A CHARACTERIZATION OF HOMOTOPICALLY LABIL POINTS

By Hiroshi NOGUCHI

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1. The concept of the labil point and the lability of a space was defined by H. Hopf and E. Pannwitz 2 . In a recent work, K. Borsuk and J. W. Jaworowski introduced an analogous concept which is called a homotopically labil point [1,pp.159-160].

The object of the present paper is to give a characterization of the homotopically labil point (in section 3). In section 2 we shall give some remarks concerning the definitions of the labil point and the homotopically labil point. In section 4 we shall show a geometrical behaviour of the homotopically labil points in 2-dimensional homogeneous complexes.

2. Let M be a space and I be the interval $0 \leqq t \leqq 1$. A mapping $f(x,t)$ which is defined in the Cartesian product $M \times I$ will be called a deformation of M whenever it satisfies the following conditions;

$$f(x,t) \in M \text{ for every } (x,t) \in M \times I$$

$$f(x,0) = x \text{ for every } x \in M.$$

The concept of the labil point owing to H. Hopf and E. Pannwitz [2, p.434] can be formulated as follows;

(2.1) DEFINITION. A point a of a space M is labil whenever for every neighbourhood U of a there exists a deformation $f(x,t)$ of M satisfying the following conditions;

(1) $f(x,t) = x$ for every $(x,t) \in (M-U) \times I$

(2) $f(x,t) \in U$ for every $(x,t) \in U \times I$

(3) $f(U,1) \neq U$.

(2.2) REMARK. We see easily that the property of being a labil point is a local one.

The concept of the lability of a space owing to H. Hopf and E. Pannwitz [2,p.434] can be formulated as follows;

(2.3) DEFINITION. A metric space M is labil if for every $\varepsilon > 0$ there exists a deformation $g(x,t)$ of M satisfying the following conditions;

(4) $\rho(x, g(x,t)) < \varepsilon$ for every $(x,t) \in M \times I$,

(5) $g(M,1) \neq M$,

where ρ denotes the metric of M .

(2.4) REMARK. It is remarked by H. Hopf and E. Pannwitz [2,p.434] that for compactum M , M is labil if, and only if, M has at least one labil point. In this statement the assumption that M is compact cannot be removed. In fact, consider in the euclidean plane the set $M = S - (1,0)$, where S denotes the unit circle having $(0, 0)$ as center. Then M is labil, but has not any labil point.

On the other hand, this example also shows that for non-compact space the lability is not a local property. In fact, M is locally homeomorphic to S, but S is evidently not labil.

The concept of the homotopically labil point owing to K. Borsuk and J. W. Jaworowski [1] can be formulated as follows;

(2.5) DEFINITION. A point a of a space M is homotopically labil whenever for every neighbourhood U of a there exists a deformation $f(x,t)$ of M satisfying the following conditions;

(6) $f(x,t) = x$ for every $(x,t) \in (M-U) \times I$,

(7) $f(x,t) \in U$ for every $(x,t) \in U \times I$

(8) $f(x,1) \neq a$ for every $x \in M$.

It is remarked by K. Borsuk and J. W. Jaworowski [1,p.160] that for metric spaces the definition 1 is equivalent to the following one;

(2.6) DEFINITION. A point a of a metric space M is homotopically labil, if for every $\varepsilon > 0$ there exists a deformation $g(x,t)$ of M satisfying the following conditions;

(9) $\rho(x, g(x,t)) < \varepsilon$ for every $(x,t) \in M \times I$

(10) $g(x,1) \neq a$ for every $x \in M$.

A point $a \in M$ will be called homotopically stabil if it is not homotopically labil.

(2.7) REMARK. K. Borsuk and J. W. Jaworowski [1,p.160] show that the property of being a homotopically labil point is a local one.

(2.8) REMARK. It is evident that a point $a \in M$ homotopically labil is necessarily labil. But even for compact absolute retract the inverse is not true. In fact, consider two triangles whose common part is a vertex a. We can easily verify that a is labil, but is not homotopically labil.

3. Now we shall investigate the properties of homotopically labil and homotopically stabil points using the concepts of algebraic topology. In the remaining part of this paper, whenever we do not state other wise, spaces which we consider are always locally finite and finite dimensional complexes. In complex A, a subcomplex which is constituted by all simplexes having the vertex a as vertex is known as the star of a, and the set of all simplexes which is contained in the star of a not having the vertex a as vertex is known as the neighbourhood complex of a.

