ON SOME BOUNDARY VALUE PROBLEM IN AN ANNULUS

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In this paper, we shall give a method to solve a boundary value problem in an annulus. Consider an annulus

$$N: \qquad 1 < |z| < R,$$

whose boundary consists of two circles

$$C_1: \quad z = e^{i\theta}, \qquad C_2: \quad z = Re^{i\theta}$$

Let $E_1 = \{E_{1n}\}$ and $E_2 = \{E_{2n}\}$ be sets of enumerable open circular arcs having no common parts on C_1 and C_2, respectively, and let their complementary sets be $\bar{E}_1 = C_1 - E_1$ and $\bar{E}_2 = C_2 - E_2$. We assume that $\bar{E}_1$ and $\bar{E}_2$ are of capacity zero. We divide $\{E_{jn}\}$, $j=1,2$, into two classes, $\{A_{jn}\}$ and $\{B_{jn}\}$, in an arbitrary way.

Problem I: To construct a function $f(z)$ which is harmonic in the annulus N, and satisfies the following boundary conditions:

$$(1)\begin{cases} \lim\limits_{r\to 1} \dfrac{\partial f(re^{i\theta})}{\partial r} = A_1(\theta) & \text{for } e^{i\theta} \text{ on } \{A_{1n}\}, \\ \lim\limits_{r\to R} \dfrac{\partial f(re^{i\theta})}{\partial r} = \dfrac{1}{R}A_2(\theta) & \text{for } Re^{i\theta} \text{ on } \{A_{2n}\}, \\ \lim\limits_{r\to 1} f(re^{i\theta}) = B_1(\theta) & \text{for } e^{i\theta} \text{ on } \{B_{1n}\}, \\ \lim\limits_{r\to R} f(re^{i\theta}) = B_2(\theta) & \text{for } Re^{i\theta} \text{ on } \{B_{2n}\}, \end{cases}$$

where $A_j(\theta)$ and $B_j(\theta)$ are the given functions bounded and continuous on respective sets.

Now we replace our problem I in an annulus with an equivalent one in a half plane. For that purpose, we cut the annulus along the negative real axis, namely we restrict θ to vary within the interval $(-\pi, \pi)$. Let this annulus with the cut be N^*. By the conformal mapping

$$(2) \qquad t = \exp\{i\pi \log z / \log R\},$$

the annulus N^* is mapped onto the domain

$$D: \quad \frac{1}{P} < \rho < P, \qquad v > 0,$$

where $t = \rho e^{i\varphi} = u + iv$ and $P = \exp(\pi^2/\log R)$. This domain D has two kinds of boundaries:

i) Segments of real axis,

$$v = 0, \quad \frac{1}{P} \leqq u \leqq P \quad ;$$

$$v = 0, \quad -P \leqq u \leqq -\frac{1}{P},$$

which correspond to the circular boundaries of N^*.

ii) Two upper semi-circles

$$\rho = \frac{1}{P}, \qquad v > 0 \quad ;$$

$$\rho = P, \qquad v > 0,$$

which correspond to the upper and lower banks of the cut respectively.

By the conformal mapping (2), our problem is transformed into the following one.

Problem II: To find a function $F(t)$ which is harmonic in D and satisfies the following boundary conditions:

1°. At the boundary points lying on the real axis, the boundary values of the function itself and of its normal derivative are equal to the values corresponding to the original ones, respectively,

$$\left\{ \lim_{\varphi\to 0} \frac{\partial F(\rho e^{i\varphi})}{\partial \varphi} = \frac{\pi}{\log R} A_1\left(-\frac{1}{\pi}\log R \log \rho\right) \quad \text{for } u = \rho \text{ on } \{a_{1n}\}, \right.$$

$$
(3)\begin{cases}
\lim\limits_{\varphi\to\pi}\dfrac{\partial F(\rho e^{i\varphi})}{\partial\varphi}=\dfrac{\pi}{\log R}A_2(-\frac{1}{\pi}\log R\log\rho) \\
\qquad\text{for } u=-\rho \text{ on } \{a_{2n}\}, \\
\lim\limits_{\varphi\to 0}F(\rho e^{i\varphi}) = B_1(-\frac{1}{\pi}\log R\log\rho) \\
\qquad\text{for } u=\rho \text{ on } \{b_{1n}\}, \\
\lim\limits_{\varphi\to\pi}F(\rho e^{i\varphi}) = B_2(-\frac{1}{\pi}\log R\log\rho) \\
\qquad\text{for } u=-\rho \text{ on } \{b_{2n}\}
\end{cases}
$$

where $\{a_{jn}\}$ and $\{b_{jn}\}$ are the images of $\{A_{jn}\}$ and $\{B_{jn}\}$, respectively, by the mapping (2).

