ON COMPACT ONE-IDEMPOTENT SEMIGROUPS

By Takayuki TAMURA

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In our previous paper (0) we considered a finite one-idempotent semigroup, i.e., a finite semigroup whose idempotent is only one, and established some theories on its structure. However most of these results are able to be extended to a compact topological one-idempotent semigroup (1). The aim of the present paper is to give some part of them. We see that if the discussions here are applied to a finite case they become simpler than those in the previous paper.

In § 1 and § 2, we introduce the concept of greatest group and topology of difference semigroup respectively. The propositions on zero-semigroups in § 3 not only play an important part in the general one-idempotent semigroup theory (§5), but form a main part of the present paper. In § 4 we discuss on types of compact zero-semigroups and give their examples. On compact enclosed extensions and compact power semigroups, we shall discuss more generally in other papers.

1 Greatest group.

We see in this paragraph that a compact one-idempotent semigroup generally contains the greatest compact group which is at the same time the least ideal.

Now a bicompact semigroup was proved to have at least one idempotent element (2), but we can prove in Lemma 1 that this holds even if a semigroup is compact (3).

Lemma 1. A compact semigroup S has at least one idempotent.

Proof. Let a be any element of S. If we set $A = \{a^n ; n=1,2,\cdots\}$, then the derived set A' (4) will be a group. Take elements b, c of A', there exist positive integer sequences $\{n_i\}$ and $\{m_i\}$, where $n_i < n_{i+1}$, $m_i < m_{i+1}$, $i = 1, 2, \cdots$, such that $a^{n_i} \to b$, $a^{m_i} \to c$ as $i \to \infty$. Then, putting $r_i = n_i + m_i$, it is easy to show that the sequence $\{a^{r_i}\}$ converges to bc because of compactness and continuity of semigroup operation. Therefore $bc = cb \in A'$; A' is a commutative subsemigroup of S. Next, we shall prove that for $b \in A'$, $c \in A'$, we can find $x \in A'$ such that $bx = c$. Letting $a^{n_i} \to b$, $a^{m_i} \to c$ as $i \to \infty$, we choose a subsequence $\{p_i\}$ of $\{m_i\}$ which satisfies $p_1 < p_2 < \cdots < p_{i-1} < p_i < \cdots$, $0 < p_{i-1} - n_{i-1} < p_i - n_i$ $(i=2,3,\cdots)$. Furthermore put $q_i = p_i - n_i$. Then $a^{p_i} = a^{n_i} a^{q_i}$, while $\{a^{q_{i'}}\}$ converges to certain $x \in S$ if $\{q_{i'}\}$ is taken adequately as a subsequence of $\{q_i\}$. After that, attending to limiting case, we get $c = bx$. Thus we have proved that $bA' = A'b = A'$ for all $b \in A'$. Take any $x_0 \in A'$. Then e which satisfies $x_0 e = x_0$ will be an idempotent. For, if we select y such that $y x_0 = e$, and multiply y with both sides of the equality $x_0 e = x_0$, then we have $y x_0 e = y x_0$, whereas $ee = e$.

Remark. Of course A' is a compact topological group, and the closure $\bar{A}$ of A is a compact one-idempotent semigroup. Our precise investigation on compact semigroups generated by one element is omitted here, for it is to be published in other paper.

Lemma 2. A compact one-idempotent semigroup S whose idempotent e is a right (left) identity is a compact group.

Lemma 2'. If a compact one-idempotent semigroup S satisfies $Se = S$ $(eS = S)$, then S is a compact group.

Proof of Lemma 2. Suppose that S has a right identity e. Since xS for any $x \in S$ is a compact subsemigroup, it contains the idempotent e by Lemma 1. This means that a right inverse of x exists. Hence S is a group. We can easily prove S to be a compact group by the below remark.

Remark. Let G be an abstract group and a compact space. Whenever the space is compact, continuity of the mapping of $G \times G$ into G, $(x,y) \to xy$, implies that the mapping of G into itself, $x \to x^{-1}$, is continuous.