(3.1) THEOREM. Let A be a complex. A point a of A is homotopically labil if, and only if, there exists a contractible neignbourhood complex.

PROOF. Necessity; We first recall that a finite complex B is contractible if, and only if, every homology group of B is zero and fundamental group of B is unity.

Hence if there exists a neighbourhood complex B of a which is not contractible, either there exists at least one cycle Z of B which is not homologous to zero in B or there exists at least one closed path z of B which is not homotopic to zero in B.

If B has a cycle Z as above and C be the star of a in A having B in the boundary, then a is linked in C with B [1,p.163]. In fact, Z is evidently homologous to zero in C. Since B is a deformation retract of $C-(a)$, Z is not homologous to zero in any complex which is contained in $C-(a)$.

By the method of K. Borsuk and J. W. Jaworowski [1,p.166] we see that a is homotopically stabil.

The same is true in the case B has a closed path z which is not homotopic to zero in B. In fact, from the same reasons as above it follows that z is homotopic to zero in C and not homotopic to zero in $C-(a)$.

If a were homotopically labil in A, then there would exists a deformation $f(x,t)$ of A such that

(11) $f(x,t) = x$
for every $(x,t) \in (M - \text{interior of } C) \times I$

(12) $f(x,t) \in$ interior of C,
for every $(x,t) \in (\text{interior of } C) \times I$

(13) $f(x,1) \neq a$ for every $x \in A$.

It follows by (12) $f(z,1) = z$. Since $f(x,t)$ is a deformation, $f(x,1)$ is then homotopic to the identity. Since z is homotopic to zero in C, then $f(z,1)$ is homotopic to zero in $f(C,1)$. But by (11), (12) and (13) $f(C,1) \subset C-(a)$. Hence $f(z,1)=z$ would be homotopic to zero in the set $C-(a)$. This contradicts the assumption that z is not homotopic to zero in $C-(a)$.

Sufficiency; If $h(x,t)$ denotes a contraction of B, $h(x,t)$ satisfies the following conditions;

$h(x,0) = x$ for every $x \in B$,

$h(x,t) \in B$ for every $x \in B$ and for every $t : 0 \leqq t \leqq 2$,

$h(x,t) = b$ for every $x \in B$ and for every $t : 1 \leqq t \leqq 2$

where b is a fixed point of B.

For each point $x \in B$, if we consider the segment $\overline{xa}$ and the point $\varphi(x,t')$ which dividing the segment $\overline{xa}$ in the ratio $t'/(1-t')$, $\varphi(x,t') = \varphi_{t'}$ for a fixed t', $0 \leqq t' < 1$,

and for every $x \in B$, is a topological mapping of B into C, by $B_{t'}$ we denote the set $\varphi_{t'}(B)$ and by B_1 we denote a.

Now putting for every point of $B_{t'}$, $0 \leqq t' < 1$

$$h'(x,t) = \varphi_{t'}\ h(\varphi_{t'}^{-1}(x), 2tt') \quad \text{for every } t,\ 0 \leqq t \leqq 1$$

and $h'(x,t) = a$ for $x \in B_1$, and for every $0 \leqq t \leqq 1$, we obtain a deformation $h'(x,t)$ of C which remains fixed each point of B such that for every t', $\frac{1}{2} \leqq t' \leqq 1$, $h'(x,t)$ maps $B_{t'}$ in a point $\varphi_{t'}(b)$ which lies on the segment $\overline{a, \varphi_{\frac{1}{2}}(b)}$

Let $\eta(x,t)$ be a deformation of $\overline{a, \varphi_{\frac{1}{2}}(b)}$ which remains fixed $\varphi_{\frac{1}{2}}(b)$ such that $\eta(x,1) \neq a$ for every $x \in \overline{a, \varphi_{\frac{1}{2}}(b)}$.