2°. On the semi-circular boundaries, $F(t)$ has the same behavior at the congruent points, i.e.

$$
(4)\begin{cases}
F(P e^{i\varphi})=F(\frac{1}{P}e^{i\varphi}), \\
\lim\limits_{\rho\to P}\rho\dfrac{\partial F(\rho e^{i\varphi})}{\partial\rho}=\lim\limits_{\rho\to 1/P}\rho\dfrac{\partial F(\rho e^{i\varphi})}{\partial\rho},
\end{cases}
$$

for any φ with $0<\varphi<\pi$.

To solve the problem II, we prolongate $F(t)$ harmonically throughout the whole upper half plane of t. This prolongation may be performed as follows. We consider the dilatation $T:\ t_1=P^2t$, and its powers T^m, ($m=0$, ± 1, ± 2, ...). Then, the whole upper half plane may be filled up with the images of the fundamental domain D by T^m. We can define a function $G(t)$ in the whole upper half plane by the relation

$$
(5)\quad G(T_t^m)=G(t),\qquad (m=\pm 1,\pm 2,\cdots)
$$

and $G(t)=F(t)$ in the fundamental domain. In fact, the function defined in such a way gives our required prolongation since the boundary condition 2° holds.

Thus our problem has been transformed into the following one in a half plane as we intended:

Problem III: To construct a function $G(t)$ which is harmonic in the upper half plane and satisfies the boundary conditions:

$$
(6)\begin{cases}
\lim\limits_{\varphi\to 0}\dfrac{\partial G(P^{2m}\rho e^{i\varphi})}{\partial\varphi}=A_1(-\frac{1}{\pi}\log R\log\rho) \\
\qquad\text{for } u=\rho \text{ on } \{a_{1n}\}, \\
\lim\limits_{\varphi\to\pi}\dfrac{\partial G(P^{2m}\rho e^{i\varphi})}{\partial\varphi}=A_2(-\frac{1}{\pi}\log R\log\rho) \\
\qquad\text{for } u=-\rho \text{ on } \{a_{2n}\}, \\
\lim\limits_{\varphi\to 0}G(P^{2m}\rho e^{i\varphi})=B_1(-\frac{1}{P}\log R\log\rho) \\
\qquad\text{for } u=\rho \text{ on } \{b_{1n}\}, \\
\lim\limits_{\varphi\to\pi}G(P^{2m}\rho e^{i\varphi})=B_2(-\frac{1}{\pi}\log R\log\rho) \\
\qquad\text{for } u=-\rho \text{ on } \{b_{2n}\}, \\
\qquad m=0,\pm 1,\pm 2,\cdots
\end{cases}
$$

To solve the problem III, we proceed in several steps.[1)]

Let $\widetilde{A}_j(\rho)$ be an arbitrary continuous function on $\{b_{jn}\}$ which is bounded and satisfies the condition

$$
(7)\quad \int_{\{b_{jn}\}}\widetilde{A}_j(\rho)\frac{d\rho}{\rho}+\int_{\{a_{jn}\}}A_j(-\frac{1}{\pi}\log R\log\rho)\frac{d\rho}{\rho}=0,
$$

$(j=1,2)$.

Denote by $\{L_j\}$ the sets $\{a_{jn}\}$ together with its congruent intervals by the transformations T^m, similarly, by $\{K_j\}$ the set $\{b_{jn}\}$ and its congruent intervals. We then define a function $\alpha_j(\alpha)$ on $\{L_j\}$ and $\{K_j\}$ such that

$$
\begin{cases}
\alpha_1(P^{2m}\rho)=A_1(-\frac{1}{\pi}\log R\log\rho) \\
\qquad\text{for } u=\rho \text{ on } \{a_{1n}\}, \\
\alpha_2(P^{2m}\rho)=A_2(-\frac{1}{\pi}\log R\log\rho) \\
\qquad\text{for } u=-\rho \text{ on } \{a_{2n}\}, \\
\alpha_1(P^{2m}\rho)=\widetilde{A}_1(\rho) \\
\qquad\text{for } u=\rho \text{ on } \{b_{1n}\}, \\
\alpha_2(P^{2m}\rho)=\widetilde{A}_2(\rho) \\
\qquad\text{for } u=-\rho \text{ on } \{b_{2n}\}, \\
\qquad m=0,\pm 1,\pm 2,\cdots
\end{cases}
$$