Proof of Lemma 2'. As $x = ye$, $y \in S$ for every $x \in S$, we get

$$xe = (ye)e = y(ee) = ye = x.$$

The idempotent e is a right identity of S, and so the proof is reduced to Lemma 2.

Making use of these lemmas, we obtain the following important theorem with respect to the greatest group.

Theorem 1. Let S be a compact one-idempotent semigroup and e be the idempotent. Then the subset $G = Se$ satisfies the following properties.

(1) G is the compact, greatest group as well as the least ideal of S, and it holds that $G = eS$.

(2) e commutes with every $x \in S$.

(3) S is homomorphic on G.

Proof. (1) Clearly G is a compact one-idempotent subsemigroup, and so $Ge = G$ where $e \in G$. By Lemma 2', G is a compact group. Since e is the identity of the group G, we have

$$Se = G = Ge = eG \subset eS$$

Similarly $eS \subset Se$; hence $G = eS$. Next, let G_1 be any group contained in S. Considering that $e \in G_1$, we get $G_1 = G_1 e \subset Se = G$. This shows that G is greatest. Finally we shall prove G to be a least ideal. Since $SG = S(Se) \subset Se = G$ and $RS \subset G$, G is an ideal. Now let Q be any ideal of S. As Q is a subsemigroup, e belongs to Q. We have $G = Se \subset SQ \subset Q$; therefore G is least.

(2) Since e commutes with every xe of G, we get

$$xe = x(ee) = (xe)e = e(xe)$$
$$= (ex)e = e(ex) = (ee)x = ex.$$

(3) Let f be the mapping of S onto G defined as $f(x) = xe$. Then, utilizing (2), we have

$$f(x)f(y) = (xe)(ye) = x(ey)e = x(ye)e$$
$$= (xy)(ee) = (xy)e = f(xy).$$

It is clear that f is continuous. Hence f is a homomorphism. Thus the proof has been completed.

In particular, if the only one idempotent is a zero written 0, i.e., $G = \{0\}$, we call S a zero-semigroup. The above theorem becomes trivial for a zero-semigroup.

2 Topology of difference semigroup.

Let us provide the condition of regularity for the topological space S. Let X be a closed ideal of a regular semigroup S, S^* be a difference semigroup of S modulo X in the sense of Rees [5], and f be the mapping of S on S^*. Of course S^* has a zero 0^*.

We shall introduce a topology into S^*. Neighbourhoods are defined as $V(0^*) = f(Y)$ where Y is any open set containing X; and, if $x^* = f(x) \neq 0^*$, as $V(x^*) = f(N(x))$ where $N(x)$ is any neighbourhood of x in S. In this definition the axioms of Hausdorff space are all fulfilled. By the way, the regularity is effective to the separation axiom. Then, since f is easily proved to be continuous and open, we have

Lemma 3. The topological semigroup S is homomorphic on the topological semigroup S^*.

If S is a one-idempotent semigroup, then S^* is a zero-semigroup. Compactness and regularity are invariant by the continuous mapping f. Thus we have

Theorem 2. A compact regular one-idempotent semigroup is homomorphic on a compact regular zero-semigroup.

3 Compact zero-semigroup.

Let a be an element of a semigroup S. If there exists $x \in S$ such that $ax = a$, a is called an r-invariant element. Likewise a an l-invariant element if $ya = a$ for some $y \in S$; an rl-invariant element if $xay = a$ for some $x, y \in S$. These are termed generally as "invariant element".

Theorem 3. If S is a compact zero-semigroup, S contains no invariant element other than a zero. Conversely if a compact semigroup S has that property, S is a zero-semigroup.