Then putting

$$h^*(x,t) = \begin{cases} h'(x,2t) & \text{for every } x \in C \text{ and for every } t,\ 0 \leqq t \leqq \frac{1}{2}, \\ h'(x,1) & \text{for every } x \in B_{t'} \text{ where } 0 \leqq t' \leqq \frac{1}{2}, \text{ and for every } t,\ \frac{1}{2} \leqq t \leqq 1 \\ \eta(h'(x,1), 2t-1) & \text{for every } x \in B_{t'} \text{ where } \frac{1}{2} \leqq t' \leqq 1 \text{ and for every } t,\ \frac{1}{2} \leqq t \leqq 1 \end{cases}$$

we obtain a deformation of C which leaves each point fixed of B and satisfies the following condition;

$$h^*(x,1) \neq a \quad \text{for every } x \in C.$$

Then putting

$$f(x,t) = \begin{cases} h^*(x,t) & \text{for every } (x,t) \in C \times I \\ x & \text{for every } (x,t) \in (A - C) \times I \end{cases}$$

we obtain a deformation satisfying the following conditions;

$f(x,t) = x$ for every $(x,t) \in (A -$ interior of $C) \times I$

$f(x,t) \in$ interior of C for every $(x,t) \in$ (interior of $C) \times I$

$f(x,1) \neq a$ for every $x \in A$.

For every neighbourhood U of a there exists the star C of a such that $U \supset C$ and C is contractible by the assumption. By the above arguments there exists a deformation of A which satisfies all conditions of the definition (2.5). This completes the proof of the theorem.

We can easily see that if there exists a contractible neighbourhood complex of a, then every neighbourhood complex of a is contractible.

By the definition (2.6) the following is obtained from Theorem (3.1).

(3.2) THEOREM. Let A be a complex. A point a of A is homotopically stabil if, and only if, there exists a neighbourhood complex of a which is not contractible.

We have easily seen that the neighbourhood complex B of a is contractible if, and only if, $C-(a)$ is contractible, where C is the star of a which has B in the boundary.

Then we can restate Theorem (3.1) and Theorem (3.2) in the following forms.

(3.3) THEOREM. Let A be a complex. A point a of A is homotopically labil if, and only if, there exists a star C of a such that $C - (a)$ is contractible.

(3.4) THEOREM. Let A be a complex. A point a of A is homotopically stabil if, and only if, there exists a star C of a such that $C-(a)$ is contractible.

4. It is evident that if a complex A has a free simplex [2,p.434], and A has homotopically labil points.

If A is 1-dimensional, the inverse is also true [2,Satz I] (In this case, the definitions (2.1), (2.5) and (2.6) are equal.)

(4.1) THEOREM. For every homogeneous 2-dimensional complex A the existence of at least one homotopically labil point is equivalent to the existence of at least one free side.

PROOF. In virtue of Theorem (3.1) the homotopically labil point a of A has a tree as the neighbourhood complex. Since the tree has at least one free vertex p, then the 1-dimensional simplex $\overline{pa}$ is the free side of A.

(4.2) REMARK. Theorem (4.1) is not true for 3-dimensional case. In fact, consider in 3-dimensional euclidean space E^3 the 2-dimensional contractible complex C^2 which is constructed by H. Hopf and E. Pannwitz [2,p.448-449. example C]. In appropriately high dimensional euclidean space which contains E^3 as subspace we choose two points a_1 and a_2 such that every segment $\overline{a_i x}$, for $i=1$, 2 and for every $x \in C^2$, does not meet each other with the exception of points a_i ($i=1$, 2) and points of C^2.

If we consider an 3-dimensional complex D^3 which is constituted by all segments $\overline{a_i x}$ for $i=1$, 2 and for every $x \in C^2$. D^3 is the required example. In fact, by Theorem (3.1) we see a_i, $i=1$, 2 are homotopically labil points of D^3 but it is evident that D^3 has not any free 2-dimensional simplex.

(4.3) REMARK. I could not solve the problem of the invariance of the homotopically stability of points under Cartesian multiplication [1, p.164]. On this problem, in virtue of Theorem (3.4), roughly speaking, we have a homotopical problem: let X and Y be contractible spaces and a be a point of X and b be a point of Y such that $X-(a)$ and $Y-(b)$ are not contractible, is $X \times Y-(a,b)$ not contractible?

It seems to me that this problem is in general very difficult.

(1) K. Borsuk and J. W. Jaworowski; On labil and stabil points, Fund. Math., 39(1952), pp.159-175..
(2) H. Hopf und E. Pannwitz; Über stetige Deformationen von Komplexen in sich, Math. Ann., 108 (1933), pp.433-465.

Mathematical Institute Waseda University, Tokyo.

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