation

$$dz/dt = f_n(z, \bar{z}),$$

we put

$$f_n(z, \bar{z}) = P_n(z, y) + iQ_n(z, y) \qquad (z = x + iy).$$

Since $P_n(x, y)$ and $Q_n(x, y)$ are homogeneous polynomials of x and y with order n, we can easily calculate the real system

$$dx/dt = P_n(x, y), \qquad dy/dt = Q_n(x, y)$$

by quadrature. The separatrices are determined by the equation

$$xQ_n(x, y) - yP_n(x, y) = 0.$$

Since $Q_n(x, 0) \not\equiv 0$, the real axis may be a separatrix. All of them are straight lines passing through the origin.

Part II

1. Case $n = 1$. We consider the equation

(2.1) $$dz/dt = az + \overline{bz} + f_2(z, \bar{z}) \qquad (|a| + |b| \neq 0).$$

Corresponding to (1.2), we have

(2.2) $$dz/dt = az + \overline{bz}.$$

Then,

$$I(0) = \frac{1}{2\pi}\oint_{|z|=r} d \arg (az + \overline{bz})$$
$$= 1 + \frac{1}{2\pi}\oint_{|z|=r} d \arg (1 + \bar{b}e^{-2i\theta}/a) \quad (a \neq 0)$$
$$= -1 + \frac{1}{2\pi}\oint_{|z|=r} d \arg (1 + ae^{2i\theta}/\bar{b}) \quad (b \neq 0).$$

Hence,

$$I(0) = 1 \quad \text{if} \quad |a| > |b|,$$
$$I(0) = -1 \quad \text{if} \quad |a| < |b|.$$

By the classical theory, the indices of node, focus, and center are equal to +1, and -1 for saddle point. We remember that indices remain invariant under any regular mapping and they are charactéristic properties of isolated sigularities. Therefore, we obtain the following criterion:

(A) The necessary and sufficient condition that the origin $z = 0$ is to be node, focus, or center for the equation (2.1) is

$$|a| > |b| .$$

(B) The necessary and sufficient condition that the origin $z = 0$ is to be saddle point for the equation (2.1) is

$$|a| < |b| .$$

Hence, the equation (2.1) in the neighborhood of the origin is topologically equivalent to the equation

(2.3) $$dz/dt = az$$

for node, focus, or center and

$$dz/dt = \overline{bz}$$

for saddle point.

We note that node is topologically equivalent to focus, but node and center are not so. However, if we consider the equation

(2.4) $$dz/dt = aiz$$

corresponding to (2.3), the solutions of (2.4) are orthogonal characteristics for those of (2.3). Hence, we can distinguish node and focus from center.

Remark: For real system, the equation corresponding to (2.1)

$$dx/dt = ax + by, \quad dy/dt = cx + dy,$$

where letters are all real. Putting $z = x + iy$, we obtain

$$dz/dt = \alpha z + \bar{\beta}\bar{z}, \quad d\bar{z}/dt = \beta z + \bar{\alpha}\bar{z}.$$

Then, consider the matrix

$$A = \begin{pmatrix} \alpha & \bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}.$$

If we consider the characteristic roots of A, we have the well-known criteria for nodes, foci, centers, and saddle points.

2. Case $n = 2$. We consider the equation

$$dz/dt = z^2.$$

The solution are the family of circles $x^2 + y^2 - cy = 0$. By the stereographical projection, the corresponding equation is

$$dz/dt = 1,$$

whose solutions are the family of lines $x =$ const. In this case, there exist no separatrices.

3. Case $n = 3$. The equation is

(2.5) $$dz/dt = z^3.$$

By means of the stereographical projection, we have

$$dz/dt = 1/z = \bar{z} / |z|^2 .$$

However, so far as we are concerned with the property of the origin, it is sufficient to consider the equation

(2.6) $$dz/dt = \bar{z}.$$

The solutions of (2.6) are the family of hyperbolas $xy = a$. By making use of the projection, the curves corresponding to $xy = a$, i.e., the solutions of (2.5) have the form

$$(x^2 + y^2)^2 = -axy,$$

which are the so-called "Lemniscates". There exist two separatrices $x = 0$ and $y = 0$ for (2.5) and (2.6). (See Fig. 1)

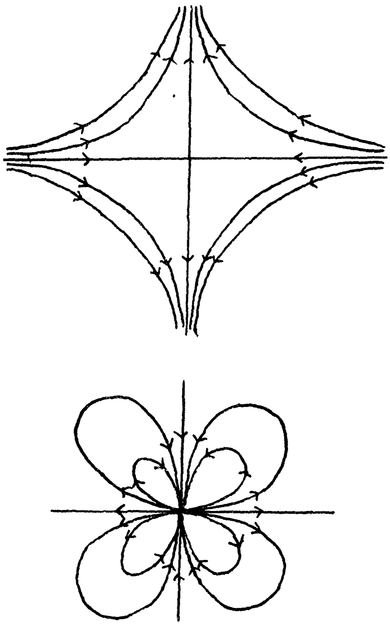


Fig. 1.

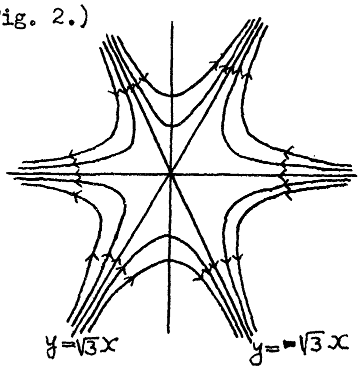
4. Case $n = 4$. The equation is

$$dz/dt = z^4.$$

The solutions are the family of curves $x^5 - 10x^3y^2 + 5xy^4 = c$. By means of (1.6), we have

$$dz/dt = z^2.$$

(See Fig. 2.)



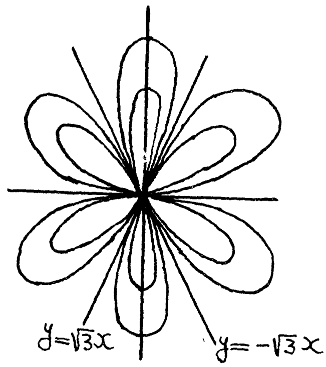


Fig. 2.

5. In general, we consider the equations

$$(2.7) \qquad dz/dt = z^n$$

$$(2.8) \qquad dz/dt = \bar{z}^n \qquad (n > 0).$$

The solutions of (2.7) start from the origin at the time $t = -\infty$ and return to it at the time $t = +\infty$. There gives rise to the nested ovals at the origin. By the stereographical projection, the point at infinity corresponds to a multiple saddle point. If n is odd and ≥ 3, the y-axis may be a separatrix, but if n is even and ≥ 4, it is not a separatrix.

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