Proof. At first we shall prove the former half of the theorem. Let us assume that a compact zero-semigroup S has an r-invariant element $a \neq 0$. Denote $X = \{x ; x \in S, ax = a\}$ for a fixed $a \neq 0$. Immediately, we see, $0 \bar{\in} X$. On the other hand, X is closed because of continuity, and if $x \in X$, $y \in X$, then $a(xy) = (ax)y = ay = a$; that is, X is a compact subsemigroup. Consequently X contains an idempotent distinct from zero. This contradicts with the definition of a zero-semigroup. The proof in the case of l-invariance is similar. When a is an rl-invariant element, that is, $xay = a$ for some $x, y \in S$, we may consider the set $X = \{x ; xay = a\}$ for a fixed $a \neq 0$.

Next, we shall prove the converse. If a compact semigroup has an idempotent x different from 0 , x is an invariant element, conflicting with the assumption. Hence S is a zero-semigroup. Thus the proof of the theorem has been accomplished.

Lemma 4. Let S be a compact zero-semigroup. If $\lim_{n\to\infty} a_n = 0$,

then $\lim_{n\to\infty} a_n x_n = 0$, and $\lim_{n\to\infty} x_n a_n = 0$

for every $\{x_n\}$.

Proof. It is sufficient to show that only one limit point of $\{a_n x_n\}$ is 0 . Let p be any limit point of $\{a_n x_n\}$, i.e., $\lim_{i\to\infty} a_{n_i} x_{n_i} = p$.

We denote by l one of limit points of $\{x_{n_i}\}$, i.e., $\lim_{k\to\infty} x_{n_{i_k}} = l$,

Then we have

$$p = \lim_{k\to\infty} (a_{n_{i_k}} x_{n_{i_k}})$$
$$= (\lim_{k\to\infty} a_{n_{i_k}})(\lim_{k\to\infty} x_{n_{i_k}}) = 0 \cdot l = 0 .$$

Therefore $\lim_{n\to\infty} a_n x_n = 0$; similarly $\lim_{n\to\infty} x_n a_n = 0$.

Lemma 5. Let $\{a_n\}$ be any sequence in the compact zero-semigroup S . Setting $b_n = a_1 a_2 \cdots a_n$, we have $\lim_{n\to\infty} b_n = 0$.

Proof. Let p be any limit point of $\{b_n\}$; $b_{n_i} \to p$, as $i \to \infty$. We rewrite $\{b_{n_i}\}$ by $\{b_i'\}$, where $b_i' = b_{n_i}$, for the sake of simplicity. Moreover set

$$b_1' = b_{n_1} = c_1$$
$$b_i' = b_{n_i} = b_{n_{i-1}} c_i \quad \text{where} \quad c_i = a_{n_{i-1}+1} a_{n_{i-1}+2} \cdots a_{n_i},$$
$$i = 2, 3, \cdots,$$

and take a subsequence $\{c_{i_j}\}$ of $\{c_i\}$ such that $c_{i_j} \to q$ ($j \to \infty$). Then $b_{i_j}' = b_{i_j - 1}' c_{i_j}$. Now, as let j attend to ∞ , we have $p = pq$. However Theorem 3 makes us to conclude that $p = 0$.

Theorem 4. If S is a compact zero-semigroup, then

$$\bigcap_{n=1}^{\infty} S^n = \{0\}.$$

Proof. Clearly $S^n \supset S^{n+1}, n = 1, 2, \cdots$. Since the sets $S^n (n = 1, 2, \cdots)$ are non-empty compact subsemigroups, the sequence of the sets has a non-empty intersection D which forms a compact subsemigroup. Let any $p \in D$, that is, $p \in S^n$ for all n , then

$$p = a_{11} = a_{12} a_{22} = \cdots = a_{1n} a_{2n} \cdots a_{nn} = \cdots ,$$

and

$$p = a_{1n} a_{2n} \cdots a_{k-1,n} b_{kn} \quad (n = 2, 3, \cdots ; k = 2, 3, \cdots, n)$$

where

$$b_{kn} = a_{kn} a_{k+1,n} \cdots a_{nn} \quad (n = k, k+1, \cdots ; k = 2, 3, \cdots, n).$$