In the first step, we shall deal with the special case where the sets $\{a_{jn}\}$ are reduced to null set, that is, the Neumann problem:

Problem III*: To find a function $G_1(t)$ which is harmonic in the upper half plane and satisfies the boundary conditions

$$(8)\quad \begin{cases} \lim\limits_{\varphi\to 0} \dfrac{\partial G_1(\rho e^{i\varphi})}{\partial\varphi} = \alpha_1(\rho) , \\ \lim\limits_{\varphi\to \pi} \dfrac{\partial G_1(\rho e^{i\varphi})}{\partial\varphi} = \alpha_2(\rho) . \end{cases}$$

This problem may be solved as follows. First, we solve a Dirichlet problem, which is to find a function $G_2(t)$ harmonic in the upper half plane, and satisfies the boundary conditions

$$(9)\quad \begin{cases} \lim\limits_{\varphi\to 0} G_2(\rho e^{i\varphi}) = \alpha_1(\rho) , \\ \lim\limits_{\varphi\to \pi} G_2(\rho e^{i\varphi}) = \alpha_2(\rho) . \end{cases}$$

Since the bounded continuous functions $\alpha_j(\rho)$ are defined on the whole real axis except for the set of points of capacity zero, this problem can be solved by a well known way. We define again another function $G_3(t)$ by

$$(10)\qquad G_3(t) = \int_{\pi/2}^{\varphi} G_2(\rho e^{i\varphi})\, d\varphi .$$

We see easily

$$(11)\qquad \lim_{\varphi\to 0}\frac{\partial G_3}{\partial\varphi} = \alpha_1(\rho) , \quad \lim_{\varphi\to\pi}\frac{\partial G_3}{\partial\varphi} = \alpha_2(\rho) ,$$

and moreover

$$(12)\qquad \Delta G_3 = \frac{1}{\rho^2}\left(\frac{\partial G_2}{\partial\varphi}\right)_{\varphi=\pi/2} .$$

As the right-hand member of (12) is a function of ρ alone, we denote it by $g(\rho)$. Next, we solve an ordinary differential equation

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dy}{d\rho}\right) = -g(\rho) .$$

a solution of which is obtained by

$$G_4(\rho) = -\int_1^{\rho}\frac{dw}{w}\int_1^{w} s\, g(s)\, ds .$$

which obviously satisfies the relation $\Delta G_4(\rho) = -g(\rho)$. Therefore

$$\Delta (G_3 + G_4) = 0 .$$

Moreover $\partial G_4/\partial\varphi = 0$. Remembering of

the relation (11), we thus get the solution of the problem III* as follows

$$(13)\quad G_1(t) = G_3(t) + G_4(t) .$$

Now we proceed to the second step. Define a function $\beta_j(\rho)$ by the relation

$$\beta_j(p^{2m}\rho) = B_j\left(-\frac{1}{\pi}\log R \log\rho\right) .$$

Let $\gamma_j(\rho)$ be the value of the function $G_1(t)$ on $\{K_j\}$. We shall solve a Dirichlet problem to find a function $G_5(t)$, harmonic in the whole t-plane except for $\{K_j\}$. On the both sides of $\{K_j\}$, G_5 should satisfy the boundary conditions

$$\lim_{\varphi\to 0} G_5(\rho e^{i\varphi}) = \lim_{\varphi\to 2\pi} G_5(\rho e^{i\varphi})$$
$$= \beta_1(\rho) - \gamma_1(\rho) ,$$
$$\lim_{\varphi\to \pi} G_5(\rho e^{i\varphi}) = \lim_{\varphi\to -\pi} G_5(\rho e^{i\varphi})$$
$$= \beta_2(\rho) - \gamma_2(\rho) .$$

By the symmetry character, we readily see

$$\frac{\partial G_5(\rho e^{i\varphi})}{\partial\varphi} = 0 \qquad \text{on } \{L_j\} .$$

Finally, construct a function

$$G(t) = G_1(t) + G_5(t) ,$$

which is evidently the solution of our problem III. Consequently, by merely transforming the variable, we have obtained a representation for our original problem I.

The uniqueness of our problem could be shown in the almost same way as L. Myrberg did for the problem in the unit circle.

REFERENCE

(1) In the following arguments, the author follows the method due to Lauri Myrberg, Über die vermischte Randwertaufgabe der harmonischen Funktionen. Ann. Acad. Sci. Fenn. 1951. 8 pp.

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