Now we denote by $\{\nu_j^{(n)}\}$ an increasing sequence of natural numbers

$$\nu_1^{(n)} < \nu_2^{(n)} < \cdots < \nu_j^{(n)} < \cdots .$$

By dint of compactness of the space we can select so suitably a sequence of sets $\{\nu_j^{(n)}\}$ of natural numbers that

$$\{\nu_j^{(1)}\} \supset \{\nu_j^{(2)}\} \supset \cdots \supset \{\nu_j^{(n)}\} \supset \cdots, \ \nu_j^{(n)} \geq n \ (j = 1, 2, \cdots)$$

and

$$a_{i\nu_j^{(n)}} \to a_i \quad \text{as } j \to \infty$$
$$(i = 1, 2, \cdots, n ; n = 1, 2, \cdots),$$

$$b_{n+1, \nu_j^{(n)}} \to b_{n+1} \quad \text{as } j \to \infty$$
$$(n = 1, 2, \cdots).$$

Then we have

$$p = a_1 a_2 \cdots a_n b_{n+1} = c_{n+1} b_{n+1} \quad (n=1,2,\cdots),$$

where

$$c_{n+1} = a_1 a_2 \cdots a_n \ (n=1,2,\cdots)$$

and $c_{n+1} \to 0 \ (n \to \infty)$ by Lemma 5, leading to $p = 0$ by Lemma 4. Thus it has been proved that $D = \{0\}$.

Corollary Let S be a compact zero-semigroup. Every sequence $\{a_n\}$ where $a_n \in S^n$ converges to 0.

Corollary Every sequence $\{a^n\}$ in S converges uniformly to 0. In other words, a compact zero-semigroup is composed of only nilpotent elements due to 6.

From Theorem 4 we have directly

Theorem 5. If S is a non-trivial compact zero-semigroup, then S is not universal i.e. $S^2 \neq S$. Here we maen by "non-trivial" that $S \supsetneqq \{0\}$.

Proof. Suppose that $S^2 = S$, then it holds $S = S^2 = \cdots = S^n = \cdots$, that is,

$$\bigcap_{m=1}^{\infty} S^m = S .$$

But Theorem 4 makes us see that $S = \{0\}$. This is contradictory with the assumption.

4 Examples.

According to Theorem 4, we obtain two types of compact zero-semigroups S : one is said to be finitely nilpotent, i.e.,

$$S^m \neq \{0\} \ (m=1,2,\cdots,n-1), \quad S^n = \{0\},$$

where n is called the null-order or n-order of S ;
the other is said to be infinitely nilpotent i.e.,

$$S^m \supsetneqq S^{m+1}, (m=1,2,\cdots), \quad \bigcap_{m=1}^{\infty} S^m = \{0\}.$$

If $aS = Sa = \{0\}$, a is called an annihilator of S.

Theorem 6. A compact finitely nilpotent zero-semigroup S has at least one annihilator different from zero.

Proof. When $n \ (n \geq 2)$ is the n-order of S, S^{n-1} contains an annihilator distinct from zero since

$$S^{n-1} - S^n \supsetneqq \{0\}.$$

Now we give some examples of compact zero-semigroup belonging to each types.

(1) finite zero-semigroup.
A finite zero-semigroup S is finitely nilpotent and the n-order of S is not greater than its d-order. (See [1] with respect to "d-order".)

(2) infinitely nilpotent zero-semigroup - 1.
Let I be the closed interval $[0, \alpha] = [x ; 0 \leq x \leq \alpha < 1]$ and Q be the set of quaternions x such that $\|x\| \leq \alpha < 1$ for a constant α. I and Q are compact zero-semigroups under the multiplications and the topologies in ordinary sense.

(3) finitely nilpotent infinite zero-semigroup.
Let E be the totality of m-rowed square matrices, $(a_{ij})_{i,j=1,2,\cdots,m}$ whose coefficients are elements of a compact ring and which have $a_{ij} = 0$ for $i \geq j$, $i,j = 1,2,\cdots,m$. Obviously E is a compact zero-semigroup with n-order m.

(4) infinitely nilpotent zero-semigroup - 2.
Let us denote by M the set of all infinite square matrices, $(a_{ij})_{i,j=1,2,\cdots}$, whose coefficients are elements of a compact ring and which have the form

$$a_{ij} = 0 \ \text{for} \ i \geq j, \ i,j = 1,2,\cdots .$$

If we define as the product of two matrices (a_{ij}) and (b_{ij}) the matrix (c_{ij}) where

$$c_{ij} = \sum_{t=1}^{\infty} a_{it} b_{tj} ,$$

and give a topology into M as usual way, then M is a compact infinitely nilpotent zero-semigroup. We notice that M contains a finitely nilpotent subsemigroup whose n-order is arbitrary.

5 One-idempotent semigroups.

Some results in the compact zero-semigroup (§3) are extended to the case of a compact regular one-idempotent semigroup.

Theorem 7. If S is a compact regular one-idempotent semigroup and G is its greatest group, then it holds that

$$S^m \supset S^{m+1}, \ m=1,2,\cdots, \quad \bigcap_{m=1}^{\infty} S^m = G.$$

Proof. We denote $D = \bigcap_{m=1}^{\infty} S^m$. Since $S^m (m=1,2,\cdots)$ are non-empty compact sets, D is a non-empty compact set. Clearly $DS \subset D$ and $SD \subset D$; that is, D is an ideal of S. On the other hand, G is the least ideal of S; hence we have $G \subset D$.

Let f be the mapping of S on the difference semigroup S^* of S modulo G. Then it follows immediately that f maps S^m to S^{*m}, that is, $(S^m)^* = (S^*)^m$ and the sequence

$$S \supset S^2 \supset \cdots \supset S^m \supset S^{m+1} \supset \cdots$$

is mapped by f to

$$S^* \supset S^{*2} \supset \cdots \supset S^{*m} \supset S^{*m+1} \supset \cdots$$

where $f(D) = \bigcap_{m=1}^{\infty} S^{*m}$ for it holds

that $f(\bigcap_{m=1}^{\infty} S^m) = \bigcap_{m=1}^{\infty} f(S^m)$.

Now, suppose $D - G \neq \phi$, there exists $p \in D - G$. Then p is mapped by f to $p^* \in \bigcap_{m=1}^{\infty} S^{*m}$ and $p^* \neq 0^*$. Nevertheless, Theorem 4 teaches us that $\bigcap_{m=1}^{\infty} S^{*m} = \{0^*\}$. These two conflict with each other. Hence it has been proved that $D = G$.

Theorem 8. If a compact regular one-idempotent semigroup S is universal, S is a compact group.

Proof. By the previous theorem, $G = \bigcap_{m=1}^{\infty} S^m$. If $S = S^2$, then we have $S = G$, which is a compact group. (cf. Theorem 1 and Lemma 2)

References

[0] T. Tamura, On finite one-idempotent semigroups, Journal of Gakugei, Tokushima Univ., Vol.4. (in press)

[1] In this paper we mean by a topological semigroup a set S that satisfies the following conditions:

i) S is a semigroup,
ii) S is a Hausdorff space,
iii) the mapping $(a,b) \to ab$ of a space $S \times S$ into a space S is continuous.

[2] K. Numakura, On bicompact semigroups, Math. Jour. Okayama Univ., Vol.1, No.1-2 (1952), pp.99-108.

[3] Compactness and bicompactness are in accordance with the definition by L. Pontrjagin, Topological groups, p.42 and p.75.

[4] If for every neighbourhood $U(x)$ of x $(\in S)$ there exists the sequence of positive integers $n_1 < n_2 < \cdots$ such that $a^{n_i} \in U(x)$, then x is called a limit point of A.

[5] D. Rees, On semigroups, Proc. Cambridge Philos. Soc., Vol.36 (1940), pp.387-400.

[6] K. Numakura, On bicompact semigroups with zero, Bull. Yamagata Univ., No.4 (1951), pp.405-412.

Gakugei Faculty, Tokushima University.